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Abstract. The moduli space $\mathcal{M}_g$ of surfaces of genus $g \geq 2$ is the space of conformal equivalence classes of closed Riemann surfaces of genus g. This space is a complex, quasi-projective variety of dimension $3g-3$. The singularity set of the moduli space, which is roughly the same as the *branch locus*, becomes increasingly complicated as the genus grows. To better understand the branch locus, the moduli space may be stratified into a finite, disjoint union of smooth, irreducible, quasi-projective subvarieties called *equisymmetric strata*. Each stratum corresponds to a collection of surfaces of the same *symmetry type*.

The topology of these strata is largely unknown. In this paper we explore the topology of the complex 1-dimensional strata, which are smooth, connected, complex curves with punctures. We are able to describe the topology of these strata explicitly, as punctured Riemann surfaces, in terms of the action of the automorphism group of the surfaces in the stratum.

1. Introduction

The moduli space $\mathcal{M}_g$ of surfaces of genus $g \geq 2$ is the space of conformal equivalence classes of closed Riemann surfaces of genus g. This space is a complex quasi-projective variety of dimension $3g-3$, whose singularity set (roughly the branch locus) becomes increasingly complicated as the genus grows. The moduli space admits a stratification into a finite, disjoint union of smooth, irreducible, quasi-projective subvarieties called *equisymmetric strata* (see Section 2.3 and references [**Bro**] and [**Ha**]). Each stratum corresponds to a collection of surfaces of the same *symmetry type*, an invariant of the automorphism group of the surface (see Section 2.3). In particular, the automorphism groups of the surfaces lying in a stratum are all isomorphic to a fixed group G.

The dimensions of the strata vary from zero for quasi-platonic surfaces to $2g-1$ for the hyperelliptic locus. Aside from these two types of strata, the topology of strata in the intermediate dimensions is largely unknown. We are going to focus on the "next step", i.e., one-dimensional strata, which are punctured Riemann surfaces. The punctures on the strata will be of two types: interior punctures corresponding to surfaces with exceptional symmetries and punctures at infinity corresponding to limiting stable, nodal Riemann surfaces in the compactification of $\mathcal{M}_g$ (see [**Be**]).

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Let $\mathcal{S}$ be a stratum of dimension one and select a surface $S \in \mathcal{S}$. There are two separate cases to consider:

Case 1: $S/\mathrm{Aut}(S)$ is the Riemann sphere and $S \to S/\mathrm{Aut}(S)$ is branched over four points. We work out this case in Section 4.

Case 2: $S/\mathrm{Aut}(S)$ is a genus one Riemann surface (elliptic curve) and $S \to S/\mathrm{Aut}(S)$ is branched over one point. This case is beyond the scope of the paper, but is a good area for future work.

Our overall strategy is based on the following idea, which works for any stratum in any moduli space. For $S \in \mathcal{S}$, we set $T = S/\mathrm{Aut}(S)$. As S varies within $\mathcal{S}$ the conformal equivalence class of T varies in some simpler moduli space $\mathcal{M}_Q$, which we call the *moduli space of quotients*. The map $q : \mathcal{S} \to \mathcal{M}_Q$ has finite fibres. So we can break up the analysis of a stratum into two components:

(1) a *continuous component*, namely $\mathcal{M}_Q$, which we also call the *horizontal component* and
(2) a *discrete component*, the fibres of q, which we also call the *vertical component*.

In our case the horizontal component is the moduli space of a sphere with four distinguished points. Once the conformal class of the orbifold surface $T = S/\mathrm{Aut}(S)$ is fixed, the vertical component is dependent on the group structure of G. The analysis of this component is a major focus of this paper.

Unfortunately, for actual computations, $q : \mathcal{S} \to \mathcal{M}_Q$ does not work well. However we can solve the problem by going to a finite "cover". We will construct a component $\widetilde{\mathcal{S}}$ of a Hurwitz space $\mathcal{H}$ and a map $\Theta : \widetilde{\mathcal{S}} \to \overline{\mathcal{S}}$, where $\overline{\mathcal{S}}$ is the closure of $\mathcal{S}$ in $\mathcal{M}_g$. The stratum cover admits a well behaved decomposition as in (1) and (2) above and we can compute the topology of $\widetilde{\mathcal{S}}$ easily. In addition, we are able to create a locally trivial, holomorphic family of surfaces $\left\{S_a : a \in \widetilde{\mathcal{S}}\right\}$, such that S_a belongs to the conformal class of surfaces represented by a. This family allows us to analyze the variation in geometry and topology of surfaces as we move around in a stratum.

To summarize, we follow these steps.

I. Find a suitable model B for $\mathcal{M}_Q$, the moduli space of quotients.
II. Build a cover $\widetilde{\mathcal{S}} \to B$ and the family $\left\{S_a : a \in \widetilde{\mathcal{S}}\right\}$.
III. Define the map $\Theta : \widetilde{\mathcal{S}} \to \overline{\mathcal{S}}$, by $a \to$ conformal class of S_a.
IV. Determine the redundancy of Θ, i.e., the failure of Θ to be 1-1.
V. Analyze the cover $\widetilde{\mathcal{S}} \to B$ to determine the topology of $\widetilde{\mathcal{S}}$.

In this paper, we implement the strategy above in a comprehensive way that works for any group G, as long as one can perform the computations in the group G. We present fairly complete results for *pure braid* strata (Definition 3.6), i.e., when the map Θ is 1-1. Our focus on the pure braid case is motivated by its simplicity of analysis and the ubiquitousness of such strata as the genus increases. We present evidence of this ubiquitousness in Example 3.12. We were not able to prove all the aspects of our program, future work is outlined in Conjecture 4.6.

We believe that our strategy is new, especially that it is comprehensive. Our main result is Theorem 4.1 for computing the topology of $\widetilde{\mathcal{S}}$ and the methods of calculation are new. Many of the technical tools to analyze redundancy appear to be new. A number of our examples are new. Checking maximality of the $(5, 5, 5, 5)$

action for A_5 in genus 37 is a new, illustrative example. The examples for prior related results have often been for specific cases, or other properties of strata, not specifically determining the topology of the strata. See [**CG**], [**CIR**], [**CIY**], and [**MSSV**]. In [**MSSV**] all the strata for genus up to 10 were determined, though only the dimensions of the strata and the signatures of the group actions.

1.1. Overview of sections. In Section 2 we introduce the necessary background for the development of the paper. In Section 3 we discuss the Hurwitz space $\mathcal{H}$, the component $\widetilde{\mathcal{S}}$ and the map $\Theta : \widetilde{\mathcal{S}} \to \overline{\mathcal{S}}$. It is no more effort to discuss general case of t branch points, so we carry out the construction for this general case. We also discuss the pure vs non-pure cases, and carry out a detailed analysis of the vertical component described above. A large number of examples are given to illustrate the ideas.

As promised, in Section 4 we discuss the orbit genus 0 case with 4 branch points, completely in the pure braid case.

Many of the computations for examples were performed using Magma [**Ma**]. The code to produce the examples is available at [**TGAS**].

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2. Preliminaries

2.1. Coverings and group actions. Let S be a compact Riemann surface. The finite group G acts conformally on S if there is a monomorphism $\epsilon : G \to \mathrm{Aut}(S)$. The quotient surface $S/G = T$ is a closed Riemann surface of genus h with a unique conformal structure so that

$$\pi_G : S \to S/G = T$$

is holomorphic. The quotient map $\pi_G : S \to T$ is branched over a finite set $B_G = \{Q_1, \ldots, Q_t\}$ such that π_G is an unbranched covering over $T^\circ = T - B_G$. We also use the notation B_π for $\pi = \pi_G$, depending on how the action is defined. Let $S^\circ = \pi_G^{-1}(T^\circ)$ so that $\pi_G : S^\circ \to T^\circ$ is an unbranched covering whose group of deck transformations are automorphisms of S°, each of which may be completed to an automorphism of S. In turn, the deck transformations $\mathrm{Gal}(\pi_G)$ may be identified with G via ϵ:

$$\epsilon : G \to \mathrm{Gal}(\pi_G) = \{\theta \in \mathrm{Aut}(S) : \pi_G \circ \theta = \pi_G\}\,. \tag{2.1}$$

For fundamental group calculations we pick $Q_0 \in T^\circ$ and let $\widetilde{Q}_0 \in S^\circ$ be some point lying over Q_0.

We summarize the foregoing in the following proposition, which may be proved using standard covering space theory (see [**Sp**] or [**Spr**]).

PROPOSITION 2.1. *The action $\epsilon : G \to \mathrm{Aut}(S)$ and covering $\pi_G : S^\circ \to T^\circ$ determines a normal subgroup $\Pi_G = \pi_1(S^\circ, \widetilde{Q}_0) \lhd \pi_1(T^\circ, Q_0)$ and an exact sequence:*

$$\Pi_G \hookrightarrow \pi_1(T^\circ, Q_0) \overset{\xi}{\twoheadrightarrow} G. \tag{2.2}$$

The epimorphism $\xi : \pi_1(T^\circ, Q_0) \twoheadrightarrow G$ is the monodromy map $\mu : \pi_1(T^\circ, Q_0) \to \mathrm{Gal}(\pi_G)$ composed with ϵ^{-1}, i.e., $\xi = \epsilon^{-1} \circ \mu$.

Conversely, given an epimorphism as in (2.2), ξ induces a cover $\pi : S \to T$ branched over B_G, such that G acts on S by deck transformations of the cover π, i.e., $\epsilon = \mu \circ \xi^{-1}$. With this action, $\pi = \pi_G$.

Remark 2.2. We call the map ξ in (2.2) a *monodromy epimorphism* or a *monodromy* for short. If $\theta \in \mathrm{Aut}(G)$ then the monodromy epimorphism $\xi' = \theta \circ \xi$ is associated with the action $\epsilon' = \epsilon \circ \theta^{-1}$. Concerning the actions, we have $\epsilon(G) = \epsilon'(G)$ and hence both actions determine the same branched covering $S \to S/G$. On the other hand, the monodromy epimorphisms ξ and $\xi' = \theta \circ \xi$ both determine the same kernel Π_G, hence the constructed covering surfaces $S \to T$ are the same.

The epimorphisms ξ and $\theta \circ \xi$ are called $\mathrm{Aut}(G)$-equivalent epimorphisms. Inequivalent epimorphisms, of which there can be many, yield distinct covering surfaces lying over (T, B_G). If we need to track the dependence of Π_G and π_G on ξ, we use the notation Π_ξ and π_ξ. We denote by

$$\xi^{\alpha G} = \{\theta \circ \xi : \theta \in \mathrm{Aut}(G)\} \tag{2.3}$$

the $\mathrm{Aut}(G)$ equivalence class of the epimorphism ξ. These classes are in 1-1 correspondence to kernels of epimorphisms and equivalence classes of covers $\pi : S \to T$ branched over B_G.

2.2. Generating vectors and signatures. Let T be a compact surface of genus h, let $B_G = \{Q_1, ..., Q_t\}$ be a finite set in T, and let $T^\circ = T - B_G$. If $Q_0 \in T^\circ$, a well-known, canonical set of generators, $\mathcal{G}$, and relations for $\pi_1(T^\circ, Q_0)$ is:

$$\left\langle \alpha_1, ..., \alpha_h, \beta_1, ..., \beta_h, \gamma_1, ..., \gamma_t : \prod_{i=1}^{h} [\alpha_i, \beta_i] \prod_{j=1}^{t} \gamma_j = 1 \right\rangle, \tag{2.4}$$

where the loop γ_j starts at the base point Q_0, encircles Q_i in a small counterclockwise loop and returns to Q_0 along the original path. The γ_j issue from Q_0, in distinct directions in cyclic counterclockwise order. Note that the ordering of the Q_j is important for the construction of $\mathcal{G}$.

Given a finite group G, a homomorphism $\xi : \pi_1(T^\circ, Q_0) \to G$ is given by a vector:

$$\begin{aligned} \mathcal{V} &= (\xi(\alpha_1), \ldots, \xi(\alpha_h), \xi(\beta_1), \ldots, \xi(\beta_h), \xi(\gamma_1), \ldots, \xi(\gamma_t)) \\ &= (a_1, \ldots, a_h, b_1, \ldots, b_h, c_1, \ldots, c_t) \in G^{2h+t}. \end{aligned} \tag{2.5}$$

The a_i, b_i and c_j must satisfy the relation implied by the presentation (2.4). If ξ is an epimorphism, i.e., $\langle a_1, \ldots, a_h, b_1, \ldots, b_h, c_1, \ldots, c_t \rangle = G$, we say that $\mathcal{V}$ is a *generating vector* for the G action on S, lying over (T, B_G) and determined by ξ. Let

$$m_j = o(c_j), \tag{2.6}$$

then $(h; m_1, \ldots, m_t)$ is called the *signature* of the action and the generating vector.

Once the generating set for $\pi_1(T^\circ, Q_0)$ is selected, the G actions on S lying over (T, B_G) with signature $(h; m_1, \ldots, m_t)$ and the associated epimorphisms are in 1-1 correspondence with the generating vectors with signature $(h; m_1, \ldots, m_t)$. For instance, mimicking (2.3) we can define $\mathrm{Aut}(G)$ classes of generating vectors

$$\mathcal{V}^{\alpha G} = \{\theta \cdot \mathcal{V} : \theta \in \mathrm{Aut}(G)\}, \tag{2.7}$$

where $(\theta, \mathcal{V}) \to \theta \cdot \mathcal{V}$ is the θ action on each component.

A surface S upon which G acts via an epimorphism ξ or generating vector $\mathcal{V}$ is called an *action surface.* The genus of an action surface can be determined from the signature of the vector using the Riemann Hurwitz formula:

$$g = 1 + \frac{1}{2}\left(|G|\,(2h+t-2) - \sum_{j=1}^{t} \frac{|G|}{m_j} \right). \tag{2.8}$$

2.2.1. *Various sets of generating vectors.* Starting from a generating vector, we define a series of sets that are important for counting epimorphisms, determining the G-Galois covers of a pair (T, B_G), and enumerating strata. Let $(g_1, \ldots, g_t)$ be a t-tuple of non-identity elements of G, e.g., the periodic part of a generating vector. Define the conjugacy classes

$$\begin{aligned} g^G &= \left\{ g^a = a^{-1}ga : a \in G \right\}, \\ C_j &= g_j^G, \ j = 1, \ldots, t, \end{aligned}$$

and a corresponding ordered list of the conjugacy classes

$$\mathcal{C} = (C_1, \ldots, C_t).$$

We will denote:

$$K_G(h; \mathcal{C}) = \tag{2.9}$$

$$\{(a_1, \ldots, a_h, b_1, \ldots, b_h, c_1, \ldots, c_t) \in G^{2h+t} : \prod_{i=1}^{h}[a_i, b_i] \prod_{j=1}^{t} c_j = 1, c_j \in C_j\}.$$

For generating vectors, we denote:

$$K_G^{\circ}(h; \mathcal{C}) = \{\text{vectors in } K_G(h; \mathcal{C}),\ \text{generating } G\}. \tag{2.10}$$

Bringing $\mathrm{Aut}(G)$ into play, for $\theta \in \mathrm{Aut}(G)$ we define:

$$\theta(\mathcal{C}) = (\theta(C_1), \ldots, \theta(C_t)) = (\theta(g_1)^G, \ldots, \theta(g_t)^G)$$

and these sets:

$$A_G(h; \mathcal{C}) = \bigcup_{\theta \in \mathrm{Aut}(G)} \theta\left(K_G(h; \mathcal{C})\right) = \bigcup_{\theta \in \mathrm{Aut}(G)} K_G(h; \theta(\mathcal{C})), \tag{2.11}$$

$$A_G^{\circ}(h; \mathcal{C}) = \{\text{vectors in } A_G(h; \mathcal{C}),\ \text{generating } G\}, \tag{2.12}$$

and

$$\overline{A_G^{\circ}}(h; \mathcal{C}) = \mathrm{Aut}(G) - \text{classes in } A_G^{\circ}(h; \mathcal{C}). \tag{2.13}$$

An n-gonal action is one for which $h = 0$. In this case we leave h out of all notation: signature $(m_1, \ldots, m_t)$ and generating vector sets $K_G(\mathcal{C})$, $K_G^{\circ}(\mathcal{C})$, $A_G(\mathcal{C})$, $A_G^{\circ}(\mathcal{C})$, and $\overline{A_G^{\circ}}(\mathcal{C})$. We call $\mathcal{C}$ the *G-signature* or *ramification type* of the n-gonal action. The set in (2.13) is our real focus of interest, since it is in 1-1 correspondence with the surfaces lying over (T, B_G) whose G-signature is any one of $\theta(\mathcal{C}) : \theta \in \mathrm{Aut}(G)$.

We now discuss some relationships among the sets in (2.9) - (2.13). Our eventual task is to compute the monodromy of $\Theta : \widetilde{\mathcal{S}} \to \overline{\mathcal{S}}$ via the braid group action on $\overline{A_G^{\circ}(\mathcal{C})}$, to be discussed in Section 2.3.1. Therefore, we need to directly construct the classes in $\overline{A_G^{\circ}(\mathcal{C})}$. Direct calculation of these sets may be difficult for larger groups, but we may be able to enumerate them. We focus on the n-gonal case though the discussion applies in the general case.

The numbers $|K_G(\mathcal{C})|$ may be calculated by using a well-known formula utilizing character tables, see [**Bre**], Theorem 14.1, or [**Jo**]. In turn, the numbers $|K_G^\circ(\mathcal{C})|$ may be computed by calculating $|K_H(C_1 \cap H, \ldots, C_t \cap H)|$ for all subgroups $H \leq G$ and then using the Möbius inversion formula for the subgroup lattice of G. We leave out the details because of space limitations of this paper.

Next, observe that two sets $K_G(\mathcal{C})$, $K_G(\mathcal{C}')$ are disjoint if the defining sequences $\mathcal{C}$ and $\mathcal{C}'$ are different. Thus, the unions in (2.11) can be made disjoint by taking a union over a transversal of the stabilizer, $\mathrm{Stab}(\mathrm{Aut}(G), \mathcal{C})$ of $\mathcal{C}$, in $\mathrm{Aut}(G)$. It follows then that

$$|A_G(\mathcal{C})| = \frac{|\mathrm{Aut}(G)|\,|K_G(\mathcal{C})|}{|\mathrm{Stab}(\mathrm{Aut}(G), \mathcal{C})|} \tag{2.14}$$

and a similar equation for $|A_G^\circ(h; \mathcal{C})|$. The stabilizer in the formula always contains the inner automorphisms of G.

Finally, the classes in $\overline{A_G^\circ}(\mathcal{C})$ are the $\mathcal{V}^{\alpha G}$, given in (2.7), and correspond to the epimorphism classes $\xi^{\alpha G}$. Since no non-trivial element of $\mathrm{Aut}(G)$ can fix any generating vector then

$$|A_G^\circ(\mathcal{C})| = |\mathrm{Aut}(G)| \left|\overline{A_G^\circ}(\mathcal{C})\right|. \tag{2.15}$$

2.2.2. *Generating vector examples.* We finish this section with a few examples of the classes $\overline{A_G^\circ}(\mathcal{C})$ for quotient genus 0 and 4 branch points. We will get some practice at finding normal forms for generating vectors, i.e., canonical, unique representatives for the classes in $\overline{A_G^\circ}(\mathcal{C})$. In later sections we use these examples as starting points for determining the topology of strata or as examples of specific concepts. By computing the classes $\overline{A_G^\circ}(\mathcal{C})$ in this section the later constructions will have shortened explanations.

Example 2.3. Let G be a cyclic group of order pqr where p, q, and r are primes. Suppose the generating vector (c_1, c_2, c_3, c_4) has signature (p, q, r, pqr), so that the action surface S has genus $\frac{1}{2}(2pqr - qr - pr - pq + 1)$. For any three c_1, c_2, c_3 with the required orders, $c_4 = (c_1c_2c_3)^{-1}$ has order pqr, so we only need to enumerate c_1, c_2, c_3. There are $p-1$ elements of order p, $q-1$ elements of order q and $r-1$ elements of order r and hence a total of $(p-1)(q-1)(r-1)$ generating vectors. The automorphism group has order $\phi(pqr) = (p-1)(q-1)(r-1)$. By (2.15) there is exactly one automorphism class of generating vectors with the given signature. Later we will see that this is a pure braid action with a single stratum.

