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Exceptional Automorphisms of (generalized) Super-Elliptic Surfaces

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Exceptional Automorphisms of (generalized) Super-Elliptic Surfaces

preliminary report

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outline

Problem history of cyclic n-gonal surfaces

n-gonal surfaces - 1

- A cyclic n-gonal surface is a smooth surface with a plane model of the form

$$y^n = f(x) = \prod_{i=1}^{r} (x - a_i)^{t_i}, \tag{1}$$

where a_i, t_i and $t = t_1 + \cdots + t_r = \deg(f)$ satisfy
  - the a_i are distinct,
  - $0 < t_i < n$,
  - n divides t (this is not the typical requirement), and
  - $gcd(n, t_1, \ldots, t_r) = 1$
- If $n = 2$ then the surface is hyperelliptic.

n-gonal surfaces - 2

- The plane model of the surface is smooth except at points the points $(a_i, 0)$ where $t_i > 1$, and at a single point at ∞ if $t > n$.
- The normalization $S^\nu \to S$ has $d_i = gcd(t_i, n)$ points lying over $(a_i, 0)$ and $n = gcd(t, n)$ points lying over ∞.
- We frequently identify S^ν and S and call S^ν the smooth model and S the plane model.
- The genus σ of (the smooth model of) S is given by

$$\sigma = \frac{1}{2}\left(2 + (r-2)n - \sum_{i=1}^{r} d_i\right).$$

n-gonal surfaces - 3

- If $\omega^n = 1$ then $(x, y) \to (x, \omega y)$ is an automorphism of S which fixes the points $(a_i, 0)$ and no others.
- Let C be the cyclic group of automorphisms obtained by letting ω range over all nth roots of unity.
- The map $\pi : S^\nu \to \mathbb{P}^1$, $(x, y) \to x$ is a quotient map for the projection $S^\nu \to S^\nu / C$, and is called the cyclic n-gonal morphism.
- The degree of ramification of π over a_i is $n_i = n/gcd(t_i, n)$. The map is unramified over ∞ because n divides t.

n-gonal surfaces - 4

Normality results using Accola's theorem on strong branching.

- Hyperelliptic case: the involution $\iota : (x, y) \to (x, -y)$ is central in $\mathrm{Aut}(S)$.
- Prime order case: $n = p$ is a prime and $\sigma > (p-1)^2$, then C is normal in $\mathrm{Aut}(S)$. If f(x) is square-free then C is central (Accola).
- Fully ramified or generalized super-elliptic case: $gcd(n, t_i) = 1$ for all i. If $\sigma > (n-1)^2$, then C is normal in $\mathrm{Aut}(S)$ (Kontogeorgis).
- Weakly malnormal case: for all $g \in \mathrm{Aut}(S)$ either $gCg^{-1} = C$ or $gCg^{-1} \cap C = \{1\}$. Then, if $\sigma > (n-1)^2$, C is normal in $\mathrm{Aut}(S)$. (Broughton-Wootton)

Automorphism groups of cyclic n-gonal surfaces

automorphism groups of cyclic n-gonal surfaces - 1

- There is a great deal of interest in the full automorphism group $A = \mathrm{Aut}(S)$ of a cyclic n-gonal surface, especially the normal case.
- In the normal case A/C is an automorphism group of the sphere, one of five types of Platonic groups.
- One "simply" solves an extension problem

$$C \to A \to K.$$

- The automorphisms can be explicitly written down as birational transformations of $\mathbb{P}^2$.

automorphism groups of cyclic n-gonal surfaces - 2

- The case $n = 2$ (hyperelliptic case) has been studied extensively: Brandt, Bujulance, Etayo, Gamboa, Gromadzki, Martinez.
- The case where $n = 3$, (cyclic trigonal surfaces): Accola, Bujalance($\times$2), Cirre, Costa, Izquierdo, Martinez, Ying.
- The case where $n = p$, for p a prime: Brandt, Gonzalez-Diez, Harvey, Wootton.
- General n where the cyclic n-gonal morphism $S \to S/C$ is fully ramified: Kontogeorgis.
- General n with weak malnormality conditions: Broughton & Wootton

cyclic n-gonal surfaces - the groups involved

Let S be a cyclic n-gonal surface, namely:

