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Vanishing Cycles and Kaleidoscopic Quadrilateral Tilings

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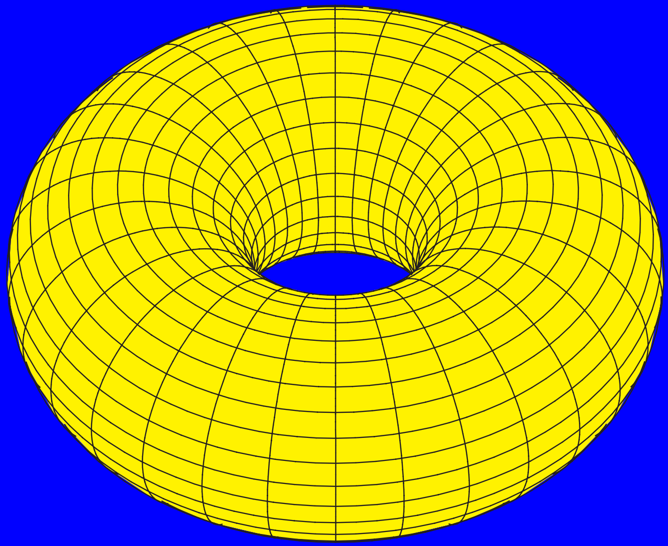

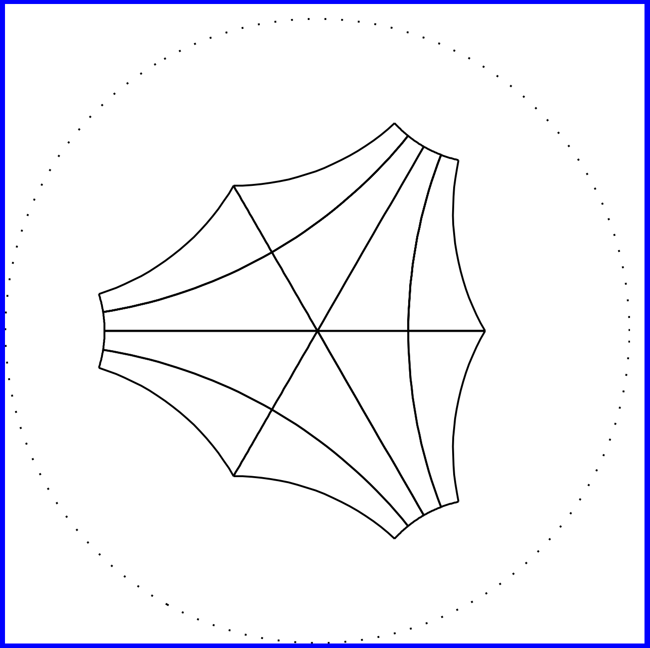

Vanishing Cycles and Kaleidoscopic Quadrilateral Tilings

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Credits

- All of this work has been done jointly with undergraduates
- Isabel Averill, Michael Burr, John Gregoire, Kathryn Zuhr

