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Full Automorphism Groups of Cyclic n-gonal Surfaces

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The Automorphism Group of a Cyclic n-gonal Surface

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parts of the talk

There are five main parts of the talk

- ¡muchas gracias!
- (very brief) historical introduction
- background and notation for the problem
- classification/computational methods
- results and results in progress

hyperelliptic surfaces - 1

- A hyperelliptic (Riemann) surface S is the smooth projective completion a complex algebraic curve of the form

$$y^2 = f(x) = \prod_{i=1}^{r}(x - a_i),$$

where the a_i are distinct.

- the genus is $\frac{r-2}{2}$ for even r

hyperelliptic surfaces - 2

- $\iota : (x, y) \to (x, -y)$ is an involution of the surface which fixes the points $(a_i, 0)$ and no others if t is even
- the map $\pi : S \to \mathbb{P}^1$, $(x, y) \to x$ is a quotient map for the projection $S \to S/\langle\iota\rangle$
- It is a very old theorem that ι is in the center of $\mathrm{Aut}(S)$ and hence the subgroup $\langle\iota\rangle$ is normal in $\mathrm{Aut}(S)$.

n-gonal surfaces - 1

- generalize the notion of hyperelliptic involution and hyperelliptic cover $\pi : S \to \mathbb{P}^1$
- A n-gonal surface S is a closed, orientable Riemann surface for which there is a degree n map

$$\pi : S \to \mathbb{P}^1$$

  called an n-gonal morphism.
- n is usually small in comparison to the genus σ
- for a general surface of genus σ the lowest value of n is σ
- If $n = 3$ the surfaces are called trigonal, and are well studied.

n-gonal surfaces - 2

- assume $\pi : S \to \mathbb{P}^1$ is an n-gonal morphism
- $\mathbb{C}(S)$ denotes the field of meromorphic functions on S
- the degree of the extension $\mathbb{C}(S)/\mathbb{C}(\mathbb{P}^1)$ is n
- The strongest results can be obtained when the extension is cyclic, and we shall focus on this case.
- In particular, cyclic n-gonal surfaces have nice equations.

n-gonal surfaces - 3

- A cyclic n-gonal surface is birationally equivalent to a complex algebraic curve of the form

$$y^n = f(x) = \prod_{i=1}^{r} (x - a_i)^{t_i}, \tag{1}$$

where a_i, t_i and $t = t_1 + \cdots + t_r = \deg(f)$ satisfy

  - the a_i are distinct
  - $0 < t_i < n$,
  - n divides t
  - $\gcd(n, t_1, \ldots, t_r) = 1$

n-gonal surfaces - 4

- If $\omega^n = 1$ then $(x, y) \to (x, \omega y)$ is an automorphism of the surface which fixes the points $(a_i, 0)$ and no others.
- Let C be the cyclic group of automorphisms obtained by letting ω range over all nth roots of unity.
- The map $\pi : S \to \mathbb{P}^1$, $(x, y) \to x$ is a quotient map for the projection $S \to S/C$, and is the cyclic n-gonal morphism.
- The degree of ramification of π at a_i is n/d_i where $d_i = (t_i, n)$
- the genus σ of S is given by

$$\sigma = \frac{1}{2}\left(2 + (r-2)n - \sum_{i=1}^{r} d_i\right)$$

n-gonal surfaces - 5

- There are analogues of the centrality of the hyperelliptic involution.
- Suppose that $n = p$ is a prime and $\sigma > (p-1)^2$ then C is normal in $\mathrm{Aut}(S)$ (Accola).
- Suppose that $(n, t_i) = 1$ for all i and $\sigma > (n-1)^2$ then C is normal in $\mathrm{Aut}(S)$. In this case the cyclic n-gonal morphism is fully ramified (Kontogeorgis).

automorphism groups of n-gonal surfaces - 1

- There is a great deal of interest in the automorphism group $A = \mathrm{Aut}(S)$ of a cyclic n-gonal surface, especially the normal case.
- In the normal case A/C is an automorphism group of the sphere, one of five types of Platonic groups.
- One "simply" solves an extension problem

$$C \to A \to K.$$

automorphism groups of n-gonal surfaces - 2

- The case $n = 2$ (hyperelliptic case) has been studied extensively: Brandt, Bujulance, Etayo, Gamboa, Gromadzki, Martinez.
- The case where $n = 3$, (cyclic trigonal surfaces): Accola, Bujalance($\times$2), Cirre, Costa, Izquierdo, Martinez, Ying.
- The case where $n = p$, for p a prime: Brandt, Gonzalez-Diez, Harvey, Wootton.
- General n where the cyclic n-gonal morphism $S \rightarrow S/C$ is fully ramified: Kontogeorgis.
- General n with certain group conditions: Broughton & Wootton

problem description

- Prove as general a statement as possible on the automorphism groups of cyclic n-gonal surfaces, especially results on normality. It is necessary to place hypotheses on how the group C sits in A.
- Classify the automorphism groups in the non-normal case.
- Work is in progress. Computational group theory methods are used very strongly.
- Some details will be deferred to the second talk.