Example 2.4. Let G be a cyclic group of order p^2q where p and q are primes. Now consider (c_1, c_2, c_3, c_4) with signature (p, p^2, q, p^2q) and action surface S of genus $\frac{1}{2}(p-1)(2pq - p + q - 1)$. By an analysis similar to the previous example, there are $(p-1)p(p-1)(q-1)$ generating vectors and the automorphism group of G has order $\phi(p^2q) = p(p-1)(q-1)$. It follows that there are $p-1$ automorphism classes of generating vectors with the given signature. If we hold c_2, c_3 fixed, vary c_1 through the $p-1$ possible values, and set $c_4 = (c_1c_2c_3)^{-1}$ we get representatives of the $p-1$ automorphism classes of generating vectors. Later we will see that these are pure braid actions with the same signature but multiple strata.

Example 2.5. Let p be an odd prime and consider the dihedral group $D_p = \langle a, b : a^2 = b^p = 1, b^a = b^{-1} \rangle$. The non-trivial elements have orders 2 and p. Let us consider actions of D_p with signature $(2,2,p,p)$, for which the genus of the action surface is $p-1$. There is a single conjugacy class $\mathcal{K}_2$ of p elements of order 2 and $\frac{p-1}{2}$ conjugacy classes $\mathcal{J}_e = \{b^e, b^{-e}\}, 1 \leq e \leq \frac{p-1}{2}$, of elements of order p. The automorphism group of G has order $p(p-1)$, with G acting freely by inner automorphisms, and the outer automorphism group is generated by $\theta_e : (a,b) \to (a, b^e)$, e as above. It follows that the automorphism group acts simply transitively on the set of $p(p-1)$ pairs $\{(x,y) : o(x) = 2, o(y) = p\}$. It further follows that in every $\mathrm{Aut}(G)$ class of vectors with signature $(2,2,p,p)$ there is exactly one with the form (a, c_2, b, c_4). Now, $c_4 = b^s$ for some $0 < s \leq p-1$, and since $c_1c_2c_3c_4 = 1$, we must have $c_2 = ab^{-(s+1)}$, which has order 2. The number of generating vectors with signature $(2,2,p,p)$ is the number of choices for c_1, c_3, c_4, namely $p(p-1)^2$. Again, from (2.15) we have $p-1$ automorphism classes of generating vectors.

Example 2.6. Let $G = \langle a, b : a^m = 1, b^n = 1, \cdots \rangle$ be a two generator group. This includes all simple groups. We always have the generating vector (a, a^{-1}, b, b^{-1}) with signature (m,m,n,n). This example is useful for constructing non-maximal actions. See Example 4.8.

Example 2.7. Let G be A_5 and let $\mathcal{K}_i$ denote the ith conjugacy class from the character table of A_5. Specifically, $\mathcal{K}_2 = (1,2)^G$ and $\mathcal{K}_3 = (1,2,3)^G$, and these are the only classes of elements of order 2 and 3, respectively. Because of this uniqueness $K_G^\circ(\mathcal{K}_2,\mathcal{K}_2,\mathcal{K}_3,\mathcal{K}_3)$ comprises all of the generating vectors with signature $(2,2,3,3)$, and it follows that $A_G^\circ(\mathcal{K}_2,\mathcal{K}_2,\mathcal{K}_3,\mathcal{K}_3) = K_G^\circ(\mathcal{K}_2,\mathcal{K}_2,\mathcal{K}_3,\mathcal{K}_3)$. By direct construction using Magma, we obtain $\left|\overline{A_G^\circ}(\mathcal{K}_2,\mathcal{K}_2,\mathcal{K}_3,\mathcal{K}_3)\right| = 18$. The genus of the action surface is 11.

2.2.3. *Non-abelian groups of order pq.* We include these examples since they are uncomplicated examples of nonabelian groups, generalizing the dihedral groups of Example 2.5. The groups were extensively studied by Wolfart and Streit in [**WS**] in the context of dessins d'enfant.

Let $p < q$ be two primes such that p divides $q-1$. It is well known that there is exactly one isomorphism class of non-abelian groups of order pq, isomorphic to $\mathbb{Z}_p \ltimes \mathbb{Z}_q$. Such a group G has a presentation $G = \langle a, b : a^p = b^q = 1, b^a = b^r \rangle$, where $1 < r < q$ and $r^p = 1 \mod q$. The non-trivial elements comprise $(p-1)q$ elements of order p and $q-1$ elements of order q. The order of the automorphism group is $q(q-1)$, consisting of the products U_uV_v, $0 \leq u < q$, $1 \leq v < q$, where $U_u : a \to a^{b^u} = ab^{u-ru}$, $b \to b$ and $V_v : a \to a, b \to b^v$.

We now determine the (p,p,q,q) generating vectors.

Example 2.8. A general (p,p,q,q) vector has the form $(a^{u_1}b^{v_1}, a^{u_2}b^{v_2}, b^{v_3}, b^{v_4})$, where $u_i \neq 0 \mod p$ and $v_3, v_4 \neq 0 \bmod q$. Using arguments similar to the dihedral case, it is easily shown that $u_2 = -u_1 \bmod p$ and $r^{u_1}v_1 + v_2 + v_3 + v_4 = 0 \bmod q$. There are $p-1$ solutions for u_1, and once u_1 is chosen, v_2, v_3, and v_4 may be chosen in $q(q-1)(q-1)$ ways. Then, $v_1 = -r^{-u_1}(v_2+v_3+v_4)$ is uniquely determined, and the total number of generating vectors classes is $(p-1)q(q-1)(q-1)/q(q-1) = (p-1)(q-1)$. Similar to the dihedral case, each class has a normal form

$$\mathcal{V}_{u,v} = (a^u, a^{-u}b^{-(v+1)}, b, b^v) \tag{2.16}$$

where $(u,v) \in \mathbb{F}_p^* \times \mathbb{F}_q^*$. The set of vectors of the form $(c_1, c_1^{-1}, c_3, c_3^{-1})$ (motivated by Example 2.6) can be chosen in $(p-1)q \times (q-1)$ ways. It is clear that this set is invariant under the action of $\mathrm{Aut}(G)$ and so there are $(p-1)q(q-1)/q(q-1) = p-1$

different classes. Indeed, the generating vectors are partitioned into $(q-1)$ $\mathrm{Aut}(G)$-invariant sets that satisfy $c_3 = c_4^s$ where $1 \leq s \leq q-1$. Each contains $p-1$ classes.

2.3. Equisymmetry. We recall some ideas on *equisymmetry* and the *branch locus*, see, e.g., [**Bro**] and [**Ha**]. Let S_0 be a distinguished surface of genus g and $\phi : S_0 \to S$ a diffeomorphism. Then, $F = \left\{\phi^{-1}\theta\phi : \theta \in \mathrm{Aut}(S)\right\}$ is a finite subgroup of MCG_g, the mapping class group of S_0. If another diffeomorphism ϕ' is chosen then the corresponding subgroup is the conjugate subgroup $F' = \left(\phi^{-1}\phi'\right)^{-1} F\phi^{-1}\phi'$ and both of F and F' are isomorphic to $\mathrm{Aut}(S)$. Thus, the surface S determines a conjugacy class of finite subgroups of MCG_g, we denote this conjugacy class by $\Sigma(S) = (F)$ and call it the *symmetry type* or *topological type* of S. Two surfaces are *equisymmetric* if their symmetry types are equal.

Conformally equivalent surfaces have the same symmetry type so that $\Sigma(S)$ is a function on the moduli space. An *equisymmetric stratum* $\mathcal{S}$ is the subset of $\mathcal{M}_g$ consisting of surfaces of the same symmetry type. We define $\Sigma(\mathcal{S}) = \Sigma(S)$ for any $S \in \mathcal{S}$. Reversing direction, for a conjugacy class (F) of finite subgroups of MCG_g, we denote by $\mathcal{S}_{(F)}$ the equisymmetric stratum consisting of surfaces S with $\Sigma(S) = (F)$.

A stratum $\mathcal{S}$ is a irreducible, locally closed, and smooth subvariety of $\mathcal{M}_g$, [**Bro**], [**Ha**]. If $\mathcal{S}$ is non-empty, its closure $\overline{\mathcal{S}}$ in $\mathcal{M}_g$ is obtained by adding strata of higher symmetry and strictly lower dimension. Specifically, suppose that $\mathcal{S}'$ is another stratum that satisfies $\Sigma(\mathcal{S}) \leq \Sigma(\mathcal{S}')$, i.e., each subgroup $F \in \Sigma(\mathcal{S})$ is contained in a subgroup $F' \in \Sigma(\mathcal{S}')$. Then $\mathcal{S}' \subset \overline{\mathcal{S}}$. For non-empty strata we have the following disjoint union decomposition of the closure in two formats:

$$\overline{\mathcal{S}} = \mathcal{S} \cup \bigcup_{\Sigma(\mathcal{S})<\Sigma(\mathcal{S}')} S', \tag{2.17}$$

$$\overline{\mathcal{S}_{(F)}} = \mathcal{S}_{(F)} \cup \bigcup_{(F)<(H)} \mathcal{S}_{(H)}. \tag{2.18}$$

In particular, the second formula shows that every surface in $\overline{\mathcal{S}_{(F)}}$ has an action of F on S, determined by the conjugacy class (F).

Let $\mathcal{T}_g$ be the Teichmüller space of surfaces of genus g. The mapping class group MCG_g acts upon $\mathcal{T}_g$ and the orbifold quotient is $\mathcal{M}_g$, with quotient map $\pi_g : \mathcal{T}_g \to \mathcal{M}_g$. The stabilizer of a point $S' \in \mathcal{T}_g$, lying over $S = \pi_g(S')$ is a subgroup in $\Sigma(S)$. Thus, the orbifold map is branched over the set $\mathcal{B}_g$ consisting of surfaces $S = \pi_g(S')$ with non-trivial automorphism group. The set $\mathcal{B}_g$ is called the branch locus and it is stratified by symmetry type. This characterization must be modified for $g = 2$ since every surface of genus 2 has a unique hyperelliptic involution. In this case $\mathcal{B}_2$ is the set of surfaces that have non-trivial automorphisms in addition to the hyperelliptic involution.

2.3.1. *Mapping class group and braid action.* Here is an alternate description of equisymmetry that is useful when constructing a surface S as a branched cover of a given, simpler surface T. Two surfaces S_1, S_2 are equisymmetric if and only if there is a diffeomorphism $\phi : S_1 \to S_2$ such that $\mathrm{Aut}(S_2) = \phi\mathrm{Aut}(S_1)\phi^{-1}$. This is summarised in the following diagram.

$$\begin{array}{ccc} S_1 & \overset{\phi}{\rightarrow} & S_2 \\ \downarrow \pi_1 & & \downarrow \pi_2 \\ T_1 & \overset{\overline{\phi}}{\rightarrow} & T_2 \end{array} \tag{2.19}$$

where the vertical arrows are quotients by the automorphism groups and the induced diffeomorphism $\overline{\phi} : T_1 \to T_2$, respects the branch sets. Starting with a diffeomorphism $\overline{\phi} : T_1 \to T_2$, respecting branch sets, $\overline{\phi}$ lifts to a covering map $\phi : S_1 \to S_2$, if and only if the relation $\ker \xi_2 = \overline{\phi}_* (\ker \xi_1)$ holds for the associated monodromy epimorphisms $\xi_1 : \pi_1(T_1^\circ) \to \mathrm{Aut}(S_1) \simeq G$ and $\xi_2 : \pi_1(T_2^\circ) \to \mathrm{Aut}(S_2) \simeq G$, where $\overline{\phi}_*$ is the isomorphism induced on fundamental groups. This relation holds if and only if there is an isomorphism $\theta \in \mathrm{Aut}(G)$ such that

$$\xi_2 = \theta \circ \xi_1 \circ \left(\overline{\phi}_*\right)^{-1}. \tag{2.20}$$

REMARK 2.9. In the above discussion, we have left out the base points for simplicity of discussion. This means that the monodromy epimorphisms (and $\overline{\phi}_*$ itself) are ambiguous up to inner automorphisms, however this ambiguity can be absorbed by the automorphism θ. The relation (2.20) can be transferred to the generating vectors, which is useful for computer calculation. Note that the action of homeomorphisms and group automorphisms commute and the actions are both left actions.

We will primarily utilize the *braid portion* of the mapping class group arising from the branch set. In the genus zero case this is the entire mapping class group. So assume in diagram (2.19) that $T_1 = T_2 = T$ and the branch sets are the same. We may construct *braid homeomorphisms* ϕ of T° as follows. A one-parameter family of homeomorphisms $h_s : T \to T,\ 0 \le s \le 1$ can be chosen so that $h_0 =$ identity, h_1 interchanges Q_j and Q_{j+1}, and h_s leaves all other branch points and the base point fixed for $0 \le s \le 1$. The family h_s can be chosen so that the action of $\overline{\phi} = h_1$ induces the following *braid operation* $\Phi_{j,j+1}$ on $\gamma_1, \ldots, \gamma_t$ and $c_1, \ldots, c_t$:

(1) $\Phi_{j,j+1} : \gamma_j \to \gamma_{j+1},\ \gamma_{j+1} \to \gamma_{j+1}^{-1}\gamma_j\gamma_{j+1}$, all other generators fixed, and
(2) $\Phi_{j,j+1}^{-1} : c_j \to c_{j+1},\ c_{j+1} \to c_{j+1}^{-1}c_jc_{j+1}$, all other generators fixed.

We view the braid operations as automorphisms $\Phi = \overline{\phi}_*$ of $\pi_1(T^\circ, Q_0)$. The braid operations on generating vectors comes from applying the epimorphism $\xi \circ \Phi^{-1} = \xi \circ (\overline{\phi}_*)^{-1}$ to the generating set $\mathcal{G}$. Hence, we need an inverse in the second equation above to get a left action. The action of Φ on the γ_j's will have the same formulas as the action of Φ^{-1} on the c_j's. The set $h_s(B_G)$ is a moving set of t points in $\widehat{\mathbb{C}}$ which equals B_G at $s = 0, 1$, i.e., a braid, and hence the name braid operations. Following Remark 2.9, the equation $\Phi = \overline{\phi}_*$ is only valid up inner automorphisms of $\pi_1(T^\circ, Q_0)$.

The *full group of braid operations*, denoted by $\mathcal{F}_t$, is generated by the $\Phi_{j,j+1}$. It is well known that in the n-gonal case the full group of braid operations induces the mapping class group action on $\pi_1(T^\circ, Q_0)$ [**Bi**], also see [**FM**]. Each braid operation Φ is associated with a permutation $\sigma = \sigma(\Phi) \in \Sigma_t$ of the t points $Q_1, \ldots, Q_t$. The permutation σ is induced by $\overline{\phi}$, namely $\overline{\phi}(Q_j) = Q_{\sigma j}$. The permutation map satisfies

$$\Phi(\gamma_j) = \gamma_{\sigma j}^{w_j^\Phi} \tag{2.21}$$

for some w_j^Φ and

$$\sigma(\Phi_1 \circ \Phi_2) = \sigma(\Phi_1) \circ \sigma(\Phi_2). \tag{2.22}$$

If σ is the identity permutation we have a *pure braid operation*. The *group of pure braid operations*, denoted by $\mathcal{P}_t$, induces the mapping class group action on $\pi_1(T^\circ, Q_0)$ consisting of homeomorphisms that fix each of the points $Q_1, \ldots, Q_t$.

Remark 2.10. It is well known that any outer automorphism of $\pi_1(T^\circ, Q_0)$ is induced by a homeomorphism of T°. Thus in constructing braid operations up to inner automorphisms we can use any assignment $\gamma_j \to \Phi(\gamma_j)$ that preserves the long relation. The *long relation* is the single relation on the right hand side of the presentation (2.4).

Remark 2.11. For calculations with 4 branch points it is useful to have a set of generators for the pure braid operations. Using Magma we may compute generators for the pure braid group $\mathcal{P}_4$ as follows. The group $\mathcal{F}_4$ has the following well known presentation

$$\langle a_1, a_2, a_3 \mid a_1a_3 = a_3a_1,\ a_1a_2a_1 = a_2a_1a_2,\ a_2a_3a_2 = a_3a_2a_3,\ a_1a_2a_3a_3a_2a_2 = 1\rangle,$$

where $a_i \leftrightarrow \Phi_{i,i+1}$. Map $\mathcal{F}_4 \to \Sigma_4$ by $a_i \to (i, i+1)$ and determine the kernel of the map using Magma. We get a set of generators for this kernel:

$$\mathcal{P}_4 \simeq \left\langle a_1^2, a_2^2, a_3^2 \right\rangle. \tag{2.23}$$

Remark 2.12. Since we will need the information later, let us record in one place a complete table of the action of the full and pure braid operators: $\Phi^{\pm 1}_{j,j+1}$ and $\Phi^{\pm 2}_{j,j+1}$ on the vector (c_1, c_2, c_3, c_4).

Map	Value	Map	Value
$\Phi^{-1}_{1,2}$	$(c_2, c_1^{c_2}, c_3, c_4)$	$\Phi_{1,2}$	$(c_2^{c_1^{-1}}, c_1, c_3, c_4)$
$\Phi^{-2}_{1,2}$	$(c_1^{c_2}, c_2^{c_1c_2}, c_3, c_4)$	$\Phi^{2}_{1,2}$	$(c_1^{c_2^{-1}c_1^{-1}}, c_2^{c_1^{-1}}, c_3, c_4)$
$\Phi^{-1}_{2,3}$	$(c_1, c_3, c_2^{c_3}, c_4)$	$\Phi_{2,3}$	$(c_1, c_3^{c_2^{-1}}, c_2, c_4)$
$\Phi^{-2}_{2,3}$	$(c_1, c_2^{c_3}, c_3^{c_2c_3}, c_4)$	$\Phi^{2}_{2,3}$	$(c_1, c_2^{c_3^{-1}c_2^{-1}}, c_3^{c_2^{-1}}, c_4)$
$\Phi^{-1}_{3,4}$	$(c_1, c_3, c_4, c_3^{c_4})$	$\Phi_{3,4}$	$(c_1, c_3, c_4^{c_3^{-1}}, c_3)$
$\Phi^{-2}_{3,4}$	$(c_1, c_2, c_3^{c_4}, c_4^{c_3c_4})$	$\Phi^{2}_{3,4}$	$(c_1, c_2, c_3^{c_4^{-1}c_3^{-1}}, c_4^{c_3^{-1}})$

2.3.2. *Topological equivalence of actions.* The discussion above can also be rephrased in terms of the well-known idea of the *topological equivalence of actions*. Two actions ϵ_1, ϵ_2 of G, on possibly different surfaces S_1, S_2 of the same genus, are *topologically equivalent* if there is an intertwining, orientation preserving, homeomorphism $\phi : S_1 \to S_2$ and an automorphism $\theta \in \mathrm{Aut}(G)$ such that

$$\epsilon_2(g) = \phi\epsilon_1(\theta(g))\phi^{-1}, \forall g \in G. \tag{2.24}$$

This is also summarized by the diagram (2.19). For monodromies of actions the relation of topological equivalence is given by (2.20). Two surfaces are equisymmetric if the actions of their automorphism groups are topologically equivalent.

Analogous to symmetry type, for an action ϵ of G on a surface, we define the *topological type of the action* $(G) = (\epsilon)$ to be the conjugacy class of subgroups

$$(G) = (\epsilon) = \left\{\phi^{-1}\epsilon(G)\phi : \phi : S_0 \to S\right\}, \tag{2.25}$$

where S_0 and ϕ are as in the definition of equisymmetry. Two actions ϵ_1, ϵ_2 are topologically equivalent if and only if $(\epsilon_1) = (\epsilon_2)$.

