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Rollups and Differential Geometry - Rose Math Seminar

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Roll-ups and Differential Geometry

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constructing cones and cylinders

constructing cones and cylinders

- a cylinder can be made by rolling up a rectangle - show and tell
- the frustum of a cone be made by rolling up the sector truncated by two concentric circles - more show and tell
- these are examples of flat surfaces - last year's talk
- this year's talk is much more practical

roll-ups and flattening

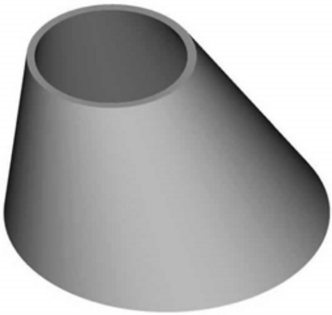
- **roll-up**: Construct a flat surface in 3-space by rolling and possibly twisting paper or metal in an inelastic fashion . We are allowed to tape (weld). Just the formula for the map, we don't actually want to get our hands dirty.
- **flattening**: Given a flat surface in 3-space determine a region in the plane when flattened. I.e., we want to construct the given surface by rolling up some region.

manufacturing design problem - 1

- phone call from local manufacturing design firm
- We want to manufacture a cut off slanted cone from a flat sheet of metal. If the cone was a normal right cone we know that we would simply cut out a sector of a circle and roll it up. However the cone is slanted. We want to know what the flattened shape looks like so that we can cut it out and roll it up to closely approximate correct final shape. We also want to minimize the amount of wasted metal after the shape is cut out.

manufacturing design problem - 2

- here is the surface



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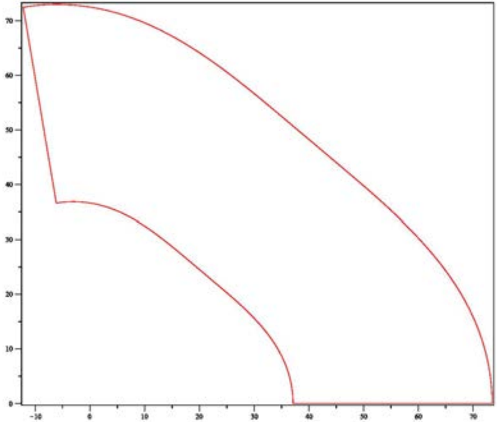
manufacturing design problem - 3

- here is the flattened surface



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manufacturing design problem - 3

- the roll-up really works - show and tell

cones - 1

- $P(s), 0 \leq s \leq L$, is a space curve in $\mathbb{R}^3$ of length L, parameterized by arclength s
- Q is any point in $\mathbb{R}^3$
- the cone C over $P(s)$ is swept out by the line segment passing through Q and $P(s)$
- frustum can be created by removing a cone from a cone - show on flattened surface
- Maple show and tell

cones - 2

- construct the roll-up map in two stages
- **A**: map from a rectangle to the cone
- **B**: map from a deformed sector to the rectangle
- the deformed sector is the flattened cone

cones - 3: the map A

- Define the cone map $A : R \to C$ from the rectangle $R = [0, L] \times [0, 1]$ by

$$A(s, t) = (1 - t)Q + tP(s)$$

- continue Maple show and tell

cones - 4: the map B

- find a sector S in the plane described in polar coordinates by

$$S = \{(r, \theta) : 0 \leq \theta \leq \Theta, 0 \leq r \leq \rho(\theta)\}$$

  for some Θ and $\rho(\theta)$ to be determined, and

- a map $B : S \to R$

$$B(r, \theta) = \left(\sigma(\theta), \frac{r}{\rho(\theta)}\right)$$

  for a function σ, to be determined

- draw picture on board

cones - 4: the roll-up map

- the composite map $A \circ B : S \to C$ given below is the desired roll-up map and S is the desired "flattened cone" region

Theorem

Suppose notation is as above, then for suitably chosen ρ and σ the composite map $A \circ B$

$$(r, \theta) \to \left(1 - \frac{r}{\rho(\theta)}\right) Q + \left(\frac{r}{\rho(\theta)}\right) P(\sigma(\theta))$$

is an isometry. I.e., every plane curve in region S is mapped to a space curve on the cone C of the same length.