set up group notation

- Let C be a cyclic group of automorphisms of S so that S/C has genus zero.
- Let $A = \mathrm{Aut}(S)$ be the group of automorphisms of S.
- Let $N = N_A(C)$ be the normalizer of C in A.
- The group $K = N/C$ acts on S/C and so must be one of the five platonic types: $\mathbb{Z}_k$, D_k, A_4, Σ_4, A_5.

strongly branched maps - 1

This is the only place where we use (algebraic) geometry. Everything else we use is essentially group theory. The concept of strong branching was developed by Accola.

- Let $\pi : S_1 \to S_2$ be a degree n map of surfaces of genus σ_1 and σ_2.
- Define the ramification R_π by
$$R_\pi = (2\sigma_1 - 2) - n(2\sigma_2 - 2).$$
- The map π is strongly branched if
$$R_\pi > 2n(n-1)(\sigma_2 + 1).$$
- If S_2 has genus zero, then π is strongly branched if
$$R_\pi > 2n(n-1)$$
or
$$\sigma_1 > (n-1)^2.$$

strongly branched maps - 2

- Let $\pi : S \to S/C$ be the cyclic n-gonal morphism of degree n derived from the defining equation 1.
- Let
$$d_i = (n, t_i),\ n_i = \frac{n}{d_i}, \textit{then}$$
-
$$R_\pi = n\sum_{i=1}^{t}\left(1 - \frac{1}{n_i}\right) = \sum_{i=1}^{t}(n - d_i)\,.$$
- The cyclic n-gonal morphism is strongly branched when there are many relatively prime power factors in the cyclic defining equation 1.

strongly branched maps - 3

Here is a useful consequence of being strongly branched.

Proposition

Let H be a group of automorphisms acting on a surface S such that $S \to S/H$ is strongly branched. Then there is a unique minimal, normal, nontrivial subgroup L of $\mathrm{Aut}(S)$ such that $L \leq H$, and $S \to S/L$ is strongly branched.

Weakly malnormal - 1

Next we develop a group theoretic concept that allows us to exploit strong branching.

Definition

Let $H \leq G$ be a pair of groups and let $N = N_G(H))$. Then H is weakly malnormal in G if for each $g \in G - N$ we have a trivial intersection $H \cap H^g = \langle 1 \rangle$.

Example

Normal subgroups are trivially weakly malnormal.

weakly malnormal - 2

Example

If $H \leq G$ is a cyclic subgroup of prime order then H is weakly malnormal in G.

Example

If $C \leq A$ is a cyclic subgroup of $A = \mathrm{Aut}(S)$ and the map $S \to S/C$ is fully ramified, then C is weakly malnormal in A.

weakly malnormal - 3

Remark

Let $H \leq G$ be a pair of groups such that H is weakly malnormal in G, but not normal. If K is a nontrivial subgroup of H, then

$$N_G(H) = N_G(K).$$

Remark

Assume the same hypotheses as above. Then the representation of G on the left or right cosets of H is faithful. For, the kernel of the representation is $\bigcap_{g \in G} H^g$.

weakly malnormal - 4

A group action of H on a surface S is called weakly malnormal if H is a weakly malnormal subgroup of $A = \mathrm{Aut}(S)$.

Proposition

Let H be a group of automorphisms acting on a surface S such that $S \to S/H$ is strongly branched and H is weakly malnormal in $A = \mathrm{Aut}(S)$. Then H is normal in A.

Proposition

If the action of a group C of order n on surface of genus $\sigma > (n-1)^2$ is weakly malnormal and S/C has genus zero then C is normal in A.

It follows the non-normal cases occur only for small genus.

weakly malnormal - 5

Remark

There are examples of cyclic 4-gonal actions on surfaces of arbitrarily high genus, but where C is not normal in A

classification steps

- Lift actions of groups on surfaces to actions of Fuchsian groups on the hyperbolic disc.
- Use computational group theory methods on
  - Fuchsian group signatures
  - monodromy of Fuchsian group pairs
  - “word maps” of Fuchsian group pairs
- Using the monodromy and word maps the pairs $C < N$ and $N < A$ may be fused together to produce A. (since the permutation representation of A the cosets of C is faithful).
- The classification comes in two parts: constrained and tight pairs.

