April 16, 2005

Enumeration of the Equisymmetric Strata of the Moduli Space of Surfaces of Low Genus

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Preliminary report

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NSF grants # DMS-9619714 and # DMS-0097804

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1 Introduction and Notation

1.1 Introduction

- Surfaces of the same genus σ are called *equisymmetric*, or are said to have the same *symmetry type*, if the two surfaces' conformal automorphism groups determine conjugate finite subgroups of the mapping class group of genus σ. See [4]

- The subset of the moduli space corresponding to surfaces equisymmetric with a given surface forms a locally closed subvariety of the moduli space, called an *equisymmetric stratum*.

- The conjugacy classes of the mapping class group determine strata but it is possible to have finite, distinct $H \subset G$ determine the same strata.

- The equisymmetric strata are smooth, irreducible locally closed, projective varieties, are finite in number, have easily computed dimensions, and do form a stratification of the moduli space.

- The stratification can be used to derive information about the cohomology of the mapping class group, and form in some sense form a "stratification by singularity" of the moduli space and hence capture some of the geometric information of the moduli space.

- The strata are in 1-1 correspondence with certain well-determined conjugacy classes of finite subgroups of the mapping class group or alternatively topological equivalence classes of orientation preserving actions of a finite group G on a surface S.

- Given a equisymmetric stratum $\mathcal{S}$ then $\overline{\mathcal{S}} - \mathcal{S} = \bigcup_j \mathcal{S}_j$ is a disjoint union of equisymmetric strata of lower dimension consisting of curves with more symmetry than just G-symmetry. We denote by $\mathcal{S} \rightarrow \mathcal{S}_j$ the relation $\mathcal{S}_j \subset \overline{\mathcal{S}}$.

1.2 Problems

1. Some problems arise:

   (a) Enumerate the strata for low genus, and the related problem.

   (b) Determine the conjugacy classes of finite subgroups of the mapping class group.

2. To understand the moduli space as a geometric object answers to the following questions would be helpful.

   (a) What do the strata look like? A genus calculation would be nice for 1-dimensional strata.

   (b) For low genus determine the adjunction relations $\mathcal{S} \to \mathcal{S}_j$.

3. There is a long history to this problem. The initial results are over 100 years old but very rapid progress had been made in the last few years because of the availability of an extensive library of Small Groups Library in GAP and MAGMA. Here is a sample of papers some of which have extensive bibliographies.

- Breuer. Characters and Automorphism, Groups of Compact Riemann Surfaces, London Math Soc. Lect. Notes, 280. CUP, 2000.

- Broughton, Classifying Finite Group Actions on Surfaces of Low Genus, JPPA 69 (1990) and

- ________, The Equisymmetric Stratification of the Moduli Space and the Krull Dimension of Mapping Class Groups, Topology and it Applications, (1990)

- G. A. Jones. Counting Subgroups of Non-Euclidean Crystallographic Groups. Math. Scand (1999)

- Magaard, Shpectorov, Völklein,The locus of curves with prescribed automorphism group, preprint 2002, Magaard home page.

- ________________, A GAP Package for Braid Orbit Computation and Applications, Experimental Mathematics, 2003

- Wooton, Counting Belyi Surfaces with many Automorphisms, Applications of Computer Algebra (ACA-2004)

2 Collaborators

Much of this work was done with undergraduates at the Rose-Hulman NSF-REU http://www.tilings.org (see next page)

- M. Haney, McLKeough, B. Smith - Divisible tilings

- R. Vinroot, R. Dirks, Sloughter, - Classification in low genus

- I Averill, J. Gregoire - Quadrilateral Classification

- Kathryn Zuhr, - Moduli for quadrilaterals

3 Notation and Facts

3.1 Notation

- Let $\mathbb{H}$ be the hyperbolic plane,
- $S = \mathbb{H}/\Pi$ is compact closed surface of genus σ, $\Pi \backsimeq \pi_1(S)$,
- M_σ be the mapping class group of S, $\mathcal{T}_\sigma$ the Teichmüller space of curves of genus σ and $\mathcal{M}_\sigma = \mathcal{T}_\sigma/M_\sigma$ the moduli space of curve of genus σ,
- G a group acting on conformally on S,
- $\eta : \Gamma \longmapsto G$ a surface-kernel epimorphism with kernel Π, uniformizing the G-action.