- S is a surface of genus σ.
- $C = \langle h \rangle$ is a cyclic group of automorphisms of S, of order n, such that S/C has genus zero.
- $A = \mathrm{Aut}(S)$ is the group of automorphisms of S.
- $N = N_A(C)$ is the normalizer of C in A.
- The group $K = N/C$ acts on S/C and so must be one of the five platonic types: $\mathbb{Z}_k$, D_k, A_4, Σ_4, A_5, if K is not trivial.
- An automorphism in $A - N$ is called *exceptional*.

goals of calculation

- Ultimately, we want to determine the automorphism group of any cyclic n-gonal surface. We will restrict our attention to generalized super-elliptic surfaces.
- The normal case $A = N$ is computable using well known extension methods for the exact sequence

$$C \hookrightarrow N \twoheadrightarrow K.$$

- Assuming that S is a generalized super-elliptic surface, $N = A$ if $\sigma > (n-1)^2$.
- For fixed n determine the finite number of cases where $N < A$ with exceptional automorphism. As $\frac{(n-1)(r-2)}{2} = \sigma \leq (n-1)^2$ then $r \leq 2n$.

Super-elliptic surfaces

super-elliptic surfaces - 1

A super-elliptic surface is a cyclic n-gonal surface, where

- $n = p$, a prime,
- $f(x)$ is square free, (implies that C will be central in A),
- p need not divide the degree of $f(x)$.

We generalize this definition to non-prime cyclic groups and relax the square-free condition.

Definition

Let S be a cyclic n-gonal surface, whose plane model satisfies the requirements given earlier. If $gcd(n, t_i) = 1$ for all t_i, or alternatively, if the degree of ramification of π over a_i equals n, then S is called a *generalized super-elliptic surface*.

super-elliptic surfaces - 2

- The definition is motivated by the papers of Kontogeorgis and Shaska. The key requirement is that C have a fully ramified action.
- We require that n divide the degree of $f(x)$ so that all ramification occurs over points of the finite plane.
- Relaxing the requirement that $f(x)$ be square-free implies that C will be only be normal instead of central (for large genus).
- The genus σ of (the smooth model of) S is given by

$$\sigma = \frac{(n-1)(r-2)}{2}.$$

super-elliptic surfaces - 3

- There is much interest – motivated by cryptography – in computing in the Jacobian of super-elliptic surfaces S for fields of prime characteristic. See the paper of Shaska and, of course, his talk. |item Throughout the remainder of the talk we use the term super-elliptic to mean the extension to general n.

Moduli spaces of cyclic n=gonal and super-elliptic surfaces

moduli spaces for n=gonal surfaces - 1

Given our cyclic n-gonal equation

$$y^n = f(x) = \prod_{i=1}^{r}(x - a_i)^{t_i},$$

call

- $(a_1, a_2, \ldots, a_r)$ the branch points of S,
- $(t_1, t_2, \ldots, t_r)$ the multi-degree of S,
- $(n_1, n_2, \ldots, n_r)$ – where $n_i = n/gcd(n, t_i)$ – the branching data or signature of the action of C on S.

moduli spaces for n=gonal surfaces - 2

- Two surfaces with branch points $(a_1, a_2, \ldots, a_r)$ and $(b_1, b_2, \ldots, b_r)$ are equivalent if there is an $L \in PSL_2(\mathbb{C})$ and a permutation $\vartheta \in \Sigma_r$, preserving multi-degree, so that

$$b_i = L(a_{\vartheta i}).$$

  for all i.
- Let Σ_T denote the group of permutation preserving the multi-degree T.

moduli spaces for n=gonal surfaces - 3

- The smooth variety

$$\mathcal{MC}_{n,T} = (\mathbb{C}^r - \mathit{diagonals})/(PSL_2(\mathbb{C}) \times \Sigma_T)$$

  of degree $r-3$ is "almost" a moduli space for the surfaces of multi-degree T.
- The action of $PSL_2(\mathbb{C})$ is only partial and exceptional automorphisms need to be taken into account.
- Each $\mathcal{MC}_{n,T}$ corresponds to a moduli space, of the same dimension, of Fuchsian groups determined by the signature $(n_1, n_2, \ldots, n_r)$.

