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The Barycenter of the Numerical Range of an Operator

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parts of the talk

There are three main parts of the talk

- Introduction, basic examples, and properties.
- Discussion of the 2-D case and the Toeplitz-Hausdorff compactness-convexity result.
- Discussion of the barycenter and proof of barycenter theorem.

notation -1

- V is a Hilbert space, but just really $\mathbb{C}^n$ for our purposes
- $X = (x_1, \ldots, x_n)$, $Y = (y_1, \ldots, y_n) \in V$ are any two vectors
- and $\langle X, Y \rangle = x_1\overline{y}_1 + \cdots + x_n\overline{y}_n$ is the standard Hermitian scalar product of X and Y
- if Y^* = conjugate transpose, then $\langle X, Y \rangle = Y^*X$ for column vectors
- $||X|| = \sqrt{\langle X, X \rangle}$
- $B_n = B(V) = \{X \in V : ||X|| \leq 1\}$ is the unit ball in V
- $\partial B_n = \partial B(V) = \{X \in V : ||X|| = 1\}$ is the unit sphere in V

notation - 2

- $A : V \to V$ is any operator, but really just an $n \times n$ matrix
- $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A.
- Recall the equation for the spectrum average

$$\frac{1}{n}\sum_{i=1}^{n}\lambda_i = \frac{1}{n}\mathrm{trace}(A)$$

- also define the map

$$f_A : \partial B_n \to \mathbb{C}, \text{by } f_A(X) = \langle AX, X\rangle.$$

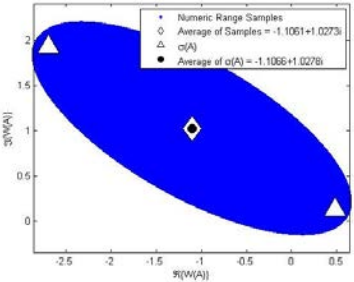
definition of numerical range

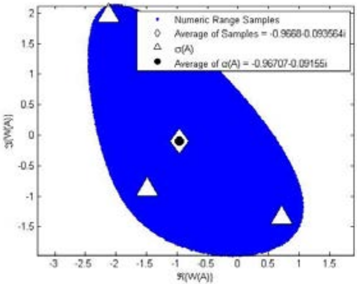
Definition

Let $A : V \to V$ be a bounded linear operator of the Hilbert space V. The numerical range $W(A)$ is the subset in the complex plane defined by

$$W(A) = \{\langle AX, X\rangle : ||X|| = 1\}$$

what does $W(A)$ look like?





simple properties

Proposition

- *For a finite dimensional space V the numerical range $W(A)$ is a compact subset of the plane .*
- *The numerical range $W(A)$ contains the eigenvalues of A.*

- The numerical range $W(A)$ is the continuous image of ∂B_n under f_A.
- Let $AX = \lambda X$ for some λ and some unit vector X. Then $\langle AX, X\rangle = \langle \lambda X, X\rangle = \lambda\langle X, X\rangle = \lambda$.

simple example

Proposition

If A is a diagonal matrix then $W(A)$ is the convex hull of the set of eigenvalues.

Proof sketch

- Assume that $A = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ and that $X = (x_1, \dots, x_n)$.
- Then $\langle AX, X \rangle = \sum_{i=1}^{n} ||x_i||^2 \lambda_i$
- As $\sum_{i=1}^{n} ||x_i||^2 = 1$ then $\langle AX, X \rangle$ is a convex linear combination of the eigenvalues.

transformation properties

The following properties are useful in $W(A)$ calculations.

Proposition

- *If U is a unitary matrix then $W(UAU^{-1}) = W(A)$.*
- *For complex constants a, b, $W(aI + bA) = a + bW(A)$.*

- For unitary U, $U^{-1} = U^*$. Setting $Y = UX$ we get

  $$\langle UAU^{-1}Y, Y\rangle = \langle UAU^{-1}UX, UX\rangle = \langle UAX, UX\rangle = \langle AX, X\rangle.$$

  As X varies completely over the sphere so does $Y = UX$.
- $\langle (aI + bA)X, X\rangle = a\langle X, X\rangle + b\langle AX, X\rangle = a + b\langle AX, X\rangle$

restriction to a subspace - 1

Restricting to a subspace, is a useful computational technique. Here is specific computational formulation.