2.4. Dimensions of the strata. Given an equisymmetric stratum $\mathcal{S}$, each Riemann surface $S \in \mathcal{S}$ has automorphism group $\mathrm{Aut}(S)$ acting with a fixed topological type. The quotient orbifold $S/\mathrm{Aut}(S)$ may be uniformized by a Fuchsian group Γ in such a way that there is a (torsion-free) surface Fuchsian group $\Pi \trianglelefteq \Gamma$, uniformizing S, such that $\mathbb{H}/\Pi \to \mathbb{H}/\Gamma$ is conformally equivalent to $S \to S/\mathrm{Aut}(S)$. The signature of Γ, $(h; m_1, ..., m_t)$, is given by the genus h of $S/\mathrm{Aut}(S)$ and the list of *branching orders* of $S \to S/\mathrm{Aut}(S)$, i.e., the *periods* of Γ. Moreover, the signature of Γ is the same for all the surfaces $S \in \mathcal{S}$. The dimension of the stratum $\mathcal{S}$ is given by the dimension of the moduli space of the Fuchsian group Γ, and the formula for the dimension depends only on the signature of Γ. Concretely, given the signature $(h; m_1, ..., m_t)$ of Γ the (complex) dimension of moduli space of Fuchsian groups with this signature is:

$$\dim \mathcal{M}_{(h;m_1,\ldots,m_t)} = 3h - 3 + t.$$

To have dimension one we must consider two cases:

- $h = 0, t = 4$: case of quotient surface of genus 0,
- $h = 1, t = 1$: case of quotient surface of genus 1.

As previously stated we only deal with the first case in the paper (Section 4).

3. Covering n-gonal strata by Hurwitz spaces

The purpose of this section is to develop models of n-gonal equisymmetric strata using Hurwitz space models. An n-gonal stratum is one for which the orbit genus is zero, and so T° is the Riemann sphere punctured at t points. Families of such surfaces have long been studied as Hurwitz spaces, see e.g., [**Na**]. We will utilize several variants of Hurwitz space models, described shortly, depending on our needs. In this section we describe a general construction for t points, and then in Section 4 we will utilize a specific reduced Hurwitz space model for four branch points.

Suppose we are given a surface S with an n-gonal G action, determined by a generating vector $(c_1, \ldots, c_t)$, and that S lies in the stratum $\mathcal{S} \subset \mathcal{M}_g$. Let $\overline{\mathcal{S}}$ be the closure of $\mathcal{S}$ in $\mathcal{M}_g$. A suitable Hurwitz space model $\mathcal{H}$ may be constructed which contains a connected component $\widetilde{\mathcal{S}} \subseteq \mathcal{H}$ with a branched covering $\widetilde{\mathcal{S}} \to \overline{\mathcal{S}}$. The Hurwitz space is determined by G and t, and the component and covering are determined by a generating vector $(c_1, \ldots, c_t)$. The construction and analysis of $\widetilde{\mathcal{S}}$ and $\widetilde{\mathcal{S}} \to \overline{\mathcal{S}}$ and their application to the analysis of $\overline{\mathcal{S}}$ is the focus of the remainder of this section. In particular, we want to create a fibre bundle model of the family of surfaces $\{S : S \in \mathcal{S}\}$ which may not exist since the moduli space is a coarse moduli space.

In this section and the next, our quotient surfaces T will have genus 0, i.e., $T = \widehat{\mathbb{C}}$. In this specific case it will be convenient to use the notation $z_0, z_1, \ldots, z_t \in \widehat{\mathbb{C}}$ instead of $Q_0, Q_1, \ldots, Q_t \in T$, which we reserve for a general quotient surface.

Remark 3.1. Given a group G and a generating vector $(c_1, \ldots, c_t)$ a Hurwitz space can always be constructed. However, the surfaces constructed may all satisfy $\epsilon(G) < \mathrm{Aut}(S)$. We discuss this in some detail in Section 4.6. Example 4.8 is a simple example of this phenomenon.

3.1. Variants of Hurwitz spaces. For our purposes, a Hurwitz space, determined by the genus g, group G and number of branch points t, is a collection of surface-map pairs (S,π), with a suitable equivalence relation, such that

- S is a surface of genus g,
- $\pi : S \to \widehat{\mathbb{C}}$ is a regular branched cover,
- the group of covering transformations of π, $\mathrm{Gal}(\pi)$, satisfies

$$G \backsimeq \mathrm{Gal}(\pi) = \{\theta \in \mathrm{Aut}(S) : \pi \circ \theta = \pi\},$$

- and there are exactly t points in the branch set $B_\pi = \{z_1, \ldots, z_t\}$ of π.

Note that, by definition, G acts on each surface S, though the specific monomorphism is not specified.

We may construct models (i.e., covers) of Hurwitz spaces in several ways, depending on how we define equivalence of pairs and how we record the branch set B_π. Our choice of models, and the relevance of the braid group, will be explained in subsequent paragraphs. First, here are two choices for pair equivalence.

(1) Standard Hurwitz space equivalence: (S_1,π_1) and (S_2,π_2) are equivalent pairs in the Hurwitz space if there is a conformal map $\phi : S_1 \to S_2$ such that $\pi_1 = \pi_2 \circ \phi$.
(2) Reduced Hurwitz space equivalence: (S_1,π_1) and (S_2,π_2) are equivalent pairs in the Hurwitz space if there is a conformal map $\phi : S_1 \to S_2$ and an automorphism α of $\widehat{\mathbb{C}}$ such that $\alpha \circ \pi_1 = \pi_2 \circ \phi$. This allows us to assume that three branch points in B_π are $0, 1, \infty$.

Second, we may choose how we record the "branch set" B_π, and the type of braid operations used in analysing the variation of B_π.

(1) Pure braid operations: Consider B_π as a varying *ordered t-tuple* $b = (z_1, \ldots, z_t)$ of t distinct points in $\widehat{\mathbb{C}}$. Points in B_π are not interchangeable, and b is called a *branch tuple.*
(2) Full braid operations: Consider B_π as a varying *set* $\overline{b} = \{z_1, \ldots, z_t\} \subset \widehat{\mathbb{C}}$ of t distinct points. All points in B_π are interchangeable, and $\overline{b}$ is called a *branch set.*

Using the standard Hurwitz space equivalence, the various types of braid groups and their actions on generating vectors are easier to describe. With the reduced equivalence the dimension of the Hurwitz space and the stratum are the same and the map $\widetilde{\mathcal{S}} \to \overline{\mathcal{S}}$ is a finite degree covering map and not a fibre bundle.

In the pure braid model the branch t-tuples are easier to work with and keeping track of the generating vectors is easier since the generating set $\mathcal{G}$ and the generating vectors are sensitive to the order of the branch points. However, we get $t!$ replications of the same quotient pair $(\widehat{\mathbb{C}}, B_\pi)$ from all the relabellings of B_π. The pure braid model is actually a cover of the Hurwitz space, whereas the full braid model will be the Hurwitz space.

By definition, for any two pairs (S_1,π_1), (S_2,π_2) in the same equivalence class in Hurwitz space, S_1, S_2 are conformally equivalent. Thus, there is a natural map

$$\Theta : \mathcal{H} \to \mathcal{M}_g,\ (S,\pi) \to \text{conformal class of } S. \tag{3.1}$$

The analysis of this map is the end goal of the rest of this section.

3.2. Building the cover of a stratum. We give an overview of the steps in constructing a stratum from Hurwitz spaces. Specific constructions for the 1-dimensional orbit genus 0 case are given in Section 4. First we construct the bundle of quotients, which corresponds to the continuous component of moduli as discussed in the introduction.

3.2.1. *Bundle of quotients.* Let $B = (\widehat{\mathbb{C}})^t - \Delta$, where Δ is the multi-diagonal of t-tuples of points in $\widehat{\mathbb{C}}$ with at least two equal components. Next, let

$$E = \left\{(z,b) \in \widehat{\mathbb{C}} \times B : z \notin b\right\},\ p : E \to B,\ p(z,b) = b. \tag{3.2}$$

Specifically, each $b \in B$ determines a set of t distinct points in $\widehat{\mathbb{C}}$ and $p^{-1}(b)$ is $\widehat{\mathbb{C}}$ with t punctures at the points determined by b. The map

$$p : E \to B \tag{3.3}$$

is a locally trivial bundle, which we call the *bundle of quotients*.

The symmetric group Σ_t acts on B by permuting component indices, namely for $\sigma \in \Sigma_t$: $\sigma\cdot(z_1,\ldots,z_t) = (z_{\sigma 1},\ldots,z_{\sigma t})$. The quotient $B_\Sigma = B/\Sigma_t$, with quotient map $b \to \bar{b}$, corresponds to the space of (unordered) sets of t points in $\widehat{\mathbb{C}}$. We can similarly define, but will barely use, the *full bundle of quotients*

$$E_\Sigma = \left\{(z,\bar{b}) \in \widehat{\mathbb{C}} \times B_\Sigma : z \notin \bar{b}\right\} \tag{3.4}$$

and bundle map

$$p_\Sigma : E_\Sigma \to B_\Sigma. \tag{3.5}$$

The *pure braid group* is defined as the fundamental group $\pi_1(B, b_0)$ for some distinguished branch set b_0. The *full braid group* is defined as the fundamental group $\pi_1(B_\Sigma, \overline{b_0})$.

3.2.2. *Action of the fundamental group.* For a general bundle $p : E \to B$ with connected base space B, we recall the action of paths in the base on the fibres. Let $F_b = p^{-1}(b)$ be the fibre lying over $b \in B$ and $\beta(s),\ 0 \le s \le 1$ a path in B with $b_1 = \beta(0), b_2 = \beta(1)$. The local triviality of the bundle allows us to construct a family of homeomorphisms

$$h_s^\beta : F_{b_1} \to F_{\beta(s)},\ h_0^\beta = \text{identity}, \tag{3.6}$$

which is well-defined up to isotopy. In turn, there is an isomorphism

$$\beta_* = h_*^\beta = (h_1^\beta)_* : \pi_1(F_{b_1}) \to \pi_1(F_{b_2}), \tag{3.7}$$

well-defined up to an inner automorphism, depending on selection of base points. If $\beta \in \pi_1(B, b_0)$, then β_* is an automorphism of $\pi_1(F_{b_0})$. We have deliberately left out the base points for fibres, since the details are messy and the inner automorphism ambiguity will not matter in our analysis. Returning to the bundles of quotients, we observe that the full group of braid operations generates all the possible β_* defined for the bundle (3.5). The pure braid operations generate all the possible β_* for the bundle (3.3).

Equation (3.7) allows us to transport monodromies and monodromy classes from $\pi_1(F_{b_1})$ to $\pi_1(F_{b_2})$ by

$$\beta_*\xi = \xi \circ \left(h_*^\beta\right)^{-1} \tag{3.8}$$

and

$$\beta_* \xi^{\alpha G} = \xi^{\alpha G} \circ \left(h_*^\beta\right)^{-1} = \left\{\theta \circ \xi \circ \left(h_*^\beta\right)^{-1} : \theta \in \mathrm{Aut}(G)\right\}. \tag{3.9}$$

Returning to the case at hand, $\pi_1(F_{b_0})$ is finitely generated, and so the epimorphisms $\pi_1(F_{b_0}) \to G$ are finite in number, and so are the kernels of the epimorphisms. The distinct kernels of these epimorphisms are in 1-1 correspondence with equivalence classes of regular coverings $\widetilde{F} \to F_{b_0}$ with Galois group isomorphic to G. The group $\pi_1(B, b_0)$ permutes these finitely many kernels. We can construct a covering space over B whose points are the finitely many kernels in $\pi_1(F_b)$, $b \in B$, and in turn a fibre bundle over B whose fibres are disjoint collections of surfaces with G action and branch set determined by $b \in B$.

3.2.3. *Bundle of actions.* Though the setup just discussed describes the geometric situation nicely, it is terrible for calculations. So we enlarge the base space to simplify things. The situation is resolved by the following lemma. We leave the proof of the lemma to the reader, though our construction following the lemma for our specific situation gives the idea of the proof. In the lemma we use the $\pi_1(B, b_0)$ action from equation (3.6) in the following way. For a covering space $p : E \to B$, and $x \in p^{-1}(b_0)$ the correspondence $(x, \beta) \to h_1^\beta(x)$ is simply path lifting.

LEMMA 3.2. *Let $r : (Y, y_0) \to (X, x_0)$ be a covering space of suitable spaces, say manifolds, with connected base X. Let H be a subgroup of the stabilizer*

$$H_{y_0} = \{\beta \in \pi_1(X, x_0) : h_1^\beta(y_0) = y_0\}. \tag{3.10}$$

Let $\overline{q} : (\widetilde{X}, \tilde{x}_0) \to (X, x_0)$ be the covering space determined by H, and consider the diagram

$$\begin{array}{ccc} \widetilde{Y} & \overset{q}{\to} & Y \\ \downarrow \widetilde{r} & & \downarrow r \\ \widetilde{X} & \overset{\overline{q}}{\to} & X \end{array} \tag{3.11}$$

defined by pullback. Then, all maps are covering spaces, q is 1-1 on fibres, and there is a continuous section $\mathfrak{s} : \widetilde{X} \to \widetilde{Y}$ with $\mathfrak{s}(\tilde{x}_0) = \tilde{y}_0$, for every $\tilde{y}_0$ satisfying $q(\tilde{y}_0) = y_0$. The subgroup $H = H_{y_0}$ determines the smallest cover for which there is a section.

If H fixes all points in the fibre $r^{-1}(x_0)$ then the covering space $\widetilde{r} : \widetilde{Y} \to \widetilde{X}$ is trivial, and there are sections $\mathfrak{s} : \widetilde{X} \to \widetilde{Y}$ for every $\tilde{y}_0$ lying over any $y_0 \in r^{-1}(x_0)$. If H is exactly the subgroup fixing all of the points in the fibre $r^{-1}(x_0)$, then the cover $\overline{q} : (\widetilde{X}, \tilde{x}_0) \to (X, x_0)$ is Galois and the action of $G_r = \pi_1(X, x_0)/H$ on $\widetilde{X}$ lifts to $\widetilde{Y}$.

We use the idea in the lemma to simplify the geometry of the surfaces lying over the points of B. Pick a specific subgroup $\Pi_{\xi_0} \triangleleft \pi_1(F_{b_0})$ among the collection of kernels of epimorphisms, and let $\mathcal{O}_{\xi_0}$ be the orbit of kernels determined by Π_{ξ_0} (orbit of monodromy classes $\xi^{\alpha G}$), under the action of $\pi_1(B, b_0)$. Let $H_{\xi_0} \leq \pi_1(B, b_0)$ be the subgroup that maps Π_{ξ_0} to itself, and let $\overline{q}_1 : B_1 \to B$ be the covering space defined by the subgroup H_{ξ_0}.

Refine the bundle $p : E \to B$ to a diagram of bundles of quotients

$$\begin{array}{ccc} E_1 & \overset{q_1}{\to} & E \\ \downarrow p_1 & & \downarrow p \\ B_1 & \overset{\overline{q}_1}{\to} & B \end{array} \tag{3.12}$$

so that the horizontal maps are covering spaces. The bottom row is the covering space defined above and the top row is the pullback of the bottom row.

Using paths in B_1 and the operators (3.7), we can transport our distinguished subgroup Π_{ξ_0} to the subgroup $\Pi_{\xi_0,b} \trianglelefteq \pi_1(F_b)$ for $b \in B_1$. According to the construction of B_1 from H_{ξ_0} the transported subgroup is independent of the transporting path. Thus, we are able to construct a regular G cover $\widetilde{F}_b \to F_b$ for each $b \in B_1$ in a compatible way so that we may extend the diagram of bundles (3.12) to get the following diagram of bundles.

$$\begin{array}{ccccc} E_2 & \overset{q_2}{\to} & E_1 & \overset{q_1}{\to} & E \\ \downarrow p_2 & & \downarrow p_1 & & \downarrow p \\ B_1 & \overset{id}{\to} & B_1 & \overset{\overline{q}_1}{\to} & B \end{array} \tag{3.13}$$

The complex analytic structure of the left hand bundle comes from lifting "tubes" of the form $p_1^{-1}(D_b)$, where D_b is a disc or ball centred at b. The tubes are lifted by using the subgroup $\iota_*(\Pi_{\xi,b}) \trianglelefteq \pi_1\left(p_1^{-1}(D_b)\right)$ where $\iota : F_b \to p_1^{-1}(D_b)$ is the inclusion map.

Finally, collapse the diagram (3.13) by composing the rows and relabelling the left column to get the following *bundle diagram of actions*:

$$\begin{array}{ccc} \widetilde{E} & \overset{q}{\to} & E \\ \downarrow \widetilde{p} & & \downarrow p \\ \widetilde{B} & \overset{\overline{q}}{\to} & B \end{array} . \tag{3.14}$$

The left hand bundle is called a *bundle of actions* or a *bundle of (punctured) surfaces.*

3.2.4. *Bundle of surfaces and Hurwitz space model.* Our covering space constructions produced punctured surfaces. However, it is easily shown that the punctures can be filled in to yield a Hurwitz space model of closed surfaces. Indeed, for each point $\tilde{b} \in \widetilde{B}$, let $S_{\tilde{b}}$ be the closed surface obtained from the fibre $\widetilde{F}_{\tilde{b}} = (\widetilde{p})^{-1}(\tilde{b})$ by filling in the punctures. Then, the family $\left\{S_{\tilde{b}} : \tilde{b} \in \widetilde{B}\right\}$ is a locally trivial family over $\widetilde{B}$, which we will call a *bundle of surfaces*.

Also, for each $\tilde{b} \in \widetilde{B}$, there is a branched G-cover $\pi_{\tilde{b}} : S_{\tilde{b}} \to \widehat{\mathbb{C}}$, branched over the branch set $\overline{q}(\tilde{b})$, and satisfying $\mathrm{Gal}(\pi_{\tilde{b}}) \simeq G$. The cover $\pi_{\tilde{b}}$ is uniquely determined by the transported epimorphism $\xi_{\tilde{b}}$. Our model of the desired component of the Hurwitz space is the collection of pairs

$$\widetilde{\mathcal{S}} = \left\{(S_{\tilde{b}}, \pi_{\tilde{b}}) : \tilde{b} \in \widetilde{B}\right\}. \tag{3.15}$$

with a natural bijection $\widetilde{\mathcal{S}} \leftrightarrow \widetilde{B}$, using the bundle of actions via the bundle of surfaces. This bijection determines a manifold structure on $\widetilde{\mathcal{S}}$, compatible with the bundle map $S_{\tilde{b}} \to \tilde{b}$. Note that the pair $(S_{\tilde{b}}, \pi_{\tilde{b}})$ is determined by a pair $(b, \xi^{\alpha G})$ as described later in Remark 3.9.

The families $\left\{S_{\tilde{b}} : \tilde{b} \in \widetilde{B}\right\}$ and $\left\{(S_{\tilde{b}}, \pi_{\tilde{b}}) : \tilde{b} \in \widetilde{B}\right\}$ serve as models for the stratum $\mathcal{S}$ and its closure $\overline{\mathcal{S}}$ in $\mathcal{M}_g$. In some ways, they may be more useful than the stratum since it allows us to analyze the variation in the geometry and topology of surfaces $S \in \mathcal{S}$ as we move around in a stratum. In particular, the behaviour at infinity is interesting.

REMARK 3.3. We may repeat all the steps in sections 3.2.1, 3.2.2, and 3.2.3 but starting with the bundle of quotients $p_\Sigma : E_\Sigma \to B_\Sigma$ defined by equations 3.4 and 3.5. The cover $\widetilde{B_\Sigma} \to B_\Sigma$ is determined by the stabilizer in $\pi_1(B_\Sigma, \overline{b_0})$ of the monodromy ξ_0 of $\pi_1(T^\circ, z_0)$ where $T^\circ = \widehat{\mathbb{C}} - \overline{b_0}$. The family

$$\widetilde{\mathcal{S}_\Sigma} = \left\{ (S_{\tilde{b}}, \pi_{\tilde{b}}) : \tilde{b} \in \widetilde{B_\Sigma} \right\} \tag{3.16}$$

is a more exact model of the Hurwitz space but is less tractable for computation.

For the remainder of the paper, the set B and the associated covers and bundles will refer to a space of distinct t-tuples of $\widehat{\mathbb{C}}$. If we wish to consider sets of t distinct points we will use the notation B_Σ and the associated constructions.