moduli spaces for n=gonal surfaces - 4

- A multi-degree $(t_1, t_2, \ldots, t_r)$ may be identified with an element of $\mathbb{Z}_n^r$ with $t_i \neq 0$ for all i, $\sum_i t_i = 0$, and $\mathbb{Z}_n = \langle t_1, t_2, \ldots, t_r \rangle$.
- $T = (t_1, t_2, \ldots, t_r)$ and $U = (u_1, u_2, \ldots, u_r)$ yield $\mathcal{MC}_{n,T} = \mathcal{MC}_{n,U}$ if there is $\omega \in \mathrm{Aut}(\mathbb{Z})$ and $\vartheta \in \Sigma_r$ such that
$$u_i = \omega(t_{\vartheta i}).$$
for all i.
- It is interesting to see how many different $\mathcal{MC}_{n,T}$ correspond to a given signature. The super-elliptic surfaces have signature $(n, n, \ldots n)$. See next slide.

moduli spaces for n=gonal surfaces - 5

Table of numbers of multi-degrees for cyclic 35-gonal surfaces with 4 branch points.

(n_1, n_2, n_3, n_4)	# inequivalent multi-degrees	$lcm(n_1, n_2, n_3, n_4)$
$(35, 35, 35, 35)$	26	35
$(35, 35, 35, 7)$	18	35
$(35, 35, 35, 5)$	13	35
$(35, 35, 7, 7)$	12	35
$(35, 35, 7, 5)$	8	35
$(35, 35, 5, 5)$	6	35
$(35, 7, 7, 5)$	2	35
$(35, 7, 5, 5)$	3	35
$(7, 7, 7, 7)$	4	7
$(7, 7, 5, 5)$	1	35
$(5, 5, 5, 5)$	3	5

Fuchsian groups

Fuchsian groups - generators, presentation and signature

- A Fuchsian group Γ, a discrete group acting on the hyperbolic plane $\mathbb{H}$, has a presentation by hyperbolic, elliptic, and parabolic generators and relations:

$$\text{generators}: \{\alpha_i, \beta_i, \gamma_j, \delta_k, 1 \leq i \leq \sigma, 1 \leq j \leq s, 1 \leq k \leq p\}$$

$$\text{relations}: \prod_{i=1}^{\sigma}[\alpha_i, \beta_i] \prod_{j=1}^{s} \gamma_j \prod_{k=1}^{p} \delta_k = \gamma_1^{m_1} = \cdots = \gamma_s^{m_s} = 1$$

- The signature of Γ is

$$\mathcal{S}(\Gamma) = (\sigma : m_1, \ldots, m_s, m_{s+1}, \ldots, m_{s+p})$$

with $m_{s+j} = \infty$, $j = 1, \ldots, p$ (the parabolic generators).

- allow for parabolic generators to account for parametric families of n-gonal surfaces, such as Fermat curves.

Fuchsian groups - invariants

Important invariants of a Fuchsian group

- The *genus of* Γ: $\sigma(\Gamma) = \sigma$ is the genus of $S = \overline{\mathbb{H}/\Gamma}$
- The area of a fundamental region: $A(\Gamma) = 2\pi\mu(\Gamma)$ where,

$$\mu(\Gamma) = 2(\sigma - 1) + \sum_{j=1}^{s+p}(1 - \frac{1}{m_j}).$$

- *Teichmüller dimension* $d(\Gamma)$ of Γ: the dimension of the Teichmüller space of Fuchsian groups with signature $\mathcal{S}(\Gamma)$ given by

$$d(\Gamma) = 3(\sigma - 1) + s + p.$$

Fuchsian group pairs

Fuchsian group pairs - index and codimension

- For finite index pair of Fuchsian groups $\Gamma < \Delta$,

$$[\Delta : \Gamma] = \mu(\Gamma)/\mu(\Delta).$$

- Also we call the quantity

$$c(\Gamma, \Delta) = d(\Gamma) - d(\Delta)$$

  the *Teichmüller codimension* of (Γ, Δ)
- These quantities are determined entirely by the signatures $\mathcal{S}(\Gamma)$ and $\mathcal{S}(\Delta)$.
- The signatures of a pair $\Gamma < \Delta$ must satisfy certain compatibility conditions.

Fuchsian group pairs - canonical generating sets

- Suppose that Γ has genus σ, s elliptic generators, and p parabolic generators and that Δ has genus τ, t elliptic generators, and q parabolic generators.
- For notational convenience, we denote the canonical generating sets of Γ and Δ, respectively, by:

$$\mathcal{G}_1 = \{\theta_1, \ldots, \theta_{2\sigma+s+p}\}$$

and

$$\mathcal{G}_2 = \{\zeta_1, \ldots, \zeta_{2\tau+t+q}\},$$

- In any calculation we will always assume that $\sigma = \tau = 0$.