- The quotient $T = S/G = \mathbb{H}/\Gamma$ is surface of genus τ, Γ has t periods $n_1, \ldots, n_t$ corresponding to branch points $P_1, \ldots P_t$ on T, We record the $n_1, \ldots, n_t$ in non-decreasing order. The *signature of* Γ is denoted $B = (\tau : n_1, \ldots, n_t)$.

- A presentation of $\Gamma = \Gamma_B$ is given by

$$\begin{aligned} \langle \alpha_i, \beta_i, \gamma_j, 1 &\leq i \leq \tau, 1 \leq j \leq r : \\ \prod_{i=1}^{\rho} [\alpha_i, \beta_i] \prod_{j=1}^{r} \gamma_j &= \gamma_1^{m_1} = \cdots = \gamma_r^{m_r} = 1 \rangle, \end{aligned}$$

- Let a_i, b_i, c_j be the images of the generators $\alpha_i, \beta_i, \gamma_j$ under the map η.
  The $(2\tau + t)$-tuple $(a_1, b_1, \ldots a_\tau, b_\tau, c_1, \ldots c_t)$ forms a generating set for G satisfying

$$\prod_{i=1}^{\tau} [a_i, b_i] \prod_{j=1}^{t} c_j = 1, \ o(c_j) = n_j.$$

  call such tuple a *generating* $(\tau : n_1, \ldots, n_t)$ - *vector* and denote it by η.

- The Riemann Hurwitz equation is also satisfied:

$$\frac{(2\sigma - 2)}{|G|} = (2\tau - 2 + t) - \sum_{j=1}^{t} \frac{1}{n_j}.$$

- Let $X(B) = X(\tau : n_1, \ldots, n_t)$ denote the set of all generating $(\tau : n_1, \ldots, n_t)$ - vectors.

3.2 Facts

- Given a triple (G, B, η), a stratum in the moduli space is determined.

- The complex dimension of the stratum determined by (G, B, η) is $3\tau + t - 3$.

- Let M_B denote the mapping class group of T preserving the branch set $\{P_1, \ldots P_t\}$ and the orders.

It may be viewed as an automorphism group of Γ_B. The group $\mathrm{Aut}(G) \times M_B$ acts on $X(B)$ by $(\omega, \phi) \cdot \eta = \omega \circ \eta \circ \phi^{-1}$.

- The finite conjugacy classes of the mapping coming from some G-action of type $(\tau : n_1, \ldots, n_t)$ are in 1-1 correspondence to the $\mathrm{Aut}(G) \times M_B$ orbits of $X(B)$.

- If $G' \subset G$ is a subgroup pair then two triples (G', B', η'), (G, B, η) are determined. There is an inclusion map $\mathcal{S} \hookrightarrow \mathcal{S}'$ which is bijective if and only if $\dim \mathcal{S} = \dim \mathcal{S}'$. This only depends on the signature.

4 Steps of a classification program

1. Determine the G's and the B's. This has been done up to genus 48 in Thomas Breuer's book [2].

2. Determine a list the Aut(G)- classes of generating vectors.

3. Determines the M_B orbits on the generating vectors in #2. The Braid package discussed in [7] works well for signatures of the form $(0 : n_1, \ldots, n_t)$ since the M_B orbits are given by the braid group action preserving the order. The action of a typical generator is given by
$(c_1, \ldots c_t) \longrightarrow (c_1, \ldots, c_{j+1}, c_{j+1}^{-1} c_j c_{j+1}, \ldots, c_t)$.

4. In the case where $\tau > 0$ the M_B action would be a bit trickier to implement though generators for the mapping class group are known [1], and translation to an action of M_B can be done.

5. These steps determine the conjugacy classes of finite subgroups of the mapping class group. There is an additional step to eliminate redundant actions for strata whose dimension is 3 or less. As noted It is possible that for a given triple (G, B, η) there is a group $G' \subset G$ and triple (G', B', η') such that $\mathcal{S} = \mathcal{S}'$.This can only happens if $\Gamma_{B'} \subset \Gamma_B$, Π is normal in both, and $3\tau' + t' - 3 = 3\tau + t - 3$.