Proposition

Let $W \subseteq V$ be subspace an let $X_1, \ldots, X_m$ be an orthonormal basis of W. Let B be the $m \times m$ matrix defined by

$$B_{i,j} = \langle AX_i, X_j \rangle$$

Then

$$W(B) \subseteq W(A).$$

restriction to a subspace - 2

Proof sketch

- Set $P = [X_1\ X_2\ \cdots\ X_m]$, then by orthogonality $P^*P = I_m$.
- Let $Z \in \partial B_m$ and $X = PZ = \sum_{i=1}^{m} z_i X_i$.
- Then $||X||$ = 1 as $\langle X, X\rangle = X^*X = Z^*P^*PZ = Z^*Z = 1$.
- $X = PZ$ defines an isometry from ∂B_m to $W \cap \partial B_n$.
- For $X \in W \cap \partial B_n$, $\langle AX, X\rangle = \langle APZ, PZ\rangle = \langle (P^*AP)Z, Z\rangle$.
- $W(P^*AP) = \{\langle AX, X\rangle : X \in W \cap \partial B_n\} \subseteq W(A)$
- The i, j entry of P^*AP is $X_i^*AX_j = \langle AX_i, X_j\rangle = B_{i,j}$

statement of result

Proposition

If A is a 2×2 matrix then $W(A)$ is a filled ellipse with the eigenvalues at the foci.

We give a proof sketch since it uses basic techniques used studying the numerical range.

proof sketch -1

Proof sketch

- Select unitary U such that UAU^{-1} is upper triangular - use Schur's Lemma. So we assume that A is upper triangular.
- Let $\tau = \mathrm{trace}(A)/2$.Then there is a unit complex scalar υ such that $\upsilon(A - \tau I)$ has eigenvalues $\pm a$ for real a. Thus, for some complex b, A has the form

$$A = \begin{bmatrix} a & 2b \\ 0 & -a \end{bmatrix}.$$

- The effect of the above transformation is a rigid motion in the plane, taking ellipses to ellipses, foci to foci and eigenvalues to eigenvalues.

proof sketch - 2

- Next, use the unitary similarity

$$\begin{bmatrix} e^{i\phi} & 0 \\ 0 & e^{i\psi} \end{bmatrix} \begin{bmatrix} a & b \\ 0 & -a \end{bmatrix} \begin{bmatrix} e^{-i\phi} & 0 \\ 0 & e^{-i\psi} \end{bmatrix} = \begin{bmatrix} a & e^{i(\phi-\psi)}b \\ 0 & -a \end{bmatrix}$$

so that we may assume that b is real non-negative.
- A typical unit vector X in $\mathbb{C}^2$ has the form

$$X = \begin{bmatrix} \cos(\theta)e^{i\phi} \\ \sin(\theta)e^{i\psi} \end{bmatrix}$$

- and so

$$\langle AX, X\rangle = \begin{bmatrix} \cos\theta e^{-i\phi} & \sin\theta e^{-i\psi} \end{bmatrix} \begin{bmatrix} a & b \\ 0 & -a \end{bmatrix} \begin{bmatrix} \cos(\theta)e^{i\phi} \\ \sin(\theta)e^{i\psi} \end{bmatrix}$$

or

$$\langle AX, X\rangle = a(\cos^2\theta - \sin^2\theta) + 2b\cos\theta\sin\theta e^{i(\psi-\phi)}$$

proof sketch - 3

- or for suitable α, β

$$\langle AX, X\rangle = a\cos(\alpha) + b\sin(\alpha)e^{i\beta}$$

$$\langle AX, X\rangle = a\cos(\alpha) + b\sin(\alpha)\cos(\beta) + ib\sin(\alpha)\sin(\beta)$$

- With some work, one can show that as α, β vary the ellipse

$$\frac{x^2}{a^2+b^2} + \frac{y^2}{b^2} \leq 1$$

  is swept out.
- The foci of this ellipse are at $-a$ and a, the eigenvalues of A.

statement of theorem

The Toeplitz-Hausdorff theorem dramatically reduces the possibilities for the shape of the numerical range of a matrix.