3.3. More on the bundle of actions. First, a couple of remarks on the bundle of actions.

REMARK 3.4. The process of constructing $\widetilde{p} : \widetilde{E} \to \widetilde{B}$ can be repeated for any of the subgroups belonging to $\mathcal{O}_{\xi_0}$. All of the covers are isomorphic. Indeed, if $\xi \in \mathcal{O}_{\xi_0}$ is connected to ξ_0 by a loop δ_1 in $\pi_1(B, b_0)$, then the bundle of subgroups over $\widetilde{B}$ may be constructed by using the paths $\delta_1\delta_2$, δ_2 a path from b_0 to an arbitrary $b \in \widetilde{B}$.

REMARK 3.5. The cover $\widetilde{B} \to B$ is the smallest cover over which we can build a bundle of surfaces from the bundle of actions $\widetilde{E} \to \widetilde{B}$. We may also construct a Galois cover $\widetilde{B}_{reg} \to B$ and a bundle $\widetilde{E}_{reg} \to \widetilde{B}_{reg}$ by letting $\widetilde{B}_{reg}$ be the cover defined by the subgroup N of $\pi_1(B, b_0)$ that fixes all the kernels in $\mathcal{O}_{\xi_0}$. This subgroup is normal and the Galois cover $\widetilde{B}_{reg} \to B$ is interesting its own right as $\widetilde{B}_{reg}$ may be completed to a quasiplatonic surface in the genus zero, four branch point case.

3.3.1. *The surfaces lying over* (T, B_G). Next we relate the points $\tilde{b} \in \widetilde{B}$ that lie over a given $b_0 \in B$, or any of the tuples σb_0, $\sigma \in \Sigma_t$, to the pure and full braid orbits of epimorphisms and generating vectors. The points $\tilde{b}$ to be considered are the full inverse image of $\overline{b_0}$ under the covering $\widetilde{B} \to B \to B/\Sigma_t$, comprising $t! \times \deg(\overline{q})$ inverse image points. These points form the discrete component of moduli analysis as discussed in the introduction. Indeed, these points parameterize the full set of topologically equivalent G-actions with fixed topological type (G) and quotient surface (T, B_G), for the specific branch set $\overline{b_0}$. Unfortunately, there may be some redundancy, though for pure braid strata this is precisely the number of topologically equivalent but conformally inequivalent surfaces lying over (T, B_G). In the non-pure braid case the analysis of the points allows us to determine which surfaces lying over a fixed pair (T, B_G) are conformally equivalent.

By definition, the points $\tilde{b} \in \widetilde{B}$ lying over $b \in B$ (diagram 3.14) are in 1-1 correspondence to the "cosets" $H_{\xi_0}\beta$ where β is any path from b_0 to b. By coset we mean that any two paths β_1, β_2 from b_0 to b are in the same H_{ξ_0} coset if and only if $\beta_1\beta_2^{-1} \in H_{\xi_0}$. These cosets are in 1-1 correspondence to the paths $\beta'\beta_b$, where β_b is a fixed path from b_0 to b, and β' runs over a transversal of the right coset space $H_{\xi_0}\backslash\pi_1(B, b_0)$. Thus, we can think of a point $\tilde{b}$ lying over b as the pair $(b, \overline{\beta})$, where $\overline{\beta} = H_{\xi_0}\beta$.

We may also think of the points $\tilde{b}$ as pairs $(b, \xi^{\alpha G})$, where $\xi^{\alpha G}$ is the monodromy class $\xi_0^{\alpha G}$ transported to $\pi_1(F_b)$ by some path β from b_0 to b, using (3.9). The monodromy classes $\xi^{\alpha G}$ are in 1-1 correspondence to the cosets $\overline{\beta} = H_{\xi_0}\beta$.

Now let us translate this discussion to generating vectors. Since all the $\tilde{b} \in \widetilde{B}$, under consideration determine the same punctured quotient T°, then the association $\xi \leftrightarrow \mathcal{V}$ is permissable, since the generating set $\mathcal{G}$ of $\pi_1(T^\circ)$ is fixed. We need to expand and refine our classes in (2.12) and (2.13) to take replications and permutations into account and to analyze the full topological equivalence class of G-actions.

For $\mathcal{C} = (C_1, \ldots, C_t)$, define $\sigma\mathcal{C} = (C_{\sigma 1}, \ldots, C_{\sigma t})$, for $\sigma \in \Sigma_t$, and define

$$J_G(\mathcal{C}) = \bigcup_{\sigma\in\Sigma_t} A_G^\circ(\sigma\mathcal{C}), \tag{3.17}$$

$$\overline{J_G}(\mathcal{C}) = \left\{\mathcal{V}^{\alpha G} : \mathcal{V} \in J_G(\mathcal{C})\right\}. \tag{3.18}$$

To explain these definitions, suppose ξ is an epimorphism corresponding to $\mathcal{V} \in A_G^\circ(\mathcal{C})$, and Φ is a braid operation, inducing σ. The vector $\Phi \cdot \mathcal{V}$ is associated to $\xi \circ \Phi^{-1}$ so $\Phi \cdot \mathcal{V}$ lies in $A_G^\circ(\sigma^{-1}\mathcal{C})$, according to equation (2.21). It follows that the group of full braid operations, $\mathcal{F}_t$ acts upon $\overline{J_G}(\mathcal{C})$ by $(\Phi, \mathcal{V}^{\alpha G}) \to \Phi \cdot \mathcal{V}^{\alpha G}$, and that the pure braid operations $\mathcal{P}_t$ map each $\overline{A_G^\circ}(\sigma\mathcal{C})$ to itself.

Finally, for any $\mathcal{V}^{\alpha G} \in \overline{A_G^\circ}(\mathcal{C})$ we define $L_{G,\mathcal{V}} : \Sigma_t \to \overline{J_G}(\mathcal{C})$, by

$$L_{G,\mathcal{V}}(\sigma) = \left\{\Phi \cdot \mathcal{V}^{\alpha G} : \Phi \in \mathcal{F}_t, \sigma(\Phi) = \sigma\right\}. \tag{3.19}$$

The set $L_{G,\mathcal{V}}(\sigma)$ is the generating vector analog of $\tilde{p}^{-1}(\sigma b_0)$. Note that $L_{G,\mathcal{V}}(\sigma)$ consists of classes in $\overline{J_G}(\mathcal{C})$, and it not a set of generating vectors, as the notation might suggest. Next, we use $L_{G,\mathcal{V}}$ to define pure braid and non-pure braid actions and strata.

Definition 3.6. Let G act n-gonally on a surface S with generating vector $\mathcal{V} = (c_1, \ldots, c_t)$ and G-signature $(c_1^G, \ldots, c_t^G)$. Let $L_{G,\mathcal{V}}(\sigma)$ be defined as in (3.19). Then, G has a *pure braid action* if and only if for any two distinct $\sigma_1, \sigma_2 \in \Sigma_t$ the sets $L_{G,\mathcal{V}}(\sigma_1)$ and $L_{G,\mathcal{V}}(\sigma_2)$ are disjoint. Otherwise the action is *non-pure*. If the G-action defines a stratum, i.e., $\epsilon(G) = \mathrm{Aut}(S)$, then we say the stratum is a *pure braid stratum* or *non-pure braid stratum* according the type of G-action.

The following proposition takes some of the mystery out of the sets $L_{G,\mathcal{V}}(\sigma)$.

Proposition 3.7. *Let $L_{G,\mathcal{V}}(\sigma)$ be as defined in* (3.19) *and let other notation be as previously defined. Then,*

(1) *Each $L_{G,\mathcal{V}}(\sigma)$ is an entire $\mathcal{P}_t$ orbit, and they all have the same cardinality.*
(2) *The two sets $L_{G,\mathcal{V}}(\sigma_1)$ and $L_{G,\mathcal{V}}(\sigma_2)$ are either equal or disjoint.*
(3) *Let $M_\sigma = \{\sigma' \in \Sigma_t : L_{G,\mathcal{V}}(\sigma'\sigma) = L_{G,\mathcal{V}}(\sigma)\}$. Then, M_σ is a subgroup of Σ_t and $L_{G,\mathcal{V}}(\sigma_1\sigma) = L_{G,\mathcal{V}}(\sigma_2\sigma)$ if and only if $\sigma_1^{-1}\sigma_2 \in M_\sigma$. Also, $M_\sigma = \sigma M_1 \sigma^{-1}$. When σ is the identity permutation, M_σ is denoted by M_1.*
(4)
$$\left|\bigcup_{\sigma\in\Sigma_t} L_{G,\mathcal{V}}(\sigma)\right| = \frac{t!}{|M_\sigma|}\left|L_{G,\mathcal{V}}(\sigma)\right|$$

i.e.,

$$\left|\mathcal{F}_t \cdot \mathcal{V}^{\alpha G}\right| = \frac{t!}{|M_\sigma|} \left|\mathcal{P}_t \cdot \sigma\mathcal{V}^{\alpha G}\right|$$

PROOF. The statements in the proposition are straightforward facts about orbit spaces, we give a few details. The group $\mathcal{F}_t$ acts transitively on the finite set $\Omega = \mathcal{F}_t \cdot \mathcal{V}^{\alpha G}$. Since $\mathcal{P}_t$ is normal in $\mathcal{F}_t$ the orbits of $\mathcal{P}_t$ form a complete block system in Ω. The blocks may be identified (with possible repetition) with the sets $L_{G,\mathcal{V}}(\sigma) = (\Phi\mathcal{P}_t) \cdot \mathcal{V}^{\alpha G} = \Phi \cdot (\mathcal{P}_t \cdot \mathcal{V}^{\alpha G})$, where $\sigma(\Phi) = \sigma$. This proves statements (1) and (2). The group $\Sigma_t = \mathcal{F}_t/\mathcal{P}_t$ acts on the block system and M_σ is the stabilizer of the block $L_{G,\mathcal{V}}(\sigma)$. This proves statement (3). The remaining statements follow from the Orbit Stabilizer Theorem. In particular, there are $|\Sigma_t/M_\sigma|$ distinct blocks, so Ω consists of $t!/|M_\sigma|$ blocks of size $|L_{G,\mathcal{V}}(\sigma)|$. □

REMARK 3.8. The subgroup order $|M_\sigma|$ and the index $t!/|M_\sigma|$ are important invariants that cannot be immediately read off from the signature or even the G-signature. Pure braid strata are ones for which $|M_\sigma| = 1$.

REMARK 3.9. We have introduced a lot of notation. We summarize the prior discussion by giving a list of equivalent sets, each determined by a $\sigma \in \Sigma_t$. The determination of when $L_{G,\mathcal{V}}(\sigma_1) \cap L_{G,\mathcal{V}}(\sigma_2)$ is non-empty determines redundancies in each line of the list, see Section 3.5. The items b_0, ξ_0, and $\mathcal{V}_0$ are as previously defined. The equivalencies are demonstrated in the prior discussion or are easily derived from that discussion.

- surfaces $S_{\widetilde{b}}$ lying over σb_0, i.e., (T, B_G) is determined by σb_0.
- $\widetilde{p}^{-1}(\sigma b_0)$
- the pairs $(\sigma b_0, \overline{\beta})$, where $\overline{\beta} = H_{\xi_0}\beta$ is a coset in $H_{\xi_0}\backslash\pi_1(B, b_0)$.
- the pairs $(\sigma b_0, \xi^{\alpha G})$, where $\xi^{\alpha G}$ is the transport of $\xi_0^{\alpha G}$ to σb_0.
- $L_{G,\mathcal{V}_0}(\sigma)$.

The point $b_0 \in B$ can be arbitrary.

3.3.2. *Braid action examples.* We look at some examples of the braid action on generating vectors, revisiting some of the examples in Section 2.2.2.

EXAMPLE 3.10. We continue Examples 2.3 and 2.4. In these examples the signatures $(m_1, \ldots, m_t)$ have distinct entries It follows that $(m_{\sigma_1 1}, \ldots, m_{\sigma_1 t})$ and $(m_{\sigma_2 1}, \ldots, m_{\sigma_2 t})$ are different for $\sigma_1 \neq \sigma_2$. It follows then that $\overline{A_G^\circ}(\sigma_1\mathcal{C}) \cap \overline{A_G^\circ}(\sigma_2\mathcal{C})$ is empty for $\sigma_1 \neq \sigma_2$ and, therefore, the actions are pure braid actions.

EXAMPLE 3.11. Let G be a cyclic group of order n, concretely represented as the additive group $\mathbb{Z}_n$. The generating vector for an arbitrary G action is a t-tuple of integers $(c_1, \ldots, c_t)$ selected from $\{1, \ldots, n-1\}$, satisfying $c_1 + \cdots + c_t = 0 \bmod n$ and $\gcd(c_1, \ldots, c_t, n) = 1$. We let $X_{n,t}^\circ$ be this set. For a cyclic group, the action of the full braid group on all generating vectors is simply permutation of the entries. Hence, the topological equivalence classes of actions in $X_{n,t}^\circ$ are the orbits in $X_{n,t}^\circ$ of the action $(e, \sigma) \cdot (c_1, \ldots, c_t) = (ec_{\sigma 1}, \ldots, ec_{\sigma t})$, where $\gcd(e, n) = 1$. This can be easily and directly computed with Magma.

In Table 1 (below) we give some sample results for $\mathbb{Z}_{10}$ which has 14 classes of actions on surfaces of genus $5, 6, 7, 8$, and 9. The genera of the surfaces giving the action are given in the column labeled $g(S)$. We may find a unique representative for each orbit as follows (in the column labeled vector). The generating vectors in an orbit are simply t-tuples of integers, we select the minimum vector under lexicographic ordering. The fifth column gives $|M_\sigma|$, a measure of how far away the action is from being a pure braid action. The group $\mathbb{Z}_{10}$ is the smallest cyclic group for which there are pure braid actions. We see from the second line that having a G-signature with distinct entries is not sufficient to guarantee a pure braid action. The last line occurs for every n, just choose the vector $(1,1,n-1,n-1)$ with signature (n,n,n,n). The genus of the surface is $n-1$ and is the largest genus surface with $\mathbb{Z}_n$ action branched over 4 points on the sphere. The corresponding action is never a pure braid action.

$\lvert\mathbb{Z}_n\rvert$	vector	signature	$g(S)$	$\lvert M_\sigma\rvert$	pure braid
10	$(1,2,3,4)$	$(10,5,10,5)$	8	1	*yes*
10	$(1,4,6,9)$	$(10,5,5,10)$	8	2	*no*
10	$(1,5,5,9)$	$(10,2,2,10)$	5	4	*no*
10	$(1,1,1,7)$	$(10,10,10,10)$	9	6	*no*
10	$(1,1,9,9)$	$(10,10,10,10)$	9	8	*no*

Table 1: Sample strata for $\mathbb{Z}_n$

$\lvert\mathbb{Z}_p\rvert$	genus of S	# strata	$\lvert M_\sigma\rvert=1$	$\lvert M_\sigma\rvert=2$	% pure
3	2	1	0	0	0.0
5	4	3	0	0	0.0
7	6	4	0	2	0.0
11	10	8	1	5	12.5
13	12	11	2	6	18.2
17	16	17	5	9	29.4
19	18	20	7	11	35.0
23	22	28	12	14	42.9
29	28	43	22	18	51.2
31	30	48	26	20	54.2
37	36	67	40	24	59.7
41	40	81	51	27	63.0
43	42	88	57	29	64.8
47	46	104	70	32	57.3
53	52	131	92	36	70.2
59	58	160	117	41	73.1
61	60	171	126	42	73.7
67	66	204	155	47	76.0
71	70	228	176	50	77.2
73	72	241	187	51	77.6
79	78	280	222	56	79.3
83	82	308	247	59	80.2
89	88	353	287	63	81.3
97	96	417	345	69	82.7
101	100	451	376	72	83.4

Table 2: Number of strata for prime cyclic groups

EXAMPLE 3.12. If $n = p$ is a prime, then the signature must be (p, p, p, p) and the genus is $p-1$. As suggested in the introduction there is an explosive growth in the number of strata as the genus increases. Using the computation method of the previous example, aided by Magma, we have calculated the number of strata, the number of strata with $M_\sigma = 1$ or 2 and the percentage of pure braid strata. The results for odd $p \leq 101$ are given in Table 2 above.

EXAMPLE 3.13. We continue Example 2.5, where

$$G = D_p = \left\langle a, b : a^2 = b^p = 1, b^a = b^{-1} \right\rangle,$$

acting upon a surface of genus $p-1$. This is a simple example of a non-abelian group action with a non-trivial pure braid action on automorphism classes of generating vectors. We will do the calculations in detail to illustrate the methods involved. Let $\mathcal{V}_s = (a, ab^{-(s+1)}, b, b^s)$, $s \in \mathbb{F}_p^*$ be any of representatives of the $p-1$ $\mathrm{Aut}(G)$ classes of $(2, 2, p, p)$ generating vectors. We shall show that $|L_{G,\mathcal{V}_s}(\sigma)| = 2$ and that $|M_\sigma| = 2$ or 4. Since the orders are independent of σ we may assume that $\sigma = 1$.

To work out the braid operations on the entire set of automorphism classes, we shall use Remark 2.12 and the equations therein, use the conjugation formulas $a^{b^e} = ab^{2e}$ and $(b^e)^a = b^{-e}$, and use the outer automorphisms $(a, b) \to (a, b^e)$. We will only need to directly compute $\Phi_{i,i+1}^{-1} \cdot \mathcal{V}_s^{\alpha G}$ for $i = 1, 3$ and $\Phi_{j,j+1}^{-2} \cdot \mathcal{V}_s^{\alpha G}$, for $j = 2, 3$.

For $\Phi_{1,2}^{-1}$ we have:

$$\Phi_{1,2}^{-1} \cdot (a, ab^{-(s+1)}, b, b^s) = (ab^{-(s+1)}, a^{ab^{-(s+1)}}, b, b^s) = (ab^{-(s+1)}, ab^{-2(s+1)}, b, b^s).$$

Conjugate the result by b^e where $2e - (s+1) = 0 \bmod p$ to get:

$$(ab^{-(s+1)}, ab^{-2(s+1)}, b, b^s) \to (ab^{2e-(s+1)}, ab^{2e-2(s+1)}, b, b^s) = (a, ab^{-(s+1)}, b, b^s).$$

Thus $\Phi_{1,2}^{-1} \cdot \mathcal{V}_s^{\alpha G} = \mathcal{V}_s^{\alpha G}$ and $\Phi_{1,2}^{-1}$ and $\Phi_{1,2}^{-2}$ fix every automorphism class setwise.

Next, for $\Phi_{2,3}^{-2}$ we have:

$$\begin{aligned} \Phi_{2,3}^{-2} \cdot (a, ab^{-(s+1)}, b, b^s) &= (a, (ab^{-(s+1)})^b, b^{ab^{-s}}, b^s) = (a, a^b b^{-(s+1)}, b^a, b^s) \\ &= (a, ab^{-(s+2)}, b^{-1}, b^s) \end{aligned}$$

Now apply $b \to b^{-1}$ to convert to standard form

$$(a, ab^{-(s+2)}, b^{-1}, b^s) \to (a, ab^{(s+2)}, b^1, b^{-s}).$$

Thus $\Phi_{2,3}^{-2} \cdot \mathcal{V}_s^{\alpha G} = \mathcal{V}_{-s}^{\alpha G}$. The map $s \to -s$ is an involutary map of the finite field $\mathbb{F}_p$ with exactly one fixed point $s = 0$, which is excluded from consideration. So the action of $\Phi_{2,3}^{-2}$ on the automorphism classes of generating vectors is an involution without fixed points.

Finally, for $\Phi_{3,4}^{-1}$ we have:

$$\Phi_{3,4}^{-1} \cdot (a, ab^{-(s+1)}, b, b^s) = (a, ab^{-(s+1)}, b^s, b),$$

since b and b^s commute. Now apply $b \to b^e$, where $es = 1 \bmod p$, to convert to standard form:

$$(a, ab^{-(s+1)}, b^s, b) \to (a, ab^{-e(s+1)}, b^{es}, b^e) = a, ab^{-(e+1)}, b, b^e) = \mathcal{V}_{s'},$$

where $s' = e = \frac{1}{s}$. The map $s \to \frac{1}{s}$ is an involutary linear fractional transformation of $\mathbb{F}_p^*$, with ± 1 as fixed points. It follows that $\Phi_{3,4}^{-1}$ is an involution with two fixed points of the automorphism classes parameterized by $\mathbb{F}_p^*$.