the function $\ell(s)$

- an important function in our analysis will be distance from Q to $P(s)$
$$\ell(s) = \|P(s) - Q\|$$
- the map of a radial line segment is to be isometric so that

$$\rho(\theta) = \ell(\sigma(\theta))$$

DE derivation and solution - 1

- the curve $r = \rho(\theta)$ is mapped isometrically to the curve $AB(r(\theta), \theta) = P(\sigma(\theta))$
- equating arclengths in three space and polar coordinates

$$\int_0^\theta \left\| \frac{d}{du} P(\sigma(u)) \right\| du = \int_0^\theta \sqrt{(\rho(u))^2 + (\rho'(u))^2} du$$

$$\int_0^\theta |\sigma'(u)| \, \|P'(\sigma(u))\| \, du = \int_0^\theta \sqrt{(\rho(u))^2 + (\rho'(u))^2} du$$

$$\int_0^\theta |\sigma'(u)| \, du = \int_0^\theta \sqrt{(\rho(u))^2 + (\rho'(u))^2} du$$

- so

$$(\sigma'(\theta))^2 = (\rho(\theta))^2 + (\rho'(\theta))^2$$

DE derivation and solution - 2

- from $(\sigma'(\theta))^2 = (\rho(\theta))^2 + (\rho'(\theta))^2$ and $\rho(\theta) = \ell(\sigma(\theta))$ we get
- $$\begin{aligned} (\sigma'(\theta))^2 &= (\rho(\theta))^2 + (\rho'(\theta))^2 \\ &= (\ell(\sigma(\theta)))^2 + (\ell'(\sigma(\theta))\sigma'(\theta))^2 \\ (1 - (\ell'(\sigma(\theta)))^2)\,(\sigma'(\theta))^2 &= (\ell(\sigma(\theta)))^2 \\ (\sigma'(\theta))^2 &= \frac{(\ell(\sigma(\theta)))^2}{(1 - (\ell'(\sigma(\theta)))^2)} \\ \sigma'(\theta) &= \pm\frac{\ell(\sigma(\theta))}{\sqrt{(1 - (\ell'(\sigma(\theta)))^2)}} \end{aligned}$$

DE derivation and solution - 3

- set $s = \sigma(\theta)$, and $\frac{ds}{d\theta} = \sigma'(\theta)$ to get

$$\frac{ds}{d\theta} = \frac{\ell(s)}{\sqrt{1 - (\ell'(s))^2}}.$$

- separate and integrate

$$\theta = \int_0^{\sigma(\theta)} \frac{\sqrt{1 - (\ell'(s))^2}}{\ell(s)} ds$$

- we can then define T, the inverse of σ, by the integral

$$T(s) = \int_0^{s} \frac{\sqrt{1 - (\ell'(u))^2}}{\ell(u)} du$$

the equations for S

Proposition

Let $P(s), \ell(s), T(s), \sigma(\theta), \rho(\theta) = \ell(\sigma(\theta)), A, B,$ and Θ be as defined above. Then

$$s = \sigma(\theta) \Leftrightarrow \theta = T(s).$$

Moreover, the sector S , which is the domain of the complete cone map,

$$AB(r, \theta) = \left(1 - \frac{r}{\rho(\theta)}\right) Q + \left(\frac{r}{\rho(\theta)}\right) P(\sigma(\theta)),$$

is defined by

$$S = \{(r, \theta) : 0 \leq \theta \leq \Theta, 0 \leq r \leq \rho(\theta)\}$$

Comment: Isometry

- as noted the map $A \circ B$ is an isometry.
- the proof is a grubby but doable computation, using Jacobians and orthonormal frames

Question: rolling up flat surfaces

- A surface is flat if its Gaussian curvature is zero. Both cylinders and cones are flat.
- Question: Given a flat surface with boundary, determine a roll-up map for the surface.
- In particular find a region determined by flattening the region.

Any Questions?