Theorem

The numerical range of $W(A)$ of a matrix A is a compact, convex subset of the plane.

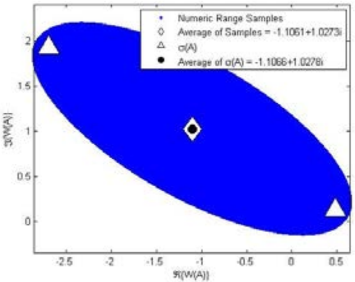
proof sketch

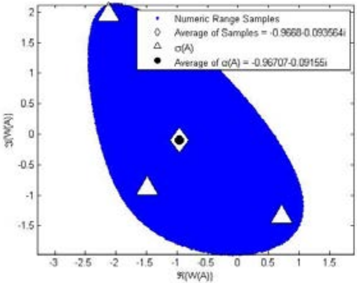
- Let X and Y be two vectors such that $\langle AX, X\rangle$ and $\langle AY, Y\rangle$ are distinct.
- Let $W \subseteq V$ be the linear span of X and Y and let $X_1, X2$ be a orthonormal basis of W
- By previous proposition, the set of values $\langle AZ, Z\rangle$ for all unit vectors Z in W is the same as the numerical range $W(B)$ of the 2×2 matrix

$$B = \begin{bmatrix} \langle AX_1, X_1\rangle & \langle AX_1, X_2\rangle \\ \langle AX_2, X_1\rangle & \langle AX_2, X_2\rangle \end{bmatrix}$$

- Thus $\langle AX, X\rangle$ and $\langle AY, Y\rangle$ are contained in an ellipse contained in $W(A)$.

another look at $W(A)$

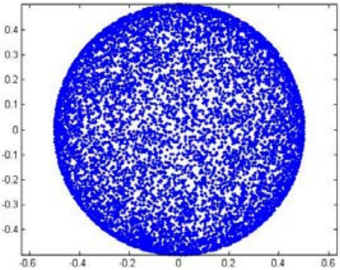


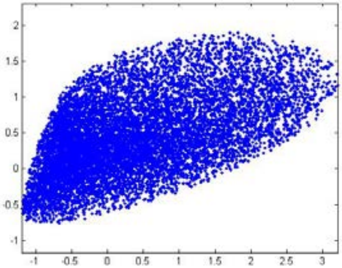


- The average of the eigenvalues appear to be at the center of $W(A)$.
- Proven to be true for the 2×2 case.

How to generate pictures

- Select a large number of vectors $X_1, X_2, \ldots, X_N$ uniformly distributed on ∂B_n
- Plot $\langle AX_i, X_i\rangle$ for N different vectors. Here are two examples.





some observations

- Points are not uniformly distributed on $W(A)$, so the standard centroid is not the right idea for the "center" of $W(A)$.
- The sample average $\frac{1}{N}\sum_{i=1}^{N}\langle AX_i, X_i\rangle$ seems be very close to spectrum average $\frac{1}{n}\sum_{i=1}^{n}\lambda_i$.
- The result above appears to hold true even if the vectors are only distributed "symmetrically".

definition of barycenter

Definition

We define the barycenter (center of mass) of $W(A)$ to be

$$BW(A) = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \langle AX_i, X_i \rangle$$

where the X_i's are chosen from the uniform distribution on the boundary of the unit ball in $\mathbb{C}^n$

uniformly distributed points

- The X_i's are uniformly distributed ∂B_n if or each closed subset U of ∂B_n,

$$\lim_{N\to\infty} \frac{\#\{i : X_i \in U\}}{N} = \frac{vol(U)}{vol(\partial B_n)},$$