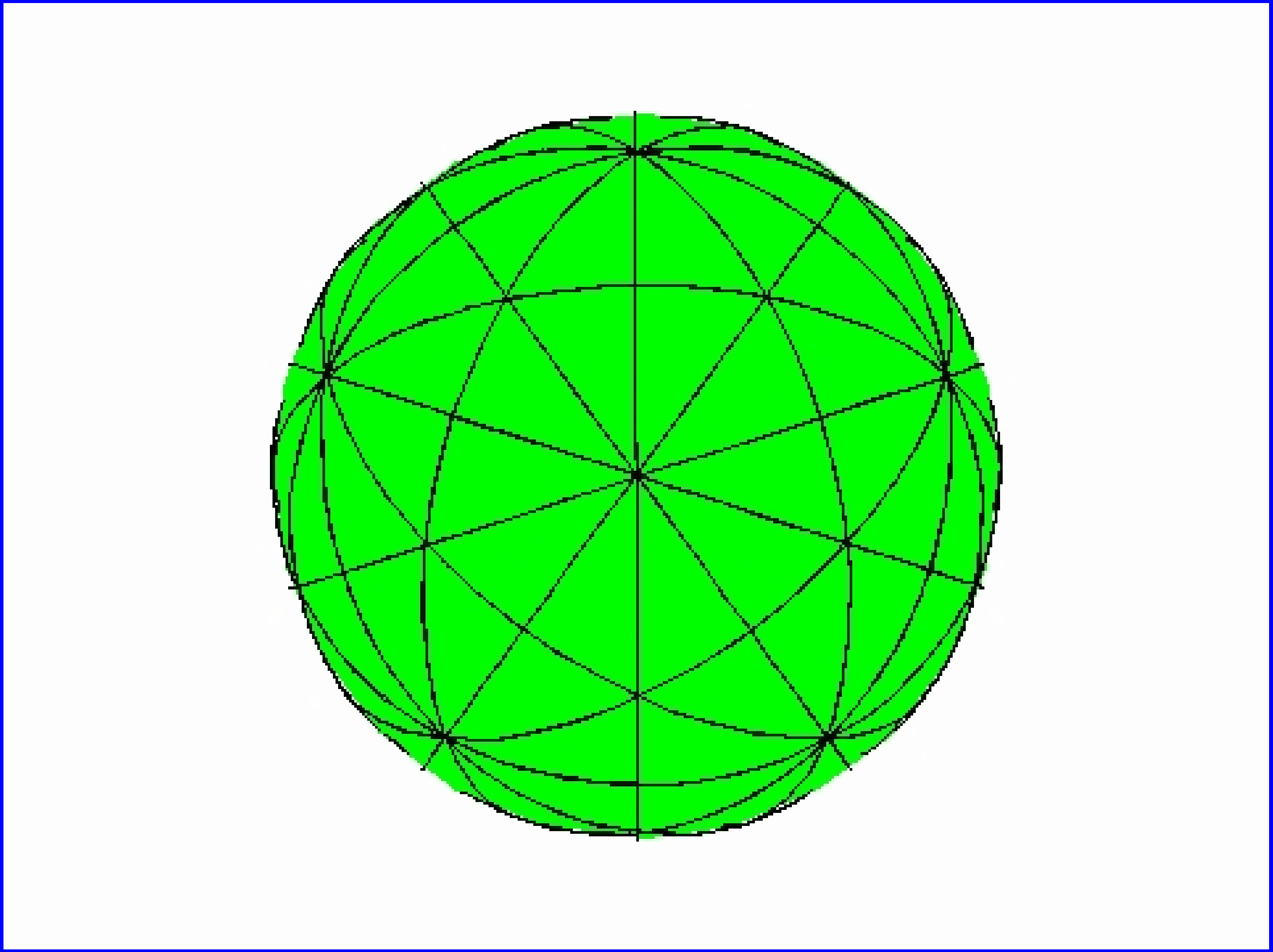
Outline - 1

- Tilings of surfaces – examples, definition
- Hyperbolic geometry
- Kaleidoscopic tilings
- Tiling group G^*
- Riemann-Hurwitz equation