We are now ready to calculate $|L_{G,\mathcal{V}}(\sigma)|$ and $|M_\sigma|$. To compute $|L_{G,\mathcal{V}}(\sigma)|$ we just need to determine the orbit $\left\langle \Phi_{1,2}^{-2}, \Phi_{2,3}^{-2}, \Phi_{3,4}^{-2} \right\rangle \cdot \mathcal{V}_s^{\alpha G}$. By calculation $\Phi_{1,2}^{-2}$ and $\Phi_{3,4}^{-2}$ act trivially on the automorphism classes, and so

$$\left\langle \Phi_{1,2}^{-2}, \Phi_{2,3}^{-2}, \Phi_{3,4}^{-2} \right\rangle \cdot \mathcal{V}_s^{\alpha G} = \left\langle \Phi_{2,3}^{-2} \right\rangle \cdot \mathcal{V}_s^{\alpha G} = \left\{ \mathcal{V}_s^{\alpha G}, \mathcal{V}_{-s}^{\alpha G} \right\}.$$

Thus, the pure braid orbits are the pairs $\left\{ \mathcal{V}_s^{\alpha G}, \mathcal{V}_{-s}^{\alpha G} \right\}$, resulting in $|L_{G,\mathcal{V}}(\sigma)| = 2$.

Now we compute M_1. By signature considerations we have $M_1 \subseteq \langle (1,2), (3,4) \rangle$. Since $\Phi_{1,2}$ acts trivially then $(1,2) \in M_1$. Next, we determine which values s allow $\Phi_{3,4}^{-1} \cdot \{s, -s\} = \{s, -s\}$ as sets. We need a solution to

$$\frac{1}{s} = s \text{ or } \frac{1}{s} = -s,$$

the latter when $\Phi_{3,4}$ switches s and $-s$. We already know that the solutions to $\frac{1}{s} = s$ are ± 1. The second equation leads to the quadratic equation $s^2 = -1$, i.e., -1 is a quadratic residue $\bmod p$, which in turn is true if and only if $p \equiv 1 \bmod 4$. It follows that $M_1 = \langle (1,2)(3,4) \rangle$ if $s \in \{1, -1\}$ or $s^2 = -1$. Otherwise, $M_1 = \langle (1,2) \rangle$.

We summarize the dihedral example in a proposition.

Proposition 3.14. *Let p be an odd prime and $G = D_p$. Then there are $\frac{p-1}{2}$ strata in $\mathcal{M}_{p-1}$ with a $(2,2,p,p)$ action of G. In every case, the pure braid orbit of automorphism classes has two points. There is always one stratum with $|M_\sigma| = 4$. If $p = 1$ mod 4 then there is a second stratum with $|M_\sigma| = 4$. All other strata satisfy $|M_\sigma| = 2$.*

Example 3.15. Continuing Example 2.7 with $G = A_5$, our Magma calculations show that there are two strata both of which satisfy $L_{G,\mathcal{V}}(\sigma) = 9$ and $|M_\sigma| = 2$.

Example 3.16. We continue Example 2.8, where $G = \langle a, b : a^p = b^q, b^a = b^r \rangle$. The calculation of the braid action is very similar the dihedral group, so we just state the results. From Example 2.8 we recall that the classes of generating (p,p,q,q) vectors are $(p-1)(q-1)$ in number with these specific representatives $\mathcal{V}_{u,v} = (a^u, a^{-u}b^{-(v+1)}, b, b^v)$ for $(u,v) \in \mathbb{F}_p^* \times \mathbb{F}_q^*$. Here are the formulas for the pure braid group action.

$$\begin{aligned} \Phi_{1,2}^{-2} &: \mathcal{V}_{u,v}^{\alpha G} \to \mathcal{V}_{u,v}^{\alpha G}, \\ \Phi_{2,3}^{-2} &: \mathcal{V}_{u,v}^{\alpha G} \to \mathcal{V}_{u,wv}^{\alpha G}, \ w = r^u, \\ \Phi_{3,4}^{-2} &: \mathcal{V}_{u,v}^{\alpha G} \to \mathcal{V}_{u,v}^{\alpha G}. \end{aligned}$$

Therefore, the pure braid group acts as $(u,v) \to (u, r^u v)$ on $\mathbb{F}_p^* \times \mathbb{F}_q^*$, i.e., $\mathcal{P}_4 \cdot \mathcal{V}_{u,v}^{\alpha G} = \left\{ \mathcal{V}_{u,wv}^{\alpha G} : w^p = 1 \, mod \, q \right\}$. The orbits all have size p.

We may calculate the group M_1 as follows. Since $M_1 \leq \langle (1,2), (3,4) \rangle$ we just need the action of $\Phi_{1,2}^{-1}$ and $\Phi_{3,4}^{-1}$. We calculate

$$\begin{aligned} \Phi_{1,2}^{-1} &: \mathcal{V}_{u,v}^{\alpha G} \to \mathcal{V}_{-u,v}^{\alpha G}, \\ \Phi_{3,4}^{-1} &: \mathcal{V}_{u,v}^{\alpha G} \to \mathcal{V}_{u,x}^{\alpha G}, \ x = v^{-1}. \end{aligned}$$

If $\Phi_{1,2}^{-1}$ stabilizes $\mathcal{P}_4 \cdot \mathcal{V}_{u,v}^{\alpha G}$ then $a^u = a^{-u}$ which cannot happen since a^u has odd order. This is a difference from the dihedral group example. It follows then that $M_1 \leq \langle(3,4)\rangle$. Next, if $\Phi_{3,4}^{-1}$ maps the orbit $\mathcal{P}_4 \cdot \mathcal{V}_{u,v}^{\alpha G}$ to itself then $v^{-1} = vw$, or $v^2 = w^{-1}$, for some w with $w^p = 1$ *mod* q. Thus $v^{2p} = 1$ and there are $2p$ solution to this equation. Thus $M_1 = \langle(3,4)\rangle$ for the $2p$ values of v satisfying $v^{2p} = 1\, mod\, q$ and is trivial otherwise.

3.3.3. *Bundle of actions and topological equivalence.* The bundle of actions captures a class of topologically equivalent actions. We have:

PROPOSITION 3.17. *Let G act n-gonally on a reference surface S_0 by ϵ_0 with corresponding monodromy ξ_0. Let $b_0 \in B$ be a base point in the bundle of quotients determined by S_0/G. Let $\widetilde{p} : \widetilde{E} \to \widetilde{B}$ be the bundle of actions determined by the epimorphism class $\xi_0^{\alpha G}$ lying over the base point b_0. Then, every surface S with G action ϵ, topologically equivalent to ϵ_0, occurs at least once in the bundle of actions. Moreover, any two fibres of the bundle of actions determine topologically equivalent actions of G.*

PROOF. First we note that, by construction, the unramified covers of punctured surfaces $\widetilde{F}_{b_1} \to F_{b_1}$ and $\widetilde{F}_{b_2} \to F_{b_2}$ define topologically equivalent actions of G on the closures, proving the second statement. Next, the supposed topological equivalence between ϵ_0 and ϵ gives rise to the following diagram where the vertical maps are quotient maps by G and the horizontal maps are homeomorphisms.

$$\begin{array}{ccc} S_0 & \overset{\phi}{\to} & S \\ \downarrow \pi_0 & & \downarrow \pi \\ T_0 & \overset{\overline{\phi}}{\to} & T \end{array} . \tag{3.20}$$

Let $b = \overline{\phi}(b_0)$ as a tuple. Let δ_1 be a path in B from b_0 to b, and $h : T_0 \to T$ the homeomorphism induced by local triviality of the bundle of quotients. The map $h^{-1}\overline{\phi}$ is a homeomorphism of T_0 fixing b_0 pointwise. By results of Birman [**Bi**] (see also [**FM**] page 245) the map $\pi_1(B, b_0) \to MCG_{0,t}$ maps onto the subgroup of $MCG_{0,t}$ fixing all punctures. The group $MCG_{0,t}$ is the mapping class group of the sphere with t punctures. It follows that there is a loop δ_2 such that the homeomorphism induced by δ_2 is $h^{-1}\overline{\phi}$. The homeomorphism induced by $\delta_2\delta_1$ is $h\left(h^{-1}\overline{\phi}\right) = \overline{\phi}$. Thus $\overline{\phi}$ is induced by a path in B and so $\pi : S \to T$ lies in the bundle of actions. □

3.4. The map to moduli space. Finally, we define the "cover" map from the bundle of actions to a closed stratum in moduli space. Recall the family $\widetilde{\mathcal{S}} = \left\{(S_{\tilde{b}}, \pi_{\tilde{b}}) : \tilde{b} \in \widetilde{B}\right\}$, defined in (3.15). Analogous to the map (3.1), we define a specific map (with the same name) to moduli space:

$$\Theta : \widetilde{\mathcal{S}} \to \mathcal{M}_g,\ \tilde{b} \to (S_{\tilde{b}}, \pi_{\tilde{b}}) \to \text{conformal class of } S_{\tilde{b}}. \tag{3.21}$$

The stratum $\mathcal{S}$ has an analytic structure (in fact is a smooth quasi-projective variety) arising from the structure on $\mathcal{M}_g$ as an orbifold quotient of the Teichmüller space $\mathcal{T}_g$. We are not going to prove anything about Θ being a morphism, with the given structure on $\mathcal{M}_g$, other than to demonstrate that it is finite-to-one over $\mathcal{S}$ in the situation in Section 4, and that the fibres have cardinality $|M_\sigma|$. Because of the space limitations of this paper we shall defer discussion of the morphism properties of Θ to a subsequent publication.

We determine the image of Θ, showing that it is the closure of a single stratum. There are two cases:

(1) For at least one $\tilde{b}_0 \in \widetilde{B}$, $\mathrm{Gal}(\pi_{\tilde{b}_0}) = \mathrm{Aut}(S_{\tilde{b}_0})$.
(2) For all $\tilde{b} \in \widetilde{B}$, $\mathrm{Gal}(\pi_{\tilde{b}}) < \mathrm{Aut}(S_{\tilde{b}})$.

The second case corresponds precisely to *non-maximal actions*. We are going to skip the detailed study of the non-maximal case, as we are primarily interested in the first case. We do show however how to detect non-maximal actions in Section 4.6. We state our result as a proposition.

PROPOSITION 3.18. *Consider the map $\widetilde{\mathcal{S}} \to \mathcal{M}_g$ given in (3.21). Assume that for some $\tilde{b}_0 \in \widetilde{B}$, $\mathrm{Gal}(\pi_{\tilde{b}_0}) = \mathrm{Aut}(S_{\tilde{b}_0})$. Then the image of Θ is $\overline{\mathcal{S}}$ where $\mathcal{S}$ is the equisymmetric stratum determined by $S_{\tilde{b}_0}$.*

PROOF. Denote the topological type of the G action on $S_{\tilde{b}_0}$ by $(G)_0$. By hypothesis $(G)_0 = \Sigma(S_{\tilde{b}_0})$, we set $\mathcal{S} = \mathcal{S}_{(G)_0}$. For any $b \in \widetilde{B}$, let $(G)_{\tilde{b}}$ be the topological type of the subgroup $\mathrm{Gal}(\pi_{\tilde{b}})$ of $\mathrm{Aut}(S_{\tilde{b}})$. By Proposition 3.17, all the G actions in the family $\widetilde{\mathcal{S}}$ are topologically equivalent, so that $(G)_{\tilde{b}} = (G)_0$ for every $\tilde{b}$. Next, for every $\tilde{b} \in \widetilde{B}$ we have $(G)_0 \leq (\mathrm{Aut}(S_{\tilde{b}}))$ and so from (2.18) we see that $S_b \in \mathcal{S}'$ for some $\mathcal{S}' \subseteq \overline{\mathcal{S}}$. It follows that $\Theta(\widetilde{\mathcal{S}}) \subseteq \overline{\mathcal{S}}$.

For the reverse inclusion, suppose that $S \in \overline{\mathcal{S}}$. From (2.17) $(G)_0 = \Sigma(\mathcal{S}) \leq \Sigma(S)$. It follows then that S has a G-action of topological type $(G)_0$. However, every surface with G action of topological type $(G)_0$ occurs somewhere in the family. Thus S lies in the image of Θ and

$$\Theta(\widetilde{\mathcal{S}}) = \overline{\mathcal{S}} = \overline{\mathcal{S}_{(G)_0}}. \tag{3.22}$$

□

3.5. Redundancy in the bundle of actions. Now that we have constructed a "cover" of a stratum closure in (3.21), there may be redundancies, i.e., a surface $S \in \overline{\mathcal{S}}$ may come from several $S_{\tilde{b}}$, $\tilde{b} \in \widetilde{B}$ perhaps infinitely many. Stated differently, the fibres of Θ are not all singletons. We now quantify the redundancy by comparing pairs (S_1, π_1), (S_2, π_2) in $\widetilde{\mathcal{S}}$, (see (3.15)) where S_1 and S_2 are conformally equivalent surfaces. The quantities $L_{G,\mathcal{V}}$ and M_σ will come into play. We give a general overview for $t \geq 4$ below and specific detail for the $t = 4$ case (where details are simpler) in Section 4.

Indeed, suppose in the family $\widetilde{\mathcal{S}} = \left\{(S_{\tilde{b}}, \pi_{\tilde{b}}) : \tilde{b} \in \widetilde{B}\right\}$ (defined in 3.15), that we have two pairs (S_1, π_1), (S_2, π_2) defined by n-gonal G actions ϵ_1 and ϵ_2 with t branch points. If $\phi : S_1 \to S_2$ is a conformal equivalence, we consider whether we can fill in the bottom row of the following diagram with a map $\overline{\phi}$, where π_i is a quotient map by $\epsilon_i(G)$, $i = 1, 2$.

$$\begin{array}{ccc} S_1 & \overset{\phi}{\to} & S_2 \\ \downarrow \pi_1 & & \downarrow \pi_2 \\ \widehat{\mathbb{C}} & \overset{\overline{\phi}}{\to} & \widehat{\mathbb{C}} \end{array} \tag{3.23}$$

If we can find such a map $\overline{\phi}$ we say that (S_1, π_1), (S_2, π_2) are *conformally equivalent pairs*.

We can also consider the diagram in terms of monodromies. According to the sentences after (3.15), the maps π_1, π_2 are determined by pairs $(b_1, \xi_1^{\alpha G})$, $(b_2, \xi_2^{\alpha G})$, where $B_{\pi_i} = \overline{b}_i$, as sets, and the $\xi_i^{\alpha G}$ are $\mathrm{Aut}(G)$ classes of monodromies on $\pi_1(\widehat{\mathbb{C}} - B_{\pi_i})$. In this case we often assume that the bottom row and the monodromies are given and we try to fill in the top row with a conformal map.

These questions can be answered with the following proposition which can easily be proven using covering space arguments.

Proposition 3.19. *Let (S_1, π_1), (S_2, π_2) be pairs in the family (3.15) and assume that $B_{\pi_i} = \overline{b}_i$ and π_i is determined by $(b_i, \xi_i^{\alpha G})$ for $i = 1, 2$. If $\phi : S_1 \to S_2$ is a conformal equivalence, then the diagram (3.23) can be completed with a map $\overline{\phi}$ if and only if $\epsilon_2(G) = \phi\epsilon_1(G)\phi^{-1}$. The map $\overline{\phi}$ is unique and conformal, since it satisfies $\overline{\phi}\pi_1 = \pi_2\phi$. The condition on the image subgroups will automatically hold if G is isomorphic to the two automorphism groups. In addition, the branch sets satisfy:*

$$B_{\pi_2} = \overline{\phi}(B_{\pi_1}) \tag{3.24}$$

as sets, and

$$\xi_2^{\alpha G} = \xi_1^{\alpha G} \circ (\overline{\phi}_*)^{-1}. \tag{3.25}$$

Next, assume we are given the two columns and bottom row of the diagram (3.23), $\overline{\phi}$ is a conformal map, and that condition (3.24) holds for the branch sets. Then, the diagram can be completed with a conformal map ϕ if and only if equation (3.25) holds.

Our first step in analyzing redundancy is to show we may assume that $B_{\pi_2} = B_{\pi_1}$ in the diagram (3.23). We formulate this in the following proposition, where we assume that we are using the standard Hurwitz model of equivalence.

Proposition 3.20. *Suppose (S, π) lies in the family (3.15) and that other notation is as in the preceding proposition. If α is an automorphism of $\widehat{\mathbb{C}}$, then $(S, \alpha\pi)$ lies in the family (3.15).*

Proof. Let $b \in B$ correspond to B_π and $\xi^{\alpha G}$ be the epimorphism class that corresponds to π. We show that we can produce $(S, \alpha\pi)$ as a member of the family by transporting the epimorphism class $\xi^{\alpha G}$ to $\pi_1(\widehat{\mathbb{C}} - \alpha(B_\pi))$. Since the automorphism group of $\widehat{\mathbb{C}}$ is connected, there is a 1-parameter family α_s, $0 \leq s \leq 1$, such that $\alpha_0 = \mathrm{Id}$ and $\alpha_1 = \alpha$. The assignment $\beta(s) = \alpha_s(b)$ defines a path in the base space B whose covering homeomorphisms $h_s : F_{\beta(0)} \to F_{\beta(s)}$ may be taken to be α_s. The epimorphism class thereby transported is $\xi_1^{\alpha G} \circ (\alpha_*)^{-1}$; this epimorphism class corresponds to $\alpha\pi$. □

Remark 3.21. The preceding proposition shows that orbits of $\mathrm{Aut}(\widehat{\mathbb{C}})$ on the family (3.15) all produce the same point in moduli space, and so that the reduced Hurwitz space is a better model for a stratum when trying to eliminate redundancy in Θ. This can be accomplished by starting with the quotient space $B/\mathrm{Aut}(\widehat{\mathbb{C}})$ and then constructing a bundle of quotients, a bundle of actions, a family of surface pairs, and then the map Θ. One could also try starting with $B_\Sigma/\mathrm{Aut}(\widehat{\mathbb{C}})$ though there are a few issues. Recalling the continuous component of moduli space analysis described in Section 1, we see that $B_\Sigma/\mathrm{Aut}(\widehat{\mathbb{C}})$ is the moduli space of quotients and $B/\mathrm{Aut}(\widehat{\mathbb{C}})$ is (branched) cover of it. However, a virtue of using the $B/\mathrm{Aut}(\widehat{\mathbb{C}})$ as a starting point is that $B \to B/\mathrm{Aut}(\widehat{\mathbb{C}})$ is a fibre bundle with a manifold base and a continuous section (see Section 4). However, because of the presence of finite sets of $\widehat{\mathbb{C}}$ with non-trivial conformal symmetries, $B_\Sigma \to B_\Sigma/\mathrm{Aut}(\widehat{\mathbb{C}})$ is not a fibre bundle, and $B_\Sigma/\mathrm{Aut}(\widehat{\mathbb{C}})$ is only an orbifold.

3.5.1. *Redundancy over a given* $(\widehat{\mathbb{C}}, B_G)$. We now assume that $B_G = B_{\pi_2} = B_{\pi_1}$, as sets in the diagram (3.23), and proceed to analyze redundancy. Since we are recording branch sets as t-tuples, the points $b_1, b_2 \in B$ determined by the surface-map pairs satisfy $b_1 = b$, $b_2 = \sigma b$ for some $b \in B$, $\sigma \in \Sigma_t$. Letting Φ denote the braid operation on $\pi_1(\widehat{\mathbb{C}} - B_G)$ determined by $\overline{\phi}$, we note that $\sigma = \sigma(\Phi)$. Let $\xi_b^{\alpha G}$ be the monodromy class determining π_1, let $\mathcal{G}$ be a standard generating set of $\pi_1(\widehat{\mathbb{C}} - B_G)$, and let $\mathcal{V}_b^{\alpha G}$ be the generating vector determined by $\xi_b^{\alpha G}$ and $\mathcal{G}$. Redundancy over $(\widehat{\mathbb{C}}, B_G)$ can arise in the three mutually exclusive ways.