- where $vol(U)$ is the volume of U computed as a subset of the ∂B_n.

integral definition

- We get an integral definition

$$BW(A) = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \langle AX_i, X_i \rangle = \int_{\partial B_n} \langle AX, X \rangle \, d\omega.$$

- Define this planar density on $W(A)$

$$\delta(z) = \lim_{r\to 0} \frac{\omega(f_A^{-1}(\Delta_r(z)))}{\pi r^2}$$

with $\Delta_r(z) = \{w \in \mathbb{C} : \|w - z\| \leq r\}$.

- Then $BW(A)$ has a planar integral definition

$$BW(A) = \int_{\partial B_n} \langle AX, X \rangle \, d\omega = \int_{W(A)} z\delta(z) dxdy$$

statement of theorem

The following theorem characterizes the barycenter.

Theorem

The barycenter $BW(A)$ of the numerical range $W(A)$ is given by:

$$BW(A) = \frac{\mathrm{tr}(A)}{n} = \frac{1}{n}\sum_{i=1}^{n} \lambda_i.$$

proof sketch -1

- From the definitions.

$$BW(A) = \int_{\partial B_n} \langle AX, X \rangle \, d\omega = \sum_{i,j} \int_{\partial B_n} a_{i,j} x_i \overline{x_j} d\omega$$

- We need only prove

$$\int_{\partial B_n} x_i \overline{x_j} d\omega = \frac{1}{n} \delta_{i,j}$$

proof sketch - 2

Now some setup

- Define the functions

$$f_i(X) = x_i\overline{x_i}, \;\; f_{i,j}(X) = x_i\overline{x_j}$$

- Note that

$$\sum_i f_i(X) = \sum_i x_i\overline{x_i} = \langle X, X\rangle = 1$$

- Also define unitary operators (transpositions and symmetries along coordinate axes)

$$U_{i,j} : (x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n) \longrightarrow (x_1, \ldots, x_j, \ldots, x_i, \ldots, x_n)$$

for any distinct i, j and

$$V_i : (x_1, \ldots, x_i, \ldots, x_n) \longrightarrow (x_1, \ldots, -x_i, \ldots, x_n)$$

proof sketch - 3

From invariance

$$\int_{\partial B_n} x_i\overline{x_i}d\omega = \int_{\partial B_n} f_i(X)d\omega = \int_{\partial B_n} f_i(U_{i,j}X)d\omega = \int_{\partial B_n} f_j(X)d\omega = \int_{\partial B_n} x_j\overline{x_j}d\omega$$

and so

$$n\int_{\partial B_n} x_i\overline{x_i}d\mu = \int_{\partial B_n} \sum_j x_j\overline{x_j}d\mu = \int_{\partial B_n} 1\,d\mu = 1$$

proving $\int_{\partial B_n} x_i\overline{x_i}d\omega = \frac{1}{n}$

proof sketch - 4

Now assuming $i \neq j$,

$$\int\limits_{\partial B_n} x_i \overline{x_j} d\omega = \int\limits_{\partial B_n} f_{i,j}(X) d\omega = \int\limits_{\partial B_n} f_{i,j}(V_i X) d\omega$$

$$\int\limits_{\partial B_n} f_{i,j}(V_i X) d\omega = \int\limits_{\partial B_n} -f_{i,j}(X) d\omega = -\int\limits_{\partial B_n} x_i \overline{x_j} d\omega.$$

and hence $\int\limits_{\partial B_n} x_i \overline{x_j} d\omega = 0$

proof sketch - 5

Remark

If the vectors are randomly chosen from any probability distribution μ on the sphere invariant under the V_i and $U_{i,j}$ then

$$BW(A) = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \langle AX_i, X_i \rangle = \int_{\partial B_n} \langle AX, X \rangle \, d\mu$$

Thank you.

Any questions?