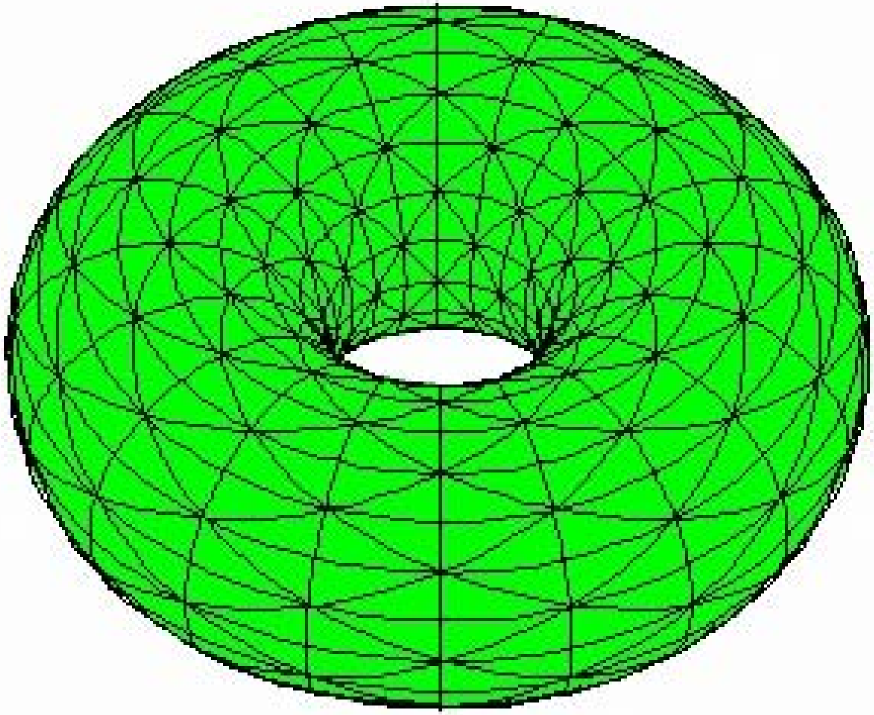
Outline - 2

- Tiling theorems
- Variation of quadrilaterals
- Vanishing cycles
- Number of vanishing cycles

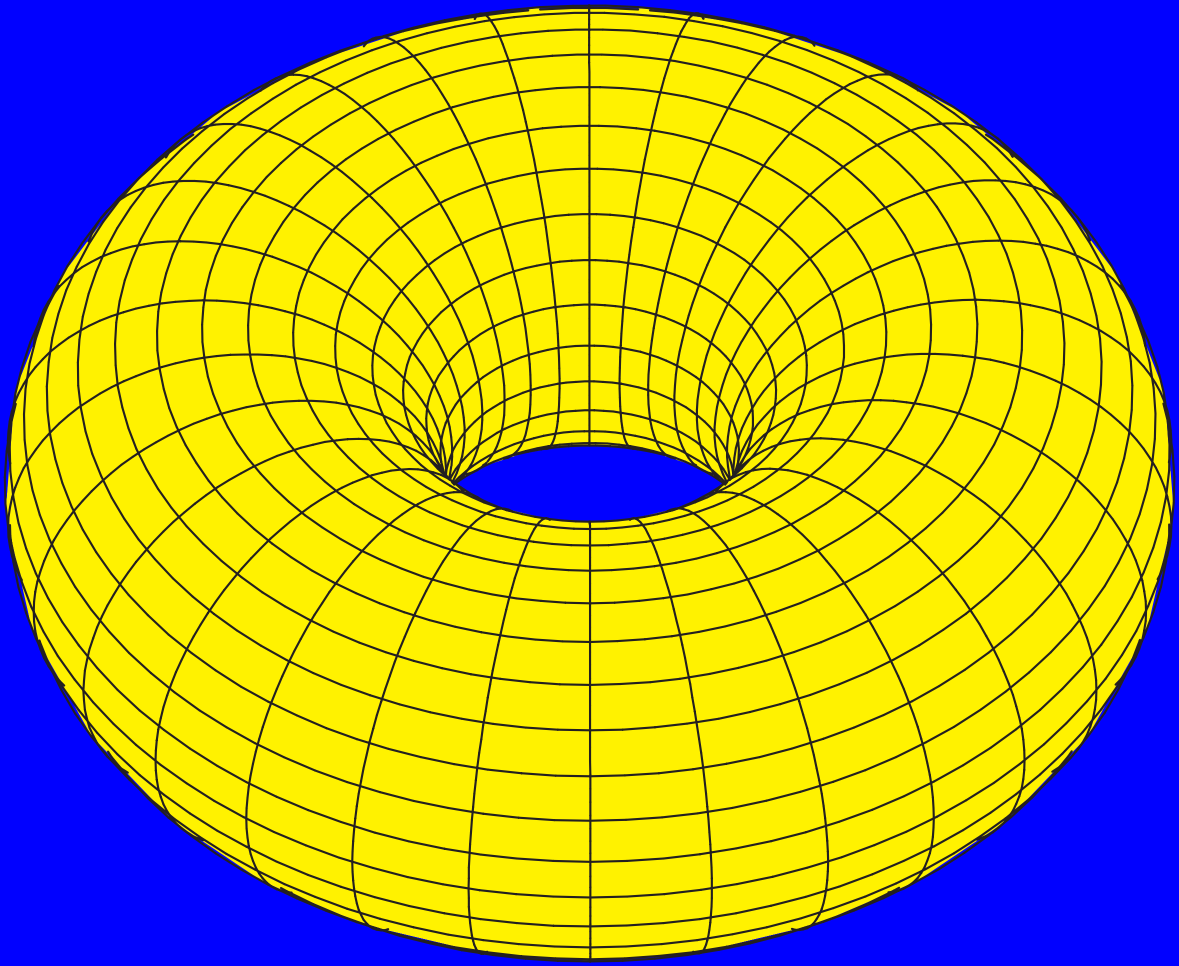
Icosahedral-Dodecahedral tiling (2,3,5) – tiling – soccer ball