I. The map $\overline{\phi}$ is the identity and $\sigma \neq 1$, so that $\sigma \in M_1$, namely $L_{G,\mathcal{V}_b}(1) = L_{G,\mathcal{V}_b}(\sigma)$.
II. The map $\overline{\phi}$ is not the identity and, consequently, $\sigma \neq 1$.
III. The surfaces are determined by different points in $\widetilde{B}$ that project to the same point in B, i.e., $\sigma = 1$.

We separate the first two issues since the first can be resolved solely from the structure of $L_{G,\mathcal{V}_b}$, whereas the second is dependent on B_G. We address issues I and II in Proposition 3.22 following and issue III in Proposition 3.23. The first proposition clarifies how the conformal equivalences are reflected in the discrete system $L_{G,\mathcal{V}_b}$.

PROPOSITION 3.22. *Suppose that surface-map pairs* (S_1, π_1), (S_2, π_2) *are chosen from the family* (3.15) *such that diagram* (3.23) *is satisfied with conformal horizontal maps. We also assume equal branch sets:* $B_G = B_{\pi_1} = B_{\pi_2}$, *and all other notation as above. Then, we have the following.*

(1) *Any redundancy falls into one of the three cases above.*
(2) *If* $\sigma(\Phi) = 1$ *then* $\overline{\phi}$ *is the identity.*
(3) *The map* $\mathcal{V}^{\alpha G} \to \Phi \cdot \mathcal{V}^{\alpha G}$ *induces a bijection* $L_{G,\mathcal{V}_b}(1) \leftrightarrow L_{G,\mathcal{V}_b}(\sigma)$. *The surfaces paired by this bijection are conformally equivalent.*
(4) *Assume only that* Φ *is induced by a conformal map* $\overline{\phi}$. *Then the two sets of surfaces* $\left\{(S_{\tilde{b}} : \overline{q}(\tilde{b}) = \sigma' b\right\}$ *and* $\left\{(S_{\tilde{b}} : \overline{q}(\tilde{b}) = \sigma\sigma' b\right\}$ *are conformally equivalent in a 1-1 fashion as in statement (3).*

PROOF. Assume the diagram (3.23) satisfies our hypothesized conditions. The proof of statement (1) is easy and left to the reader. To prove statement (2) we observe that $\sigma(\Phi) = 1$ implies that $\overline{\phi}$ fixes all the points of B_G. This means that $\overline{\phi}$ fixes 4 or more points and so must be the identity. To prove statement (3), pick any $\mathcal{V} \in L_{G,\mathcal{V}_b}(1)$ with ξ and $\pi : S \to \widehat{\mathbb{C}}$ the corresponding monodromy and branched cover. Then choose for the right hand side of the diagram the branched cover $\pi' : S' \to \widehat{\mathbb{C}}$ determined by $\xi' = \xi \circ (\overline{\phi})^{-1}$, which in turn is determined by $\mathcal{V}' = \Phi \cdot \mathcal{V}$. In the diagram (3.23) replace the surface-map pairs (S_1, π_1), (S_2, π_2) by (S, π), (S', π'), respectively. By the second half of Proposition 3.19 the surfaces S and S' are conformally equivalent. The proof of statement (4) is similar to that of statement (3). □

Proposition 3.23. *Let $\tilde{b} \in \widetilde{B}$ lie over $b \in B$, and suppose that $G \simeq \mathrm{Aut}(S_{\tilde{b}})$. Then $S_{\tilde{b}}$ cannot be conformally equivalent to any other $S_{\tilde{b}'}$ where $\tilde{b}'$ also lies over b.*

Proof. Seeking a contradiction, suppose that the two surfaces $S_{\tilde{b}}$ and $S_{\tilde{b}'}$ are conformally equivalent. We may plug them into the diagram (3.23) for S_1 and S_2. By Proposition 3.19 and the condition on the automorphism group, the diagram may be completed to a commutative diagram with holomorphic rows. However, $\overline{\phi}$ is induced by a pure braid operation Φ since both $\tilde{b}$ and $\tilde{b}'$ lie over the same b. Since $\sigma(\Phi) = 1$ from statement 3 of Proposition 3.22 it follows that $\overline{\phi}$ must be the identity. The epimorphism classes defining the covers are the same by equation (2.20) and so $\tilde{b}$ and $\tilde{b}'$ are actually the same point. □

3.5.2. *Summary of redundancy.* We summarize informally how Propositions 3.22 and 3.23 shed some light on redundancies over a given B_G. Let b be some t-tuple representing B_G, and $\mathcal{V}$ a generating vector for t points. Let $\mathcal{N}_{B_G}$ be the subgroup of $\mathcal{F}_t$ induced by automorphisms of $\widehat{\mathbb{C}}$ that fix $\overline{b} = B_G$ as a set. Note that $\mathcal{N}_{B_G}$ can vary as the branch set B_G varies. The subgroup $\langle \mathcal{N}_{B_G}, \mathcal{P}_t \rangle$ is of finite index in $\mathcal{F}_t$ and the map $\mathcal{F}_t \to \Sigma_t$ is injective when restricted to $\mathcal{N}_{B_G}$. Using the terminology of the proof of Proposition 3.7, let $\mathcal{O}$ be an orbit $\langle \mathcal{N}_{B_G}, \mathcal{P}_t \rangle$ of the blocks of $\Omega = \mathcal{F}_t \cdot \mathcal{V}^{\alpha \mathcal{G}}$ determined by the orbits of $\mathcal{P}_t$ on Ω. Then, by Proposition 3.22, for any two $L_{G,\mathcal{V}}(\sigma_1)$, $L_{G,\mathcal{V}}(\sigma_2) \in \mathcal{O}$, with $\sigma_1 b \neq \sigma_2 b$, the surfaces over $\sigma_1 b$ are in 1-1 conformal correspondence with the surfaces lying over $\sigma_2 b$. This characterizes "horizontal redundancy" as we move over the set B.

The "vertical redundancy" i.e., redundancy within a fibre $\tilde{p}^{-1}(b)$ is governed by Proposition 3.23. If a surface $S_{\tilde{b}}$ maps to a point $\Theta(\tilde{b}) \in \mathcal{S}$ then $G = \mathrm{Aut}(S_{\tilde{b}})$ and $S_{\tilde{b}}$ is not conformally equivalent to any other surface lying over b. However, this does not guarantee that all the surfaces lying over b are conformally inequivalent, even though it should be generically true. Resolving this issue is beyond the scope of this paper.

3.5.3. *Example of redundancy.* We finish this section by giving an explicit example of lifting a conformal automorphism of (T, B_G) to a conformal equivalence of different surfaces lying over (T, B_G).

Example 3.24. Let p be a prime, let $a_i \in \mathbb{C}$ be distinct, and let $n_1, \ldots, n_t$ be integers satisfying $1 \leq n_j < p$ such that p divides $n_1 + \cdots + n_t$. The projective closure of the plane curve defined by $y^p = f(x) = (x - a_1)^{n_1} \cdots (x - a_t)^{n_t}$ may be smoothed by normalization to get a branched cover $\pi : S \to \widehat{\mathbb{C}}$, $(x, y) \to x$. The branch points are the a_j and π is unramified over ∞ since p divides $n_1 + \cdots + n_t$. The covering transformations $\mathrm{Gal}(\pi)$ can be realized as the p'th roots of unity $(x, y) \to (x, uy)$, $u \in U_p = \{u \in \mathbb{C} : u^p = 1\}$. Using a little complex analysis we see that the epimorphism $\xi : \pi_1(T^\circ) \to U_p$ corresponding to π is given by $\xi(\gamma_j) = \exp(2\pi i \frac{n_j}{p})$.

Now let us be specific and suppose that our equation is

$$y^p = f(x) = x^{n_1}(x-1)^{n_2}(x+1)^{n_3}(x-a)^{n_4}, \tag{3.26}$$

where $a \neq 0, \pm 1$. This surface branches over the (ordered set of) points $(0, 1, -1, a)$. Consider the transformation $z = \frac{z-a}{az-1}$. It maps $0 \leftrightarrow a$, $1 \leftrightarrow -1$, and so is an automorphism of $\widehat{\mathbb{C}}$ permuting the specific branch points. We need to transform the equation (3.26) to the following equation

$$w^p = (z-a)^{n_1}(z+1)^{n_2}(z-1)^{n_3}z^{n_4}. \tag{3.27}$$

which is branched over $(a,-1,1,0)$. The following substitutions effect the transformation

$$z = \frac{x-a}{ax-1}, \text{ and } w = C\frac{y}{(ax-1)^d},$$

where $n_1+n_2+n_3+n_4 = pd$, and $C^p = (1-a^2)^{n_1}(1+a)^{n_2}(1-a)^{n_3}$.

4. Case: Orbit genus 0 and 4 branch points

In this section we consider Riemann surfaces S such that $\mathrm{Aut}(S) \simeq G$, a given finite group, and such that the quotient map $S \to S/G$ is a cover of the Riemann sphere $\pi_G : S \to \widehat{\mathbb{C}}$ branched over four points z_1, z_2, z_3, z_4. We adopt all the notation of the preceding sections.

Let $\mathcal{S}$ be the n-gonal stratum of dimension 1 determined by S. We shall outline an approach to determine the topology of $\mathcal{S}$ following the discussion in the previous section. We shall determine the topology of the "best" Hurwitz model $\widetilde{\mathcal{S}}$ and make good progress on identifying the redundancy of the map $\Theta : \widetilde{\mathcal{S}} \to \overline{\mathcal{S}}$. We follow these steps.

I. Construct a bundle of quotients $p : E \to B$ and a bundle of actions $\tilde{p} : \tilde{E} \to \tilde{B}$, corresponding to a reduced Hurwitz space. This is done in Section 4.1.
II. Build a reduced Hurwitz space model $\left\{(S_{\tilde{b}}, \pi_{\tilde{b}}) : \tilde{b} \in \widetilde{B}\right\}$ of $\widetilde{\mathcal{S}}$ with a map $\widetilde{\mathcal{S}} \leftrightarrow \widetilde{B} \to B$. This is done in Section 4.1.
III. Determine the action of the fundamental group $\pi_1(B, b_0)$ and the monodromy $\widetilde{\mathcal{S}} \leftrightarrow \widetilde{B} \to B$. From this we determine the topology of $\widetilde{\mathcal{S}}$: genus and number of punctures. This is done in Section 4.2.
IV. As in Section 3.5, define the map $\Theta : \widetilde{\mathcal{S}} \to \overline{\mathcal{S}}$, and analyze the redundancies of Θ. This is done in Section 4.4.

4.1. Reduced Hurwitz space. First we discuss a specific implementation of the reduced Hurwitz space for four branch points. Using the Möbius transformation

$$L(z) = \frac{z_2 - z_3}{z_2 - z_1}\frac{z - z_1}{z - z_3}, \tag{4.1}$$

and the cross-ratio

$$\lambda = L(z_4) = \frac{z_2 - z_3}{z_2 - z_1}\frac{z_4 - z_1}{z_4 - z_3}, \tag{4.2}$$

we may associate a branch tuple $B_G = (z_1, z_2, z_3, z_4)$ to a standard branch tuple $(0, 1, \infty, \lambda)$ by

$$z_1 \leftrightarrow 0, z_2 \leftrightarrow 1, z_3 \leftrightarrow \infty, \text{ and } z_4 \leftrightarrow \lambda. \tag{4.3}$$

By the invariance of the cross-ratio, if $\alpha \in \mathrm{Aut}(\widehat{\mathbb{C}})$ and $z_i' = \alpha(z_i)$ then (z_1', z_2', z_3', z_4') determines the same λ.

We may therefore assume that all our branched covers have the form:

$$\pi_{G,\lambda} : S \to S/G \xrightarrow{L} \widehat{\mathbb{C}},$$

branched over $\{0, 1, \infty, \lambda\}$. We set

$$T_\lambda^\circ = \widehat{\mathbb{C}} - \{0, 1, \infty, \lambda\},$$

the conformal structures of T_λ° and $S^\circ \xrightarrow{\pi_{G,\lambda}} T_\lambda^\circ$ are determined by λ. Thus, all branched covers $\pi_{G,\lambda}$ under consideration are captured by the sets

$$\Lambda = \widehat{\mathbb{C}} - \{0, 1, \infty\} \text{ and } B_\Lambda = \{(0, 1, \infty, \lambda) : \lambda \in \Lambda\}\,. \tag{4.4}$$

The varying family of punctured quotient surfaces $\{T_\lambda^\circ : \lambda \in B\}$ can be put together into a bundle of quotients as in Section 3.2. Namely,

$$E_\Lambda = \left\{(z, \lambda) \in \widehat{\mathbb{C}} \times B : z \notin \{0, 1, \infty, \lambda\}\right\}$$

and

$$p : E_\Lambda \to B_\Lambda,\ (z, \lambda) \to (0, 1, \infty, \lambda) \leftrightarrow \lambda,$$

gives us a locally trivial bundle over B with typical fibre T_λ°. We then construct the analogous bundle of actions as in (3.14), the Hurwitz model $\widetilde{\mathcal{S}}$, and the map Θ given in (3.21).

Unless we need to distinguish the standard and reduced Hurwitz cases, we shall simplify notation to be consistent with our previous discussion by identifying λ with $\{0, 1, \infty, \lambda\}$ and identifying B with Λ and B_Λ so that $\lambda \in B$ makes sense.

4.2. Fundamental group action and topology of $\widetilde{\mathcal{S}}$. In our new setting let us be a bit more specific about the action of the fundamental group. It suffices to describe the action on generating vectors. To this end, let $\lambda_0 \in B$ be a base point, $(c_1, \ldots, c_4)$ a generating vector for some surface $S \to \widehat{\mathbb{C}}$ branched over $(0, 1, \infty, \lambda_0)$, and $\mathcal{C}$ the G-signature of the action. Since B is the Riemann sphere punctured at $0, 1$, and ∞, the fundamental group $\pi_1(B, \lambda_0)$ is generated by counterclockwise loops β_0, β_1 and β_∞ around 0, 1, ∞, respectively, with product 1. The action $(\beta_0)_*$, $(\beta_1)_*$, and $(\beta_\infty)_*$ on $K_G^\circ(\mathcal{C})$ and the various other derived sets in (2.10) - (2.13) are given in Table 3 following. It is essentially the pure braid action described in Section 2.3.1. The table has been adjusted by inner automorphisms so that each of the operators fixes c_1 and the product is the identity. This does not affect the action on $\overline{A_G^\circ}(\mathcal{C})$. The column entitled Φ-formula relates the action of the β's to the braid action introduced in Section 2.3.1. The purpose of the inverse applied to the formula is explained in that section. For x in any group $Ad_x(y) = xyx^{-1}$, the inner automorphism induced by x.

	c_1	c_2	c_3	c_4	Φ-formula
$(\beta_0)_*$	c_1	$c_2^{c_4}$	$c_3^{c_4}$	$c_4^{c_1^{-1}}$	$\left(Ad_{c_1} \circ \Phi_{2,3}^{-2}\right)^{-1}$
$(\beta_1)_*$	c_1	$c_2^{c_1^{-1} c_4^{-1} c_1}$	c_3	$c_4^{c_1 c_3}$	$\left(\Phi_{2,3}^2 \circ Ad_{c_1^{-1}} \circ \Phi_{3,4}^2\right)^{-1}$
$(\beta_\infty)_*$	c_1	c_2	$c_3^{c_4^{-1} c_3^{-1}}$	$c_4^{c_3^{-1}}$	$\left(\Phi_{3,4}^{-2}\right)^{-1}$

Table 3. Action of $\pi_1(B, \lambda_0)$

4.2.1. *Topology of the Hurwitz space model.* Next, we determine the topology of $\widetilde{\mathcal{S}}$ by computing the monodromy $\widetilde{\mathcal{S}} \to B$. Let $\mathcal{O}$ be an orbit of $\pi_1(B, \lambda_0)$ on the $\mathrm{Aut}(G)$-classes $\overline{A_G^\circ}(\mathcal{C})$, and let $\{\mathcal{O}_{0,i} : i\}$, $\{\mathcal{O}_{1,j} : j\}$, $\{\mathcal{O}_{\infty,k} : k\}$ be the orbit decomposition of $\mathcal{O}$ with respect to the cyclic subgroups $\langle \beta_{0*} \rangle$, $\langle \beta_{1*} \rangle$, $\langle \beta_{\infty*} \rangle$. We have the following:

THEOREM 4.1. *Let the notation for $\pi_1(B, \lambda_0)$ orbits of $\overline{A_G^\circ}(\mathcal{C})$ be as above. The covering surface $\widetilde{\mathcal{S}}$ and the covering map $\widetilde{\mathcal{S}} \to B$ are completely described by the following:*

1. *The degree of the covering $\widetilde{\mathcal{S}} \to B$ is the size of the orbit $|\mathcal{O}| = m$.*
2. *Let $\mathcal{O} = \{o_1, ..., o_m : o_i \in \overline{A_G^\circ}(\mathcal{C})\}$. The monodromy of the covering $\widetilde{\mathcal{S}} \to B$ is*
$$\omega : \pi_1(B, \lambda_0) \to \Sigma_{|\mathcal{O}|}$$
defined by $\omega(\beta)(i) = j$ if for each $(c_1, ..., c_4) \in o_i$ we have that $\beta_*(c_1, ..., c_4) \in o_j$, where $\beta \in \pi_1(B, \lambda_0)$.
3. *The sets of local degrees above $0, 1, \infty$ are the sets $\{|\mathcal{O}_{0,i}| : i\}$, $\{|\mathcal{O}_{1,j}| : j\}$, $\{|\mathcal{O}_{\infty,k}| : k\}$, respectively.*
4. *The Riemann surface $\widetilde{\mathcal{S}}$ is a surface of genus*
$$\rho = 1 - \frac{(h_0 + h_1 + h_\infty) - m}{2} \tag{4.5}$$
with $h_0 + h_1 + h_\infty$ punctures, where h_0, h_1, h_∞ are the number of orbits in $\{\mathcal{O}_{0,i} : i\}$, $\{\mathcal{O}_{1,j} : j\}$, $\{\mathcal{O}_{\infty,k} : k\}$ respectively.

PROOF. The covering space $\widetilde{\mathcal{S}} \to B$ is really just the covering space $\widetilde{B} \to B$. By construction, the degree of this unramified covering is the index of $H_{\underline{\xi_0}}$ in $\pi_1(B, b_0)$ which is $m = |\mathcal{O}|$. The cover may be completed to a branched cover $\overline{\widetilde{\mathcal{S}}} \to \widehat{\mathbb{C}}$, which is in fact a Belyi function. The remainder of the proof is a standard application of the monodromy of branched covers and the Riemann-Hurwitz equation. □

4.2.2. *Sample calculations of $\widetilde{\mathcal{S}} \to B$.*

EXAMPLE 4.2. If G is abelian then the pure braid action is trivial on generating vectors according to Table 3. It follows that the cover has degree 1 and hence is an isomorphism to the thrice punctured sphere. Since $m = h_0 = h_1 = h_\infty = 1$, then the formula (4.5) gives the correct answer of genus zero with three punctures.

EXAMPLE 4.3. Our remaining example is a bit lengthy with the results presented in both Tables 4 and 5. We complete the discussion of the dihedral family D_p (Example 3.13), the pq family $\mathbb{Z}_p \ltimes \mathbb{Z}_q$ (Example 3.16), and then consider an interesting selection of A_5 actions. The A_5 examples are then reconsidered in Section 4.6 as examples of potential non-maximal actions.

Tables 4 and 5 are linked, each row of Table 4 corresponds to a row in Table 5 and vice versa. In the two tables, column two either has a signature such as $(2, 2, 3, 3)$ or a G-signature such as $[4, 4, 5, 5]$ in the case of A_5. The square brackets indicate a G-signature and the numbers within indicate which conjugacy classes are selected. For A_5 we use the ordering $\mathcal{K}_2$, $\mathcal{K}_3$, $\mathcal{K}_4$, $\mathcal{K}_5$ with representatives $(1, 2)$, $(1, 2, 3)$, $(1, 2, 3, 4, 5)$ and $(1, 5, 4, 3, 2)$, respectively. The classes $\mathcal{K}_4$ and $\mathcal{K}_5$ consist of elements of order 5 so that $(5, 5, 5, 5)$ is the signature corresponding to $[4, 4, 4, 4]$, $[4, 4, 4, 5]$, and $[4, 4, 5, 5]$. The two classes $\mathcal{K}_4$ and $\mathcal{K}_5$ are switched by Σ_5, the automorphism group of A_5. Therefore, for example, the G-signatures $[4, 4, 4, 5]$ and $[5, 5, 5, 4]$ determine the same sets $\overline{A_G^\circ}(\mathcal{C})$. Moreover, permutation of the entries of a G-signature are induced by the action of the full braid group. Therefore, the totality of the behaviour of the $(5, 5, 5, 5)$ actions of A_5 is captured in the three chosen G-signatures.