Fuchsian groups - covering groups

- The inclusion of groups acting on S

$$C \hookrightarrow N \hookrightarrow A$$

- gives us a diagram of covering Fuchsian groups

$$\begin{array}{ccccc} \Gamma_C & \hookrightarrow & \Gamma_N & \hookrightarrow & \Gamma_A \\ \downarrow \eta & & \downarrow \eta & & \downarrow \eta \\ C & \hookrightarrow & N & \hookrightarrow & A \end{array} \tag{2}$$

- for an exact sequence

$$\Pi \hookrightarrow \Gamma_A \overset{\eta}{\twoheadrightarrow} A$$

- such that Π is torsion free and $S = \mathbb{H}/\Pi$

Fuchsian group - signatures

- If S/C has genus zero then each of the Fuchsian groups $\Gamma = \Gamma_A, \Gamma_N, \Gamma_C$ in the covering diagram 2 have genus zero and have signatures of the form

$$\mathcal{S}(\Gamma) = (0; m_1, m_2, \ldots, m_r)$$

  where the map $\mathbb{H} \to \mathbb{H}/\Gamma$ is branched over r points with ramification indices m_i for $1 \leq i \leq r$.
- The signatures of $\Gamma_A, \Gamma_N, \Gamma_C$ must satisfy certain relationships.

Fuchsian group - canonical generating sets

- Each of the Fuchsian groups $\Gamma = \Gamma_A, \Gamma_N, \Gamma_C$ has a canonical generating set of the form $\mathcal{G} = \{\zeta_1, \ldots \zeta_r\}$ satisfying
$$\zeta_1^{m_1} = \zeta_2^{m_2} = \cdots = \zeta_r^{m_r} = \prod_{j=1}^{r} \zeta_j = 1.$$
- The generating sets of $\Gamma_A, \Gamma_N, \Gamma_C$ must satisfy certain relationships.

Fuchsian group pairs - monodromy

A pair of Fuchsian groups $\Gamma_1 < \Gamma_2$ can be described in two ways: the monodromy vector of the pair, and the word map of the pair. Additional details are given in the lecture on Fuchsian group pairs.

- Let $\Gamma_1 < \Gamma_2$ be a Fuchsian group pair of index m and suppose Γ_2 has canonical generating set $\mathcal{G} = \{\zeta_1, \ldots \zeta_r\}$
- Let π_i be the permutation determined by ζ_i acting on the cosets of Γ_2/Γ_1.
- $\mathcal{P} = (\pi_1, \pi_2, \ldots, \pi_r)$ is a called the monodromy vector of the pair.
- The monodromy vector satisfies
    - the cycle type of π_i is determined by the signatures of Γ_1 and Γ_2.
    - $\pi_1 \pi_2 \cdots \pi_r = 1$,
    - $\langle \pi_1, \pi_2, \ldots, \pi_r \rangle$ is a transitive subgroup of Σ_m

Fuchsian group pairs - word maps

- Let $\Gamma_1 < \Gamma_2$ be as before and suppose that Γ_1 and Γ_2 have canonical generating sets

$$\mathcal{G}_1 = \theta_1, \ldots \theta_s$$

and

$$\mathcal{G}_2 = \zeta_1, \ldots \zeta_r,$$

respectively.

- The word map of the pair $\Gamma_1 < \Gamma_1$ is a set of words $\{w_1, \ldots, w_s\}$ in the generators in $\mathcal{G}_2$ such that

$$\theta_i = w_i(\zeta_1 \ldots, \zeta_r), i = 1, \ldots, s$$

Fuchsian group pairs - monodromy and word maps

The monodromy and word maps can be used interchangeably (with lots of computational group theory).

- Given a monodromy vector of a pair $\Gamma_1 < \Gamma_2$, the word map may be constructed (see talk on Fuchsian group pairs).
- Given a word map the monodromy of the pair may be computed using the Todd-Coxeter algorithm.

Classification Steps - 1

- Determine the signatures the pairs $\Gamma_C < \Gamma_N$, $\Gamma_N < \Gamma_A$
- Determine monodromy groups for $\Gamma_C < \Gamma_N$ (easy) and $\Gamma_N < \Gamma_A$ (this requires the primitive group classification).
- Compute word maps for $\Gamma_C < \Gamma_N$ and $\Gamma_N < \Gamma_A$.
- Compose word maps and compute monodromy for $\Gamma_C < \Gamma_A$.
- The above representation is faithful and C is the stabilizer of a point.
- Determine if the stabilizer is cyclic.