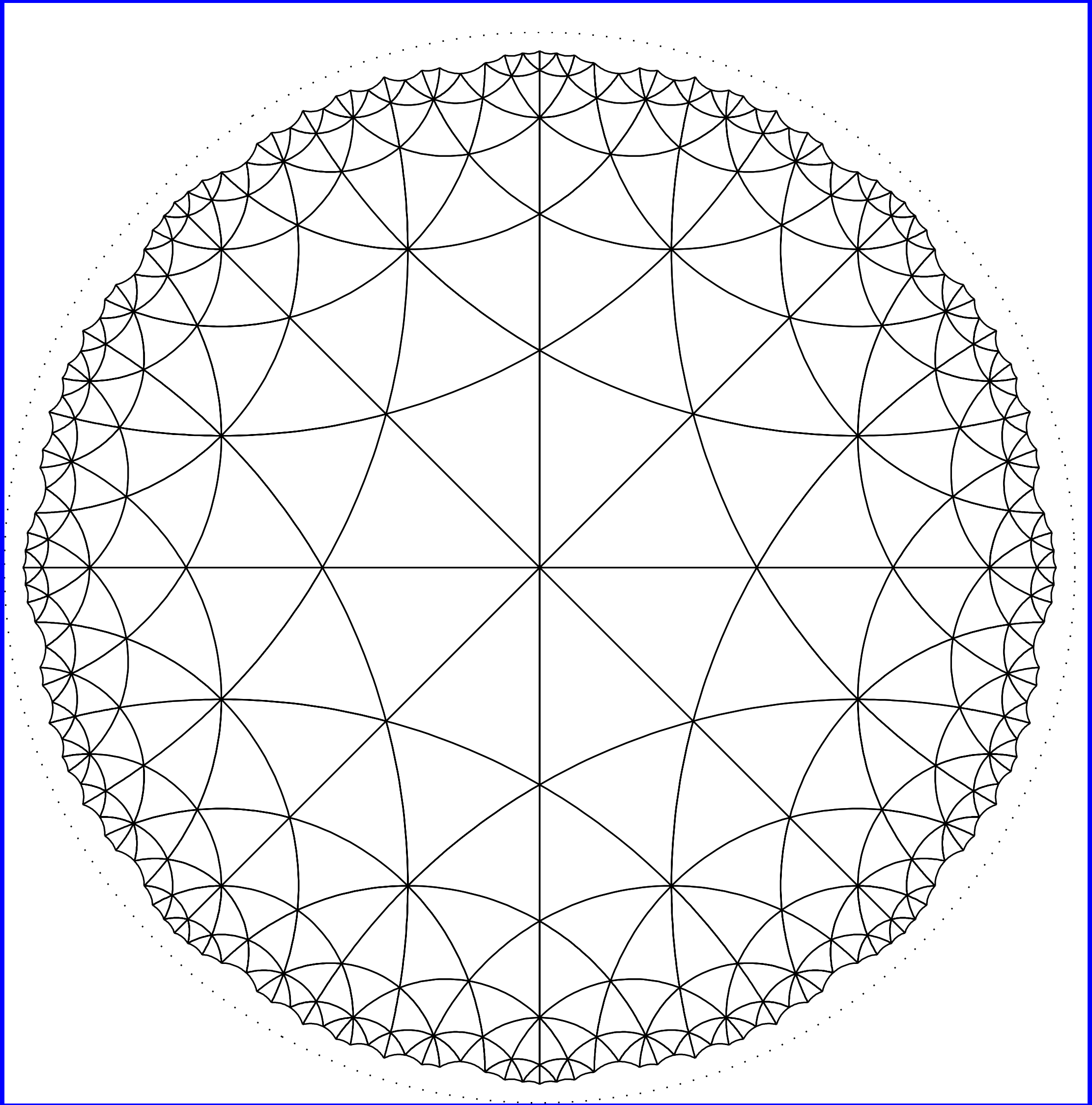
(2,4,4) -tiling of the torus



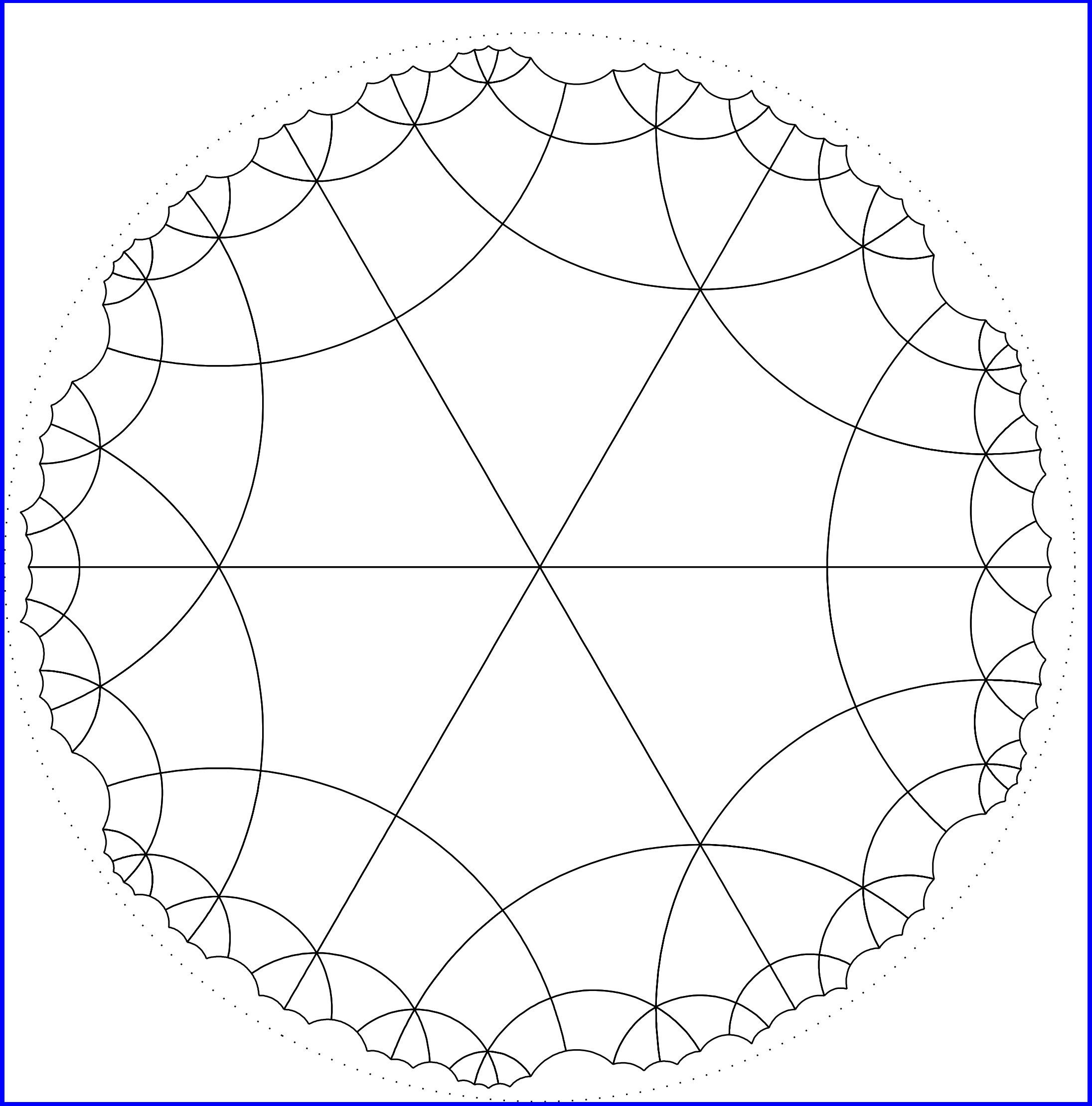
(2,2,2,2) -tiling of the torus



(3,3,4) -tiling of the hyperbolic plane



(2,2,3,3) -tiling of the hyperbolic plane



Tiling: definition

- Let **S** be a surface of genus σ .
- **<u>Tiling:</u>** Covering by polygons "without gaps and overlaps"
- **<u>Kaleidoscopic:</u>** Symmetric via reflections in edges.
- **<u>Geodesic:</u>** Edges in tiles extend to geodesics in both directions
- terminology: (l,m,n)-triangle, (k,l,m,n)-quadrilateral