In Table 4 we summarize the action of $\pi_1(B,\lambda_0)$ and the monodromy information for our selected examples. For the dihedral groups and the groups of order pq we used the results in Examples 3.13 and 3.16. For the A_5 examples, we used Magma to compute the classes in $\overline{A_G^\circ}(\mathcal{C})$ and then calculate the Table 3 monodromy action of $\pi_1(B,\lambda_0)$ as described in Theorem 4.1. In each row of the table we consider the totality of generating vectors for which the monodromy action is the "same" and so several strata may be included. Note that there are two different types of actions for the G-signature $[4,4,5,5]$ of A_5.

The columns in Table 4 refer to the monodromy information described in Theorem 4.1. In the columns labeled with a β we either give the actual cycle on $\mathcal{O}$ or its cycle structure. The next column gives the order of $\mathrm{Mon}(\beta)=\langle\beta_0,\beta_1,\beta_\infty\rangle$. The last column gives the orbit counts h_0,h_1,h_∞.

Group	sig / $\mathcal{C}$	$\lvert\mathcal{O}\rvert$	β_0	β_1	β_∞	$\lvert\mathrm{Mon}(\beta)\rvert$	(h_0,h_1,h_∞)
D_p	$(2,2,p,p)$	2	$(1,2)$	$(1,2)$	id	2	$(1,1,2)$
$\mathbb{Z}_p\ltimes\mathbb{Z}_q$	(p,p,q,q)	p	p	p	1^p	p	$(1,1,p)$
A_5	$(2,2,2,3)$	9	$1\cdot3\cdot5$	$1\cdot3\cdot5$	$1\cdot3\cdot5$	181440	$(3,3,3)$
A_5	$(2,2,3,3)$	9	$1\cdot3\cdot5$	$1\cdot3\cdot5$	$1\cdot3\cdot5$	181440	$(3,3,3)$
A_5	$[4,4,4,4]$	10	$1^2\cdot3\cdot5$	$1^2\cdot3\cdot5$	$1^2\cdot3\cdot5$	1814400	$(4,4,4)$
A_5	$[4,4,4,5]$	4	$(1,4,2)$	$(1,3,4)$	$(2,4,3)$	12	$(2,2,2)$
A_5	$[4,4,5,5]$	5	$2\cdot3$	$2\cdot3$	$1^2\cdot3$	120	$(2,2,3)$
A_5	$[4,4,5,5]$	2	$(1,2)$	$(1,2)$	id	2	$(1,1,2)$

Table 4. $\pi_1(B,\lambda_0)$ action

In Table 5 we list some information about the strata for each case. The column labeled $g(S)$ is the genus of the surface upon which G acts. The column labeled $\rho(\widetilde{\mathcal{S}})$ is the genus of the $\widetilde{\mathcal{S}}$ computed from formula (4.5). The column labeled #puncs is $h_0+h_1+h_\infty$, the number of punctures. The column labeled $\#\mathcal{V}^{\alpha\mathcal{G}}$ is the number of generating vector classes for which the monodromy information holds. Finally, the column labeled $\#\widetilde{\mathcal{S}}$ is the number of strata for which the monodromy structure is valid. It is computed from the formula $\#\mathcal{V}^{\alpha G}=\#\widetilde{\mathcal{S}}\times\lvert\mathcal{O}\rvert$. The quantity $\lvert\mathcal{O}\rvert$ is in Table 4.

Group	sig / $\mathcal{C}$	$g(S)$	$\rho\left(\widetilde{\mathcal{S}}\right)$	#puncs	$\#\mathcal{V}^{\alpha G}$	$\#\widetilde{\mathcal{S}}$
D_p	$(2,2,p,p)$	$p-1$	0	4	$p-1$	$\frac{p-1}{2}$
$\mathbb{Z}_p\ltimes\mathbb{Z}_q$	(p,p,q,q)	$(p-1)(q-1)$	0	$p+2$	$(p-1)(q-1)$	$\frac{(p-1)(q-1)}{p}$
A_5	$(2,2,2,3)$	6	1	9	18	2
A_5	$(2,2,3,3)$	11	1	9	18	2
A_5	$[4,4,4,4]$	37	0	12	10	1
A_5	$[4,4,4,5]$	37	0	6	4	1
A_5	$[4,4,5,5]$	37	0	7	5	1
A_5	$[4,4,5,5]$	37	0	4	2	1

Table 5. Strata information

REMARK 4.4. From the Tables 4 and 5 it can be seen that the genus of $\widetilde{\mathcal{S}}$ can be positive and the monodromy group can be complex. For the non-abelian pq groups, after a rotation that fixes 0 and switches 1 and ∞, the map $\widetilde{\mathcal{S}} \to B$ is $z \to z^p$.

4.3. The anharmonic group and reduced Hurwitz models. To determine the redundancies of the map Θ we need to know the action of the anharmonic group, namely the transformations of $\widehat{\mathbb{C}}$ determined by a permutation σ of the coordinates of $B_G = (z_1, z_2, z_3, z_4)$. Applying σ to the coordinates of B_G we obtain a new cross-ratio by the following formula:

$$\lambda' = \overline{\sigma}(\lambda) = \frac{z_{\sigma 2} - z_{\sigma 3}}{z_{\sigma 2} - z_{\sigma 1}} \frac{z_{\sigma 4} - z_{\sigma 1}}{z_{\sigma 4} - z_{\sigma 3}}$$

The transforms are given in Table 6 below. The third column gives the values of λ for which the permutation is induced by an automorphism $\overline{\phi}$ of $\widehat{\mathbb{C}}$ mapping $\{0, 1, \infty, \lambda\}$ to itself as a set.

Permutations σ	λ'	$\lambda' = \lambda$
$id, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)$	λ	all λ
$(1,2), (3,4), (1,4,2,3), (1,3,2,4)$	$1-\lambda$	$\frac{1}{2}$
$(1,3), (2,4), (1,2,3,4), (4,3,2,1)$	$1/\lambda$	-1
$(1,4), (2,3), (1,3,4,2), (1,2,4,3)$	$\lambda/(\lambda-1)$	2
$(1,2,3), (4,3,2), (4,2,1), (1,3,4)$	$(\lambda-1)/\lambda$	$\zeta = \frac{1}{2} \pm \frac{\sqrt{3}}{2}$
$(3,2,1), (2,3,4), (1,2,4), (4,3,1)$	$1/(1-\lambda)$	$\zeta = \frac{1}{2} \pm \frac{\sqrt{3}}{2}$

Table 6. Anharmonic action

Next, we need to study the permutations in the first row of Table 6. For every λ there is an automorphism of $\widehat{\mathbb{C}}$ that maps $\{0, 1, \infty, \lambda\}$ to itself and induces the permutation. Since these are the permutations that do not affect the cross-ratio we call these permutations σ and the corresponding automorphism $\overline{\phi}$ the anharmonic kernel action. The non-identity operations are listed in Table 7.

Permutation σ	$\overline{\phi}(0, 1, \infty, \lambda)$	$\overline{\phi}$
$(1,2)(3,4)$	$(1, 0, \lambda, \infty)$	$\frac{\lambda z - \lambda}{z - \lambda}$
$(1,3)(2,4)$	$(\infty, \lambda, 0, 1)$	$\frac{\lambda}{z}$
$(1,4)(2,3)$	$(\lambda, \infty, 1, 0)$	$\frac{z-\lambda}{z-1}$

Table 7. Anharmonic kernel action

4.4. Redundancies of Θ in the reduced model. We are now ready to discuss the redundancies of Θ in the reduced Hurwitz space setting. Most of the work has already been done in the standard setting. However, we need to show that the calculations in the standard and reduced Hurwitz model yield the same results. To distinguish the two settings, we now specify that B is the space of general quadruples $Z = (z_1, z_2, z_3, z_4)$ and B_Λ consists of quadruples $(0, 1, \infty, \lambda)$, see (4.4).

Next, we define a matrix L_Z and map $\mathcal{L}$. Our matrix is based on formula (4.1),

$$L_Z = \begin{bmatrix} z_2 - z_3 & -z_1(z_2 - z_3) \\ z_2 - z_1 & -z_3(z_2 - z_1) \end{bmatrix}. \tag{4.6}$$

The determinant of this matrix is $(z_2 - z_1)(z_3 - z_1)(z_3 - z_2) \neq 0$, and so L_Z determines an element $\overline{L_Z}$ of $PGL_2(\mathbb{C})$. We may take limits if any of z_1, z_2, z_3 is infinite and still get an element of $PGL_2(\mathbb{C})$. Our map is:

$$\mathcal{L} : B \to B_\Lambda \to \Lambda : (z_1, z_2, z_3, z_4) \to (0, 1, \infty, \lambda) \to \lambda, \tag{4.7}$$

where λ is given in formula (4.2). We shall use the matrix formula and the map to show that calculations in the standard and reduced Hurwitz model yield the same results. The essential properties we need are in the following proposition.

PROPOSITION 4.5. *Let B, Λ, B_Λ $\overline{L_Z}$, and $\mathcal{L}$ be as above. Then*

(1) The map $B \to \mathrm{Aut}(\widehat{\mathbb{C}}) \times \Lambda$: $Z \to (\overline{L_Z}, \mathcal{L}(Z))$ is a birational homeomorphism.
(2) The map in (1) induces an isomorphism

$$\pi_1(B) \simeq \pi_1(\mathrm{Aut}(\widehat{\mathbb{C}})) \times \pi_1(\Lambda) \simeq \mathbb{Z}_2 \times \pi_1(\Lambda).$$

In this isomorphism $\pi_1(\mathrm{Aut}(\widehat{\mathbb{C}}))$ acts trivially on the bundle of quotients.

PROOF. For statement (1) we first observe that the map $Z \to \overline{L_Z}$ is a rational map to $PGL_2(\mathbb{C})$. Thus, $Z \to (\overline{L_Z}, \mathcal{L}(Z))$ is rational. The inverse map is

$$(L, \lambda) \to (L^{-1}(0), L^{-1}(1), L^{-1}(\infty), L^{-1}(\lambda)),$$

also rational. For statement (2) we have

$$\pi_1(B) = \pi_1(\mathrm{Aut}(\widehat{\mathbb{C}}) \times \Lambda) = \pi_1(PGL(2, \mathbb{C}) \times \Lambda) = \mathbb{Z}_2 \times \pi_1(\Lambda). \tag{4.8}$$

The action of $\pi_1(\mathrm{Aut}(\widehat{\mathbb{C}}))$ on the bundle of quotients is induced as follows. Let $L_s, 0 \leq s \leq 1$ be a loop in $\mathrm{Aut}(\widehat{\mathbb{C}})$ based at the identity. The path induced in B is $(L_s^{-1}(0), L_s^{-1}(1), L_s^{-1}(\infty), L_s^{-1}(\lambda_0))$ for the base point λ_0. The covering isotopy of the path is L_s^{-1} and so $L_1^{-1} =$ identity induces the action on fundamental group of the fibre. The action therefore is trivial. Thus, we lose nothing in passing to the reduced Hurwitz space. □

Now consider the diagram.

$$\begin{array}{ccccc}
\widetilde{\mathcal{S}} & \xrightarrow{\widetilde{\mathcal{L}}} & \widetilde{\mathcal{S}_\Lambda} & \xrightarrow{\Theta} & \mathcal{S} \\
\updownarrow & & \updownarrow & & \\
\widetilde{B} & \xrightarrow{\widetilde{\mathcal{L}}} & \widetilde{B_\Lambda} & & \\
\downarrow \overline{q} & & \downarrow \overline{q}_\Lambda & & \\
B & \xrightarrow{\mathcal{L}} & B_\Lambda \ . & &
\end{array} \tag{4.9}$$

The left column is the bundle of actions and bundle of surface-map pairs constructed with the standard Hurwitz space and the right column is the same tower constructed with the reduced Hurwitz space model. The lower vertical maps are covering spaces and the upper maps are isomorphisms. The horizontal map at the base has been defined and the remaining horizontal maps are created by lifting. Some remarks about the diagram are in order:

(1) The map $\widetilde{\mathcal{L}}$ really does exist.
(2) The map $\widetilde{\mathcal{L}}$ is bijective on fibres.
(3) The map Θ factors as advertised.

The proof is basically covering space theory. For (1) we observe that the covers $\overline{q}$ and $\overline{q}_\Lambda$ are defined by subgroups stabilizing an epimorphism through the action of the fundamental groups on epimorphisms. It is not hard to show that

$$\mathcal{L}_* \overline{q}_*(\pi_1(\widetilde{B})) = \overline{q}_{\Lambda *}(\pi_1(\widetilde{B_\Lambda}))$$

since $\mathcal{L}_* : \pi_1(B) \to \pi_1(B_\Lambda)$ is surjective and the kernel acts trivially in the $\pi_1(B)$ action on the bundle of quotients. This implies that $\widetilde{\mathcal{L}}$ exists. For statement (2) it suffices to observe that the degrees of $\overline{q}$ and $\overline{q}_\Lambda$ are preserved by $\widetilde{\mathcal{L}}$. This follows from the surjectivity of $\mathcal{L}_*$ and the construction of the covers. Statement (3) follows from the definition of Θ and the fact that for every $(z_1, \ldots, z_4)$ in the inverse image of $(0, 1, \infty, \lambda)$ there is a holomorphic map $\overline{\phi} : \widehat{\mathbb{C}} - \{z_1, \ldots, z_t\} \to \widehat{\mathbb{C}} - \{0, 1, \infty, \lambda\}$, which lifts to conformal equivalences of the surfaces lying over the punctured sphere.

Next we recall that the redundancies of Θ are "horizontal", namely caused by surfaces lying over conformally equivalent orbit spaces. Now that we have factored out the action of $\mathrm{Aut}(\widehat{\mathbb{C}})$, redundancies can only arise by recording B_G as a tuple instead of a set. However, as a result of having generically induced permutations, (Section 4.5) the possible redundancy is reduced from 24 possibilities to 6, which will be the six different values λ' in Table 6, the anharmonic group action table. So, finally we can state as a conjecture the following redundancy characterization in the reduced Hurwitz case. Note for $t > 4$ there are $t!$ possible redundancies in both the standard and reduced Hurwitz models, so t=4 is a peculiar case.

Conjecture 4.6. *Let $\mathcal{S}$ be an equisymmetric stratum corresponding to a maximal action of G, defined by a generating vector class $\mathcal{V}^{\alpha G}$ for some specific generating vector $\mathcal{V} = (c_1, c_2, c_3, c_4)$. Let M_1 (see Proposition 3.7) be the subgroup of Σ_4 determined by the system $L_{G,\mathcal{V}}(\sigma)$, $\sigma \in \Sigma_4$. Then we have the following.*

1. *Horizontal redundancy is determined by the orbits of the group M_1V_4/V_4 acting via the anharmonic group. The subgroup and the action can be read off from the anharmonic action table, Table 6. The subgroup V_4 is the Klein 4 group of generically induced permutations.*
2. *There is no vertical redundancy, at least over the stratum.*
3. *In the pure braid case there is no redundancy and the map Θ is 1-1 over the stratum.*

We only state our result as a conjecture since we have not yet fully resolved the issues below. The section on redundancy in the standard model case gives a lot of evidence that the conjecture is "mostly true".

- What is the nature of the map $\Theta : \widetilde{\mathcal{S}} \to \overline{\mathcal{S}}$ as a morphism?
- A rigorous proof that there is no vertical redundancy over the stratum.
- Identifying the interior punctures on the stratum, namely the special points which are fixed points of the anharmonic group, and points of extra symmetry in $\overline{\mathcal{S}}$ corresponding to non normal overgroups.

4.5. Induced permutations and normal extension of actions. Because we utilize branch tuples instead of branch sets we will frequently need to consider the case $B_{\pi_1} = B_{\pi_2} = B_G$ in diagram (3.23). Write $B_G = \{z_1, \ldots, z_t\}$ and assume that equation (3.24) holds, and that $\overline{\phi}$ is a conformal map (we don't worry about ϕ yet). Then $\overline{\phi}$ determines a permutation σ in Σ_t, determined by the ordering $z_1, \ldots, z_t$. For a given σ two different cases arise.

- For every $B_G = (z_1, \ldots, z_t)$ there is a conformal map $\overline{\phi}$ such that $\overline{\phi}(B_G) = B_G$ and $\overline{\phi}$ induces the conjugacy class of σ. We say that σ is generically induced.
- There is at least one B_G for which there is no conformal map $\overline{\phi}$ such that $\overline{\phi}(B_G) = B_G$ and $\overline{\phi}$ induces σ. We say that σ is exceptionally induced.

For $t = 3$ every permutation is generically induced since $\mathrm{Aut}(\widehat{\mathbb{C}})$ is triply transitive. For $t = 4$ the non-identity permutations which are generically induced are $(1,2)(3,4)$, $(1,3)(2,4)$, and $(1,4)(2,3)$ and constitute the anharmonic kernel action (Table 7 in Section 4.3). Each of these permutations is induced by an order 2 rotation $\overline{\phi}$ switching the branch points in pairs. All remaining permutations are exceptionally induced. The generically induced permutations are linked to non-maximal actions, as we show in an example later in the section

Remark 4.7. Though we do not need this fact for the paper it turns out that for $t > 4$, there are no non-identity generically induced permutations. Generically induced permutations are a peculiarity of four branch points.

Using these ideas, we want to characterize symmetry jumping, namely when a surface $S \in \overline{\mathcal{S}}$ belongs to one of the strata $\mathcal{S}'$ in $\overline{\mathcal{S}} - \mathcal{S}$. i.e., $\epsilon(G) < \mathrm{Aut}(S)$. We give a characterization for normal inclusions. We start with the diagram below based on diagrams (2.19) and (3.23)

$$\begin{array}{ccc} S_1^\circ & & S_2^\circ \\ \downarrow \pi_1 & & \downarrow \pi_2 \\ T_1^\circ & \overset{\overline{\phi}}{\rightarrow} & T_2^\circ \end{array},$$

where we do not assume that there is a ϕ in the top row, but that there is a conformal map $\overline{\phi}$ so that the rest of the diagram commutes. We assume that $T_1 = T_2 = T = \widehat{\mathbb{C}}$. Next, assume $B_{\pi_1} = B_{\pi_2}$ so that $T_1^\circ = T_2^\circ = T^\circ$, and that the monodromies satisfy $\xi_2^{\alpha G} = \xi_1^{\alpha G}$, namely $\xi_2 = \theta \circ \xi_1$ for some $\theta \in \mathrm{Aut}(G)$. Since $\xi_1^{\alpha G} = \xi_2^{\alpha G}$ the kernels are the same and, hence, the surfaces lying over T_1° and T_2° are identical. Thus, $S_1^\circ = S_2^\circ = S^\circ$ and $\pi_2 = \pi_1 = \pi$. Since $\pi_2 = \pi_1 \circ \overline{\phi}$ in the diagram, then equation (2.20) holds. We rewrite this as $\xi_1^{\alpha G} = \xi_2^{\alpha G} = \xi_1^{\alpha G} \circ \overline{\phi}_*$. From covering space theory, after filling in the punctures, and dropping the indices, our diagram now becomes

$$\begin{array}{ccc} S & \overset{\phi}{\rightarrow} & S \\ \downarrow \pi & & \downarrow \pi \\ \widehat{\mathbb{C}} & \overset{\overline{\phi}}{\rightarrow} & \widehat{\mathbb{C}} \end{array}, \tag{4.10}$$

where ϕ is conformal, also from covering space theory. We see that ϕ normalizes the action of G. If $\overline{\phi}$ is not the identity then ϕ cannot lie in $\epsilon(G)$, and the group $\langle \phi, \epsilon(G) \rangle$ strictly contains $\epsilon(G)$, so that $S \in \overline{\mathcal{S}} - \mathcal{S}$.