Classification Steps - 2

The classification steps for $\Gamma_C < \Gamma_N$ requires the signatures for the K acting on S/C.

Group	Signature
$\mathbb{Z}_k$	(k, k)
D_k	$(2, 2, k)$
A_4	$(2, 3, 3)$
S_4	$(2, 3, 4)$
A_5	$(2, 3, 5)$

Table: Groups of Automorphisms and Signatures of $\mathbb{P}^1$

Theorem on signatures - 1

Theorem

If the action of C on S is weakly malnormal, then Γ_N has at most 3 additional periods to Γ_A. If Γ_A and Γ_N have the same number of canonical generators, then they appear in Singerman's list. The signatures for Γ_A and Γ_N appear as a pair in following where (a_1, a_2, a_3) or (k, k) is the signature of $K = \Gamma_N / \Gamma_C$. The signature for Γ_C is automatically determined from Γ_N.

Theorem on signatures - 2

Case	Signature of Γ_N	Signature of Γ_A
$0A$	$(0; a_1m_1, a_2m_2, a_3m_3, n_1, \ldots, n_r)$	$(0; b_1, b_2, b_3, n_1, \ldots, n_r)$
$0B$	$(0; km_1, km_2, n_1, \ldots, n_r)$	$(0; b_1, b_2, n_1, \ldots, n_r)$
$1A$	$(0; a_1m_1, a_2m_2, a_3m_3, n_1, \ldots, n_r)$	$(0; b_1, b_2, n_1, \ldots, n_r)$
$1B$	$(0; km_1, km_2, n_1, \ldots, n_r)$	$(0; b_1, n_1, \ldots, n_r)$
$2A$	$(0; a_1m_1, a_2m_2, a_3m_3, n_1, \ldots, n_r)$	$(0; b_1, n_1, \ldots, n_r)$
$2B$	$(0; km_1, km_2, n_1, \ldots, n_r)$	$(0; n_1, \ldots, n_r)$
$3A$	$(0; a_1m_1, a_2m_2, a_3m_3, n_1, \ldots, n_r)$	$(0; n_1, \ldots, n_r)$

Table: Signatures for Γ_A and Γ_N

Fuchsian group classification results

- The possible pairs $\Gamma_N < \Gamma_A$ satisfying the signature constraints break up into two families: constrained and tight pairs (more detail in the talk on Fuchsian group pairs).
- There are 202 constrained pairs yielding two known examples of *p*-gonal surfaces.
- There 597 potential tight pairs $\Gamma_N < \Gamma_A$ each of which yields a family of Fuchsian group pairs.
  - Each family above could yield a finite number of exceptional pairs and a finite number of families all of which satisfy the signature theorem.
  - some sample families have been found and the auto-classification is underway.

cyclic *n*-gonal groups from constrained pairs

$\mathcal{S}(\Gamma_A)$	$\mathcal{S}(\Gamma_N)$	$\mathcal{S}(K)$	$\|\Gamma_A/\Gamma_N\|$	$\|C\|$	genus	Group
$(2,4,5)$	$(4,4,5)$	$(4,4)$	6	5	4	Σ_6
$(2,3,7)$	$(3,3,7)$	$(3,3)$	8	7	3	$PSL_2(7)$

cyclic *n*-gonal groups from tight pairs

$\mathcal{S}(\Gamma_A)$	$\mathcal{S}(\Gamma_N)$	$\lvert\Gamma_A/\Gamma_N\rvert$	$\lvert C\rvert$	genus	Group
$(2,3,4n)$	$(2,2,3,n)$	4	$n=2$	2	$GL(2,3)$
$(2,3,3n)$	$(3,n,3n)$	4	$n=4$	3	$SL(2,3)/CD$
$(2,3,2n)$	$(2,n,2n)$	3	$n\geq 5$	$\frac{(n-1)(n-2)}{2}$	$\Sigma_3\times(\mathbb{Z}_n\times\mathbb{Z}$
$(2,2,2,n)$	$(2,2,n,n)$	2	$n\geq 3$	$(n-1)^2$	$V_4\times(\mathbb{Z}_n\times\mathbb{Z}$
$(2,4,2n)$	$(2,2n,2n)$	2	$n\geq 3$	$(n-1)^2$	$D_4\times(\mathbb{Z}_n\times\mathbb{Z}$

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