Hyperbolic geometry -1

- Refer to tiling pictures
- Points, lines and angles
- Reflections

Hyperbolic geometry -2

- formula for area of a triangle

$$\pi - \left(\frac{\pi}{l} + \frac{\pi}{m} + \frac{\pi}{n} \right)$$

$$= \pi \left(1 - \frac{1}{l} - \frac{1}{m} - \frac{1}{n} \right)$$

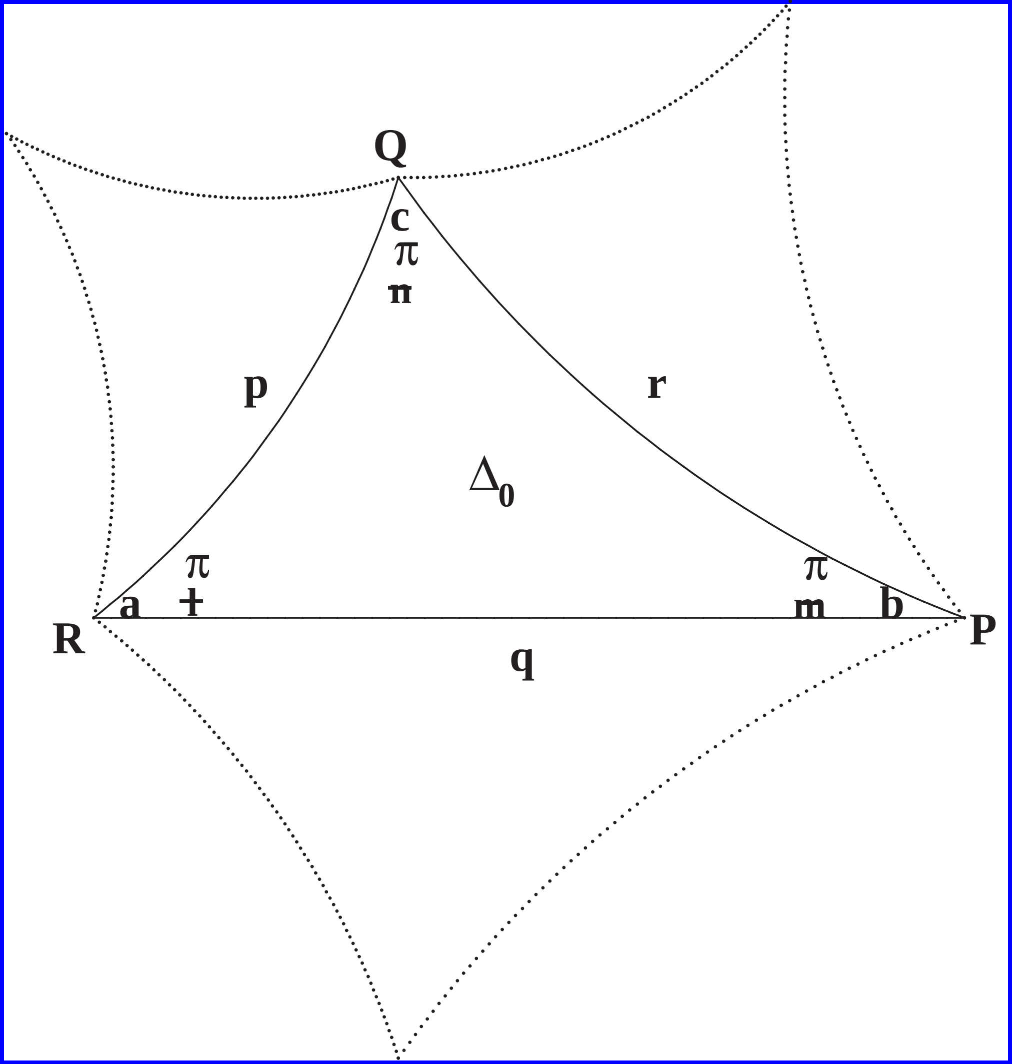
Hyperbolic geometry -3

- formula for area of a quadrilateral

$$2\pi - \left(\frac{\pi}{k} + \frac{\pi}{l} + \frac{\pi}{m} + \frac{\pi}{n}\right)$$

$$= \pi\left(2 - \frac{1}{k} - \frac{1}{l} - \frac{1}{m} - \frac{1}{n}\right)$$

The tiling group - triangle - 1



The tiling group - triangle - 2

Full Tiling Group for triangle (a finite group)