6. Greenberg, and later Singerman determined the cases in which there was a possible $\Gamma_{B'} \subset \Gamma_B$.

$3\tau + t - 3$	0	1	2	3
Families of cases	7	2		
Exceptional cases	7	1	1	1

Families of cases are one in which depend on a parameter e.g., $\Gamma_{(0:d,d,d)} \subset \Gamma_{(0:2,3,2d)}$ with index 6. An example of an exceptional case is $\Gamma_{(0:4,4,5)} \subset \Gamma_{(0:2,4,5)}$. For a given genus the variable in the families of cases are finite in number. A nice description of the cases is given in [8].

7. We may handle a case such as $\Gamma_{(0:4,4,5)} \subset \Gamma_{(0:2,4,5)}$ by noting that if $(\gamma_1, \gamma_2, \gamma_3)$ is a generating triple for $\Gamma_{(0:2,4,5)}$ then $(\gamma_2, \gamma_3^2\gamma_2\gamma_3^{-2}, \gamma_3^{-1}\gamma_2\gamma_3\gamma_2^{-1}\gamma_3)$ may be taken as a generating triple for $\Gamma_{(0:4,4,5)}$.

8. Then the action $(G, (0:2,4,5), (c_1, c_2, c_3))$ determines a generating vector $(c_2, c_3^2c_2c_3^{-2}, c_3^{-1}c_2c_3c_2^{-1}c_3)$ of a triple $(H, (0:4,4,5), (c_2, c_3^2c_2c_3^{-2}, c_3^{-1}c_2c_3c_2^{-1}c_3)$ which may be compared to a given $(G', (0:4,4,5), (c_1', c_2', c_3'))$.

9. Often there is a unique candidate for (G', B', η'), so there is nothing to calculate. A contrary example is given by the two inequivalent actions $(\mathbb{Z}_7, (0:7,7,7), (x, x^2, x^4))$ and $(\mathbb{Z}_7, (0:7,7,7), (x, x, x^5))$ in genus 3. One embeds in the unique D_7 action and the other in the unique $PSL_2(7)$ action .

5 Results

- Complete lists for genus 2 and 3 have been calculated piecemeal in the past in [3] and a series of papers starting in [6].

- Recently, in [8] genus 2 and 3 results have been recomputed, using GAP . The surfaces with "large automorphism groups" up to genus 10 have been calculated in the same paper. Similar calculations have been done in [9].

- The complete classification of 0-dimensional strata up to genus 25 has been calculated. The number of actions and the number of strata in the following table.

6 Curves over $\mathbb{R}$

- Here we are interested in surfaces with reflections and so a non-empty real form.

- The fixed point subsets of all complex conjugations define a tiling on the surfaces (show pictures)

- Surfaces with triangular tilings up to genus 25 and quadrilaterals tilings up to 13 have been classified.

- Pairs of inclusions $\Gamma_{(0:a,b,c)} \subset \Gamma_{(0:k,l,m,n)}$ defined by inclusions of triangles into quadrilateral have been classified. There are 34 families of pairs and 27 exceptional pairs. See the next two pages for examples of families and exceptional pairs.

- Most of the complex strata contain a real curve (at least in the 0 and 1-dimensional cases).

- A connected complex stratum will yield multiple real strata for example $S_4, (0 : 2, 2, 2, 3)$ in genus 3 has 9 separate components.

7 References

1. Birman, Braids, Links and Mapping class groups, Annals of Math Studies, PUP, 1974.

2. Breuer. Characters and Automorphism, Groups of Compact Riemann Surfaces, London Math Soc. Lect. Notes, 280. CUP, 2000.

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6. A. Kuribayashi, K. Komiya, On the structure of the automorphism group of a compact Riemann surface of genus 3. Bull. Fac. Sci. Engrg. Chuo Univ. 23 (1980), and subsequent papers.

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genus	actions	strata
2	11	3
3	25	8
4	28	12
5	31	11
6	33	18
7	37	20
8	36	18
9	75	33
10	79	37
11	58	22
12	60	40
13	73	42
14	44	29
15	88	53
16	80	50
17	123	59
18	63	43
19	124	73
20	59	37
21	158	75
22	111	67
23	44	28
24	101	75
25	182	113
total	1723	966