

# Rose-Hulman Institute of Technology

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October 1, 2008

## Billiards and Flat Surfaces - Sabbatical Report Part I

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# Billiards and Flat Surfaces - Sabbatical Report Part I

S. Allen Broughton - Rose-Hulman Institute of Technology

Rose Math Seminar, October 1, 2008

# sabbatical at IU - 1

## My sabbatical at Indiana University

- Sabbatical at Indiana University Spring 2008 - Thank you Rose-Hulman.
- Sabbatical partly to learn new areas of mathematics related to surfaces.
- Also to come up with problems suitable for undergraduate research.

# sabbatical at IU - 2

## Two reports

- First talk (today) to give an introduction to flat surfaces via the historical motivation of billiard trajectories.
- Second talk (much later) to give a bit more depth on flat surfaces and present problems suitable for undergraduate research.

# parts of today's talk

There are three main parts of the talk:

- motivation through billiard problems
- definition of flat surfaces
- properties of flat surfaces and relation to billiards

# square - 1

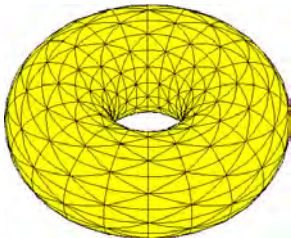
## First example - square billiard table

- consider a square billiard table and taking a shot
- move to Matlab show and tell (`billiard4_key.m` and `billiard4_pause.m`)
- trajectories and square tiling of the plane (board work)

# square - 2

## Making a compact flat surface from square tiling

- square tiling of the plane tiling may be converted to a "flat torus" using translation identifications of the plane.
- show process on the board.
- picture of a flat torus



# square - 3

## Equivalent representations of a billiard trajectories

- billiard trajectory on a square table
- line in the plane
- geodesic on the flat torus



# square - 4

## On a square table

- The billiard closes up if and only if the generated line in the plane has rational slope.
- Irrational lines have dense trajectories on the table and on the flat torus.

# pentagon - 1

## Billiard trajectories on pentagonal table

- move to Matlab show and tell (`billiard_reg4_key.m` and `billiard_reg_pause.m`)
- development of a pentagonal billiard table in the plane (show on board)
- construct flat surface from development by translation

# flat surfaces - definition

Here is an informal definition of a flat surface.

## Definition

Let  $P_1, \dots, P_n$  be a sequence of polygons such that every side of every polygon is matched with exactly one side (same edge length) of another polygon. The match may be to another side of the same polygon. The compact space obtained by gluing the polygons together via the matching is called a *flat surface*

# flat surfaces - examples

Here are some examples of flat surfaces.

## Example

- any of the platonic surfaces (show models)
- the surface formed from the development of a convex table whose corner angles are rational multiples of  $\pi$
- identify the opposite sides of a regular polygon with an even number of sides

# flat surfaces geometry - 1

A flat surface has three types of points:

- interior points of polygons
- hinge points where two polygons meet along an edge
- cone points at the vertices of polygons
- The first two types of points are regular points on the surface. A neighbourhood of a hinge point can be made to look like a flat piece of plane by flattening.
- The cone points are usually considered to be singular. They cannot be flattened unless the total angle is  $2\pi$ .

# flat surfaces geometry - 2

The local geometry is determined as follows:

- Each regular point has a neighbourhood with the regular flat plane geometry, flattening a hinge as needed.
- Cone points need a measure of non-regularity, called the cone angle.
- If  $\alpha_1, \dots, \alpha_s$  are the angles at a cone point  $v_j$ , then the (total) cone angle at  $v_j$  is

$$\theta_j = \sum_{i=1}^s \alpha_i.$$

- A cone point is regular if and only the cone angle equals  $2\pi$  (lies flat).

# flat surfaces geometry - 3

Here are some examples of cone angles.

## Example

- A cube has 8 cone points with cone angle  $3\pi/2$ .
- An icosahedron has 12 cone points with cone angle  $5\pi/3$ .
- The torus has no singular cone points.

# Euler's formula

## Theorem

*Suppose a flat surface has genus  $g$  and  $v$  cone points with cone angles  $\theta_j$ . Then*

$$\sum_{j=1}^v \theta_j = 2\pi(2g - 2 + v).$$



# Euler's formula

## Proof.

- We may assume all polygons are triangles, giving the surface a triangulation with  $v$  vertices,  $e$  edges, and  $f$  faces.
- By Euler's formula  $2 - 2g = v - e + f$ .
- Since all faces are triangles  $2e = 3f$ .
- Thus,  $2 - 2g = v - 3f/2 + f = v - f/2$ , so  $f/2 = 2g - 2 + v$ .
- Now count all angles in all triangles in two ways:

$$\sum_{j=1}^v \theta_j = \text{sum of all angles} = \pi f = 2\pi(2g - 2 + v).$$



# classification of geodesics on surfaces

## Theorem

*Suppose we have a flat surface then a geodesic must be one of the following types:*

- *a closed geodesic (periodic billiard trajectory)*
- *a saddle connection (connect two cone points)*
- *an infinite dense geodesic from a cone point (dense one way billiard trajectory)*
- *an infinite dense geodesic with no cone points (dense two way billiard trajectory)*

# Any Questions?