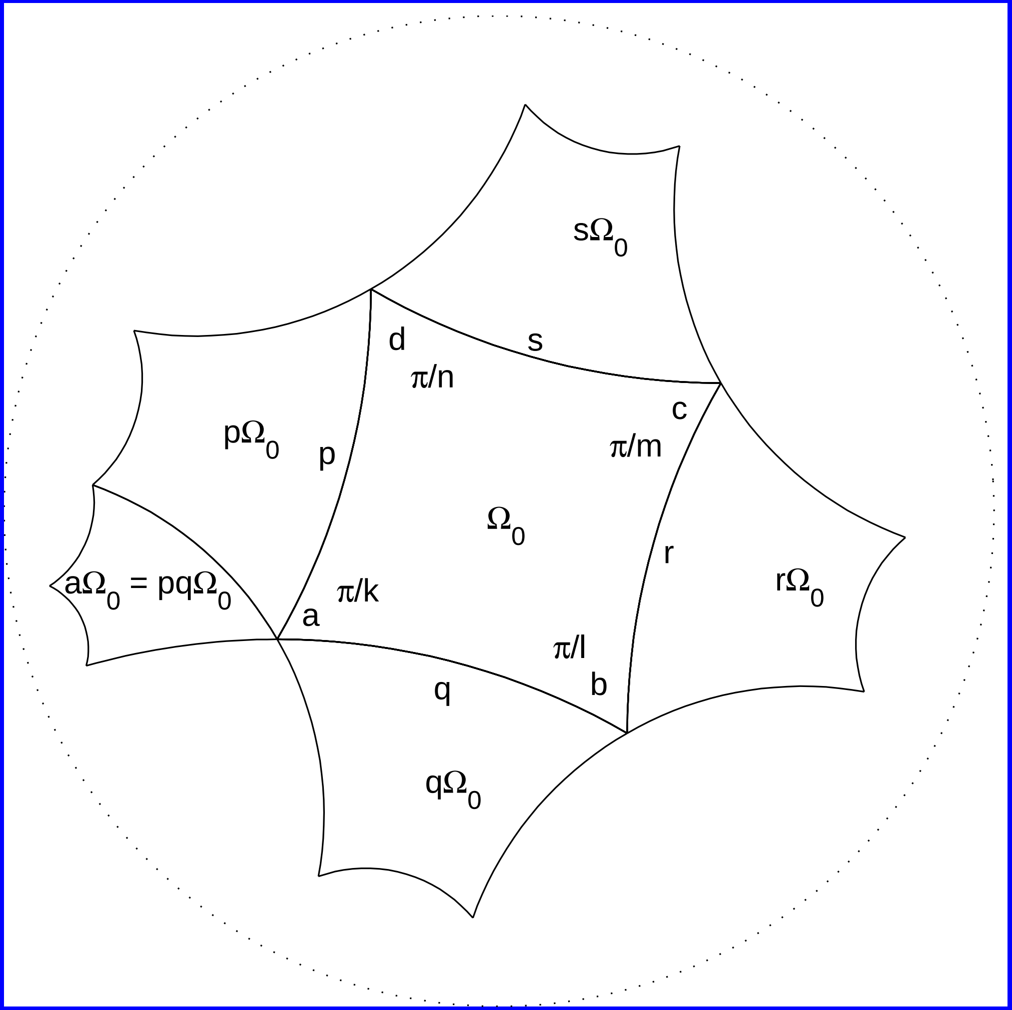
$$G^* = \langle p, q, r \rangle$$

Group Relations

$$p^2 = q^2 = r^2 = 1.$$

$$(pq)^l = (qr)^m = (rp)^n = 1.$$

The tiling group - quadrilateral - 2



The tiling group - quadrilateral - 2

Full Tiling Group for quadrilateral (a finite group)

$$G^* = < p, q, r, s >$$

Group Relations

$$p^2 = q^2 = r^2 = s^2 = 1.$$

$$(pq)^k = (qr)^l = (rp)^m = (ps)^m = 1.$$

Riemann Hurwitz equation - triangles

Let S be a surface of genus σ and $|G^*|$ the number of triangles:

$$\frac{4\sigma - 4}{|G^*|} = 1 - \frac{1}{l} - \frac{1}{m} - \frac{1}{n}$$

Riemann Hurwitz equation - quadrilaterals

Let S be a surface of genus σ and $|G^*|$ the number of quadrilaterals:

$$\frac{4\sigma - 4}{|G^*|} = 1 - \frac{1}{k} - \frac{1}{l} - \frac{1}{m} - \frac{1}{n}$$

Tiling theorem - triangles

A surface S of genus σ has a tiling with tiling group:

$$G^* = < p, q, r >$$

if and only if

- the group relations hold, and
- the Riemann Hurwitz equation holds.

Therefore Tiling Problems can be solved via group computation.

Tiling theorem - quadrilaterals

A surface S of genus σ has a tiling with tiling group:

$$G^* = < p, q, r, s >$$

if and only if

- the group relations hold, and
- the Riemann Hurwitz equation holds.

Variation of quadrilaterals

- Matlab show
- Parameter space for the quadrilateral looks like the real line
- As you go to infinity in the parameter space the lengths of pairs of opposite side goes to infinity

Construction of vanishing cycles

- The perpendicular bisectors of opposite sides of a quadrilateral, $B_{p,r}$ and $B_{q,s}$ generates a geodesic on the surface, called a vanishing cycle
- Let $D_{p,r} = <p,r>$ and $D_{q,s} = <q,s>$ These groups map the vanishing cycles to themselves just like a dihedral group maps a circle to itself

Vanishing cycles do vanish

- The hyperbolic length of $B_{p,r}$ goes to zero as the lengths of p,r go to infinity
- The hyperbolic length of $B_{q,s}$ goes to zero as the lengths of q,s go to infinity
- Maple “show and tell” of a vanishing cycle in a family of surfaces

Number of vanishing cycles

- The number of p,r vanishing cycles is

$$|G^*|/|D_{p,r}|$$

- The number of q,s vanishing cycles is

$$|G^*|/|D_{q,s}|$$