Example 4.8. Assume that $G = \langle a, b : ab = ba, \ldots \rangle$ is a finite abelian group. Let $B_G = (z_1, z_2, z_3, z_4)$ be some branch set. Consider the generating vectors $\mathcal{V}_1 = (a, a^{-1}, b, b^{-1})$ and $\mathcal{V}_2 = (a^{-1}, a, b^{-1}, b)$ and the corresponding epimorphisms ξ_1 and ξ_2 determined by B_G and the $\mathcal{V}_i$. Then the actions of G defined by the ξ_i are never maximal.

First we note that inversion automorphism $\iota : x \to x^{-1}$ satisfies $\xi_2 = \iota \circ \xi_1$. Furthermore, there is an involution $\overline{\phi}$ of $\widehat{\mathbb{C}}$ satisfying $\overline{\phi} : z_1 \leftrightarrow z_2$, $z_3 \leftrightarrow z_4$. Consider the full braid operation Φ that operates on generating vectors by

$$\Phi : (c_1, c_2, c_3, c_4) \to (c_2, c_1^{c_2}, c_4, c_3^{c_4}).$$

Then $\overline{\phi}_* = \Phi \circ \Psi$, where Ψ is a pure braid operation. Now Ψ acts trivially on generating vectors from an abelian group so that the action $\overline{\phi}_*$ on abelian generating vectors is the involution $(c_1, c_2, c_3, c_4) \leftrightarrow (c_2, c_1, c_4, c_3)$. We are now in the situation described in the paragraphs above so we conclude that $\overline{\phi}$ lifts to an automorphism ϕ of S such that $\langle \phi, G\rangle$ strictly contains G. Since $(1, 2)(3, 4)$ is generically induced we can always construct the extra automorphism ϕ .

4.6. Non-maximal actions. Non-maximal actions do occur but there are a limited number of types. Methods for completely solving this problem are given in [**BCC**], based on Singerman's list in [**Si**]. The paper of Ries [**Ri**] may also be used. First we need to make a brief excursion to Fuchsian groups and surface kernel maps which are analogues of monodromy maps.

Given an G action on S it may be uniformized by a *surface kernel map* or *surface kernel epimorphism* given in sequence form:

$$\Pi \hookrightarrow \Gamma \overset{\eta}{\twoheadrightarrow} G.$$

The groups $\Pi \lhd \Gamma$ are Fuchsian groups satisfying $S \backsimeq \mathbb{H}/\Pi$ and $\Pi \backsimeq \pi_1(S)$. The map η is surjective and the action on S is defined by $\epsilon = \overline{\eta}^{-1}$ where $\overline{\eta} : \Gamma/\Pi \to G$ is the quotient isomorphism and Γ/Π is naturally a subgroup of $\mathrm{Aut}(S)$. Now let us assume that Γ has signature $(n_1, \ldots, n_4)$ so that is has presentation

$$\Gamma = \langle \gamma_1, \ldots, \gamma_4 : \gamma_1\gamma_2\gamma_3\gamma_4 = \gamma_1^{n_1} = \cdots = \gamma_4^{n_4}\rangle \,.$$

The surface kernel map is determined by a generating vector $\mathcal{V} = (c_1, \ldots, c_4)$ with signature $(n_1, \ldots, n_4)$, defined by $c_j = \eta(\gamma_j)$.

If ψ is any automorphism of S then it lifts to an automorphism $\widetilde{\psi}$ of $\mathbb{H}$, normalizing Π, and such that ψ is the induced map of $\widetilde{\psi}$ acting upon $S \backsimeq \mathbb{H}/\Pi$. Now suppose that Δ is a Fuchsian group properly containing Γ and normalizing Π. Then Δ/Π is a subgroup of $\mathrm{Aut}(S)$ strictly containing $\epsilon(G)$. We say that Δ extends the action of G. In turn, if there is a group H such that $\epsilon(G) < H \le \mathrm{Aut}(S)$ then there is a corresponding group Δ with $\Pi \lhd \Delta$ and $H = \Delta/\Pi$.

We will have a non-maximal action if the dimension of the moduli spaces of Δ and Γ both equal 1. (See definition of dimension in Section 2.4.) Then, according to "Singerman's list"[**Si**], $\Gamma \lhd \Delta$ and the only possibilities are

Case	sig Γ	sig Δ	$\lvert\Delta/\Gamma\rvert$	Δ/Γ	restriction
A	(m, m, n, n)	$(2, 2, m, n)$	2	$\mathbb{Z}_2$	$m + n > 4$
B	(n, n, n, n)	$(2, 2, 2, n)$	4	$V_4 = \mathbb{Z}_2 \times \mathbb{Z}_2$	$n > 2$

Notice that case B is two steps of case A. Since $\epsilon(G) \lhd H$ then the conjugation of $H/\epsilon(G)$ will induce one or two automorphisms α, β on G. These can be expressed in terms of generating vectors e.g.,

$$\alpha\mathcal{V} = \mathcal{V}^{\alpha} = (w_1(\mathcal{V}), w_2(\mathcal{V}), w_3(\mathcal{V}), w_4(\mathcal{V}))\,, \tag{4.11}$$

where the $w_i(\mathcal{V})$ are words in $c_1, \ldots, c_4$, and similarly for $\mathcal{V}^{\beta}$. The form of the required automorphisms were determined in [**BCC**] and are reproduced in the following table.

Case	sig Γ	sig Δ	$\mathcal{V}^\alpha$	$\mathcal{V}^\beta$
A	(m,m,n,n)	$(2,2,m,n)$	$\left(c_2, c_1, c_4^{c_1}, c_3^{c_2^{-1}}\right)$	*none*
B	(n,n,n,n)	$(2,2,2,n)$	$\left(c_2, c_1, c_4^{c_1}, c_3^{c_2^{-1}}\right)$	$\left(c_4, c_3^{c_4}, c_2^{c_1^{-1}}, c_1\right)$

The result in [**BCC**] (among many other cases) is that the G action may be extended by Δ if and only if there are automorphisms of G that act as the proposed automorphisms in the table. Specifically, we must have some $\theta \in \mathrm{Aut}(G)$ such that $\theta\mathcal{V} = \mathcal{V}^\alpha$ and similarly for $\mathcal{V}^\beta$.

Remark 4.9. The extension criterion does not depend on the automorphism class of a vector. Assume that $\mathcal{V}' = \theta\mathcal{V}$ for some $\theta \in \mathrm{Aut}(G)$ and that (4.11) holds for $\mathcal{V}$. Then

$$\begin{aligned}\theta\alpha\theta^{-1}\mathcal{V}' &= \theta\alpha\theta^{-1}\theta\mathcal{V} = \theta\alpha\mathcal{V}\\ &= (\theta w_1(\mathcal{V}), \theta w_2(\mathcal{V}), \theta w_3(\mathcal{V}), \theta w_4(\mathcal{V}))\\ &= (w_1(\theta\mathcal{V}), w_2(\theta\mathcal{V}), w_3(\theta\mathcal{V}), w_4(\theta\mathcal{V}))\\ &= (\mathcal{V}')^\alpha .\end{aligned}$$

Example 4.10. The abelian example 4.8 is an ad hoc use of case A.

Example 4.11. We consider the $(5,5,5,5)$ actions of A_5 discussed in Example 4.3. A crude way for checking for automorphisms that fulfill the requirements of α and/or β is to select a generating vector $\mathcal{V}$ from each class, compute $\mathcal{V}^\alpha$ and $\mathcal{V}^\beta$ then check if $\theta\mathcal{V} = \mathcal{V}^\alpha$ or $\theta\mathcal{V} = \mathcal{V}^\beta$ as we run through all 120 elements in $\theta \in \Sigma_5$. If we have a way of computing normal forms for automorphism classes of generating vectors we just check that $\mathcal{V}$ and $\mathcal{V}^\alpha$ have the same normal form and the similarly for $\mathcal{V}^\beta$. Here are the results:

G-Sig (# classes)	$\#PBO$	$\#FBO$	maximal	$\lvert\Delta/\Gamma\rvert$
$[4,4,4,4]$ (1 total)	10	10	no	4
$[4,4,4,5]$ (4 total)	4	16	yes	
$[4,4,5,5]$ (3 total)	5	15	no	2
$[4,4,5,5]$ (3 total)	2	6	no	2

Here is what the columns mean. The first column has a seed G-signature that gets moved around by the full braid group. Recall that a 4 in the G-signature specifies an element from the conjugacy class $\mathcal{K}_4$ containing $(1,2,3,4,5)$ and a 5 specifies an element from $\mathcal{K}_5$, the other class of elements of order 5. Any outer automorphism of A_5 switches the classes. The number in parenthesis is the number of classes in $\overline{A_G^\circ}(\mathcal{C})$ obtained by permuting the entries of the given G-signature. The next two columns are the cardinalities of the pure braid orbit (PBO) and the full braid orbit (FBO). The last two columns indicate whether the actions are maximal or non-maximal, and if non-maximal, the size of Δ/Γ.

The last two rows correspond to two different pure braid orbits with the same G-signature.

We observe that there is a mixture of maximal and non-maximal actions, and conclude that computing strata is trickier than one might initially guess.

4.7. A non-abelian pure braid example. For pure braid strata there is no redundancy in the mapping $\Theta : \widetilde{\mathcal{S}} \to \overline{\mathcal{S}}$ at least over the stratum. So assuming Conjecture 4.6 we can at least determine the genus of the stratum and the number of punctures at infinity.

Despite the ubiquity of pure braid strata in the prime cyclic case it takes a while to find one for a non-abelian group. What works well is to find four conjugacy classes which are not equivalent under automorphisms. Rather than an exhaustive search we picked a simple group with at least 4 different orders. The action of the group $\mathrm{PSL}_2(7)$ with signature $(2,3,4,7)$ is a good candidate as long as there are generating vectors, it did not disappoint.

Example 4.12. For $\mathrm{PSL}_2(7)$ with signature $(2,3,4,7)$ the monodromy information is as follows: The size of the pure braid orbit is $|\mathcal{O}| = 42$. The cycle structures of β_0, β_1, and β_∞ are $3^4 \cdot 4^4 \cdot 7^2$; $1^2 \cdot 2^4 \cdot 3^6 \cdot 7^2$; and $2^4 \cdot 3^4 \cdot 4^2 \cdot 7^2$, respectively. The order of the monodromy group is quite large.

$$|\mathrm{Mon}(\beta)| = 2^{38} \cdot 3^{19} \cdot 5^9 \cdot 7^6 \cdot 11^3 \cdot 13^3 \cdot 17^2 \cdot 19^2 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 41$$

The stratum information is as follows. The genus of the action surface is 66. The stratum cover has genus 4 with 36 punctures at infinity. There are 42 classes in $\overline{A^\circ_G}(\mathcal{C})$, and hence a single stratum.

4.8. Surfaces with two p-gonal morphisms, a non-pure example. Let p be an odd prime. The family $\mathcal{F}_{(p-1)^2}$ of cyclic p-gonal Riemann surfaces S of genus $g = (p-1)^2$ admitting two p-gonal morphisms forms an equisymmetric stratum given by the action of $G = D_p \times D_p$ on the surfaces with n-gonal signature $(2,2,2,p)$, it is a punctured Riemann surface (see [**CIY, Go**]). It was proven in [**CIY**], in a direct manner, that $\mathcal{F}_{(p-1)^2} = \widehat{\mathbb{C}} \setminus \{0,1,\infty\}$ for all odd primes p. We use Theorem 4.1 and the method to calculate a quotient presented in this section to show in a different way that $\mathcal{F}_{(p-1)^2} = \widehat{\mathbb{C}} \setminus \{0,1,\infty\}$.

We will use the following presentation of G:

$$\begin{aligned} G &= D_p \times D_p = \langle a, b, s, t \,:\, a^p = b^p = s^2 = t^2 \\ &= [a,b] = [s,b] = [t,a] = (sa)^2 = (tb)^2 = (st)^2 = 1 \rangle \end{aligned}$$

In [**CIY**] it was shown that there are 3 classes of epimorphisms ξ_1, ξ_2, ξ_3 corresponding to these representative $(2,2,2,p)$ generating vectors

$$\begin{aligned} \mathcal{V}_1 &= (s, t, stab, (ab)^{-1}), \\ \mathcal{V}_2 &= (s, stab, t, ba^{-1}), \\ \mathcal{V}_3 &= (stab, s, t, ba). \end{aligned}$$

The classes are determined by applying the reduction map

$$G \to \langle s \rangle \times \langle t \rangle : s \to s,\ t \to t,\ a, b \to 1$$

to the generating vectors:

$$\mathcal{V}_1 \to (s, t, st, 1),\ \mathcal{V}_2 \to (s, st, t, 1),\ \text{and } \mathcal{V}_2 \to (st, s, t, 1).$$

The "missing" reduced vectors may be recovered by applying the switch automorphism $\theta_0 : s \leftrightarrow t, a \leftrightarrow b$. Also, by using the reduction map we see that $\pi_1(B, \lambda_0)$ fixes all three classes. The three classes constitute a single orbit under the full braid action.

Let $\widetilde{\mathcal{S}}$ be the stratum cover defined by ξ_1, or the vector $\mathcal{V}_1 = (s, t, stab, (ab)^{-1})$, which we write in general form as (c_1, c_2, c_3, c_4). Since $\pi_1(B, \lambda_0)$ fixes all generating vector classes the map $\widetilde{\mathcal{S}} \to B$ has degree 1. We shall see which pairs $\lambda, \lambda' \in B$ yield the same surfaces when lifted to $\widetilde{\mathcal{S}}$. To this end, suppose that $B_G = \{z_1, z_2, z_3, z_4\}$, with cross-ratio λ. If we make an alternate ordering of the z_i then a λ' will be determined as in Table 6. Assume for concreteness that the two lists are $(z_1, z_2, z_3, z_4) \leftrightarrow \lambda$ and $(z_2, z_1, z_3, z_4) \leftrightarrow \lambda'$, so that $\lambda' = 1 - \lambda$. Using standard braid operations, we may assume that the induced map on $\pi_1(T^\circ, z_0)$ is given by:

$$(\Phi_{1,2})_* : (\gamma_1, \gamma_2, \gamma_3, \gamma_4) \to (\gamma_2, \gamma_2^{-1}\gamma_1\gamma_2, \gamma_3, \gamma_4).$$

and the surface corresponding to λ' is defined by $\xi_1 \circ (\Phi_{1,2})_*^{-1}$, with generating vector $(t, s, stab, (ab)^{-1})$. The surfaces corresponding to λ and $\lambda' = 1 - \lambda$ will be conformally equivalent if and only if there is an automorphism $\theta \in \mathrm{Aut}(G)$ satisfying:

$$\xi_1 = \theta \circ \xi_1 \circ (\Phi_{1,2})_*^{-1}.$$

Transferring to generating vectors, we see that we should select $\theta = \theta_0$.

The classes of c_3^G and c_4^G are not $\mathrm{Aut}(G)$ equivalent to each other or to c_1^G and c_2^G. The conjugacy classes c_1^G and c_2^G can be interchanged by automorphisms. Therefore, the only non-trivial permutation of $\{(z_1, z_2, z_3, z_4)\}$ yielding a conformally equivalent surface for λ and λ' is $z_1 \leftrightarrow z_2$.

The discussion above shows that $\widetilde{\mathcal{S}_1} \to \overline{\mathcal{S}}$ is the $2:1$ map of $\widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$ to $\widehat{\mathbb{C}} \setminus \{0, \infty\}$ under the action of $\langle 1 - \lambda \rangle$. Picking $q : \lambda \to 4\lambda(1-\lambda)$ for a quotient map we see that q is unramified over $\widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$, with preimage $\widehat{\mathbb{C}} \setminus \{0, 1, \infty, \frac{1}{2}\}$. The completed map $q : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is doubly ramified at $\frac{1}{2}$ and ∞. The stratum is

$$\mathcal{S} = \overline{\mathcal{S}} - \left\{ q\left(\frac{1}{2}\right) \right\} = \widehat{\mathbb{C}} - \{0, 1, \infty\}$$

verifying the claim that $\mathcal{S} = \mathcal{F}_{(p-1)^2}$ is the Riemann sphere with three punctures.

4.8.1. *Final Remark.* We finish the paper with a remark linking the strata covers $\widetilde{\mathcal{S}}$ to Belyi surfaces.

Remark 4.13. By construction, the map $\widetilde{\mathcal{S}} \to B$ can be completed to a Belyi function. This means that the completion of $\widetilde{\mathcal{S}}$ carries a dessin d'enfant and that it is defined over a number field. This brings to mind some questions: Is the stratum itself a Belyi curve? Does the dessin d'enfant or the field of definition have any importance for the stratum. What information does the monodromy group hold?

References

[Be] L. Bers, *On spaces of Riemann surfaces with nodes.* Bull. Amer. Math. Soc. **80** (1974), no. 6, 1219-1222.

[Bi] J.S. Birman, *Braids, links, and mapping class groups*, Annals of Mathematics Studies, No. 82, Princeton University Press, Princeton, N.J. (1974).

[Bre] T. Breuer, *Characters and automorphism groups of compact Riemann surfaces*, London Mathematical Society Lecture Note Series, **280**, Cambridge University Press, Cambridge, (2000).

[Bro] S.A. Broughton, *The equisymmetric stratification of the moduli space and the Krull dimension of mapping class groups.* Topology Appl. **37** (1990) 101–113.

[TGAS] S.A. Broughton, *Tilings, Geometry, and Automorphisms of Surfaces* https://tilings.org/autosurf

[BCC] E. Bujalance, F.J. Cirre, and M. Conder, *On Extendability of Group Actions on Compact Riemann Surfaces*, Transactions of the American Mathematical Society, **355**, No. 4 (2003), 1537–1557

[CG] A.F. Costa; V. González-Aguilera; *Limits of equisymmetric 1-complex dimensional families of Riemann surfaces.* Math. Scand. **121** no. 1, (2017) 26–48.

[CIR] A.F. Costa, M. Izquierdo, and G. Riera, *One-dimensional Hurwitz spaces, modular curves, and real forms of Belyi meromorphic functions*. Int. J. Math. Math. Sci. **2008** (2008), 1–18.

[CIY] A.F. Costa, M. Izquierdo, and D. Ying, *On cyclic p-gonal Riemann surfaces with several p-gonal morphisms*. Geom. Dedicata **147** (2010), 139–147.

[FM] B. Farb, D. Margalit, *A Primer on Mapping Class Groups*, Princeton University Press, Princeton, N. J. (2012).

[Go] G. González-Díez, *On prime Galois covering of the Riemann sphere*, Ann. Mat. Pure Appl. **168** (1995) 1–15.

[Ha] W. Harvey, *On branch loci in Teichmüller space*, Trans. Amer. Math. Soc. **153** (1971) 387–399.

[Jo] G.A. Jones, *Enumeration of homomorphisms and surface-coverings* Q. J. Math., **46** (2) (1995), 485–507

[MSSV] K. Magaard, T. Shaska, S. Shpectorov, H.C. Volklein, *The locus of curves with prescribed automorphism group* (Communications in Arithmetic Fundamental Groups), RIMS 1267, (2002), 112–141.

[Ma] Magma. *Magma computational algebra system*, Computational Algebra Group, University of Sydney.

[Na] S.M. Natanzon *Hurwitz spaces.* Topics on Riemann surfaces and Fuchsian groups (Madrid, 1998), 165–177, London Math. Soc. Lecture Note Ser., **287**, Cambridge Univ. Press, Cambridge, 2001.

[Ri] J.F.X. Ries. *Subvarieties of moduli space determined by finite groups acting on surfaces*, Trans. Amer. Math. Soc. **335** (1993) 385–406.

[Si] D. Singerman, *Finitely maximal Fuchsian groups*, J. London Math. Soc. **6** (2), (1972), 29–38.

[Sp] E. Spanier, *Algebraic Topology*, Springer (1966).

[Spr] G. Springer, *Introduction to Riemann Surfaces*, Chelsea Publishing Co., 2nd edition (1957).

[WS] J. Wolfart and M. Streit, *Characters and Galois invariants of regular dessins*, Revista Matematica Complutense vol. XIII, num. 1, (2000) 49–81

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