Fuchsian group pairs - monodromy

- The pair $\Gamma < \Delta$ determines a permutation or monodromy representation of Δ on the cosets of Γ

$$\rho : \Delta \to \Sigma_m$$

  where m is the index of Γ in Δ.
- Write

$$\mathcal{P} = (\pi_1, \pi_2, \dots, \pi_{2\tau+t+q})$$

  for $\pi_i = \rho(\zeta_i) \in \Sigma_m$, to construct the *monodromy vector* of the pair.
- The cycle types and other properties of $\mathcal{P}$ are determined by signatures $\mathcal{S}(\Gamma)$ and $\mathcal{S}(\Delta)$ and the relations on the generators.
- $M(\Delta, \Gamma) = \rho(\Delta) = \langle \pi_1, \pi_2, \dots, \pi_{2\tau+t+q} \rangle$ is called the *monodromy group* of the pair.

Fuchsian group pairs - word maps and monodromy

- The *word map* of the inclusion $\Gamma \hookrightarrow \Delta$ is a set of words $\{w_1, \ldots, w_{2\sigma+s+p}\}$ in the generators in $\mathcal{G}_2$ such that

$$\theta_i = w_i(\zeta_1 \ldots, \zeta_{2\tau+t+q}), i = 1, \ldots, 2\sigma + s + p$$

- Given a word map for the inclusion $\Gamma \hookrightarrow \Delta$ a monodromy vector $\mathcal{P}$ is easily calculated using the Todd-Coxeter algorithm.
- Given monodromy vector $\mathcal{P}$ of a genus zero pair $\Gamma < \Delta$ (both groups), then the word map of the pair may be calculated, by an easily implemented algorithm.

Fuchsian group pairs - example part 1

- Suppose we have these signatures

$$\mathcal{S}_1 = (0; 2, 2, 2, 5), \mathcal{S}_2 = (0; 2, 4, 5)$$

- We want a pair $\Gamma < \Delta$ with

$$\mathcal{S}(\Gamma) = \mathcal{S}_1, \mathcal{S}(\Delta) = \mathcal{S}_2$$

- Find a compatible monodromy vector in Σ_6

$$\pi_1 = (1, 3)(4, 6), \pi_2 = (1, 2)(3, 5, 4, 6), \pi_3 = (1, 2, 3, 4, 5),$$

note that $M(\Delta, \Gamma) = A_6$.

Fuchsian group pairs - example part 2

- Define $\rho\colon \Delta \to \Sigma_6$ by $\rho\colon \zeta_i \to \pi_i, i = 1 \ldots 3$.
- Γ is the stabilizer of a point for the permutation action of Δ on $\{1, \ldots, 6\}$
- From the algorithm, a generating set for Γ is
  - $\theta_1 = (\zeta_1\zeta_2)\zeta_1(\zeta_1\zeta_2)^{-1}$
  - $\theta_2 = \zeta_2\zeta_1\zeta_2^{-1}$
  - $\theta_3 = \zeta_2^2$
  - $\theta_4 = (\zeta_2^{-1}\zeta_1^{-1}\zeta_2^{-1}\zeta_1\zeta_3\zeta_1)\zeta_3(\zeta_2^{-1}\zeta_1^{-1}\zeta_2^{-1}\zeta_1\zeta_3\zeta_1)^{-1}$

Constrained and tight pairs - 1

The following concept is introduced to account for families of pairs.

Definition

Let $\rho : \Delta \to \Sigma_m$ be as previously defined.

- A pair $\Gamma < \Delta$ is called *constrained* if Δ has no parabolic generators and $o(\zeta_i) = o(\rho(\zeta_i)) = o(\pi_i)$ for each elliptic generator ζ_i.
- A pair $\Gamma < \Delta$ is called *tight* if Δ has at least one parabolic generator and $o(\zeta_i) = o(\rho(\zeta_i)) = o(\pi_i)$ for each elliptic generator ζ_i.

Remark

The definition depends only on the cycle types, and hence only on the signature pair.

Constrained and tight pairs - 2

Proposition

Let $\Gamma < \Delta$ be a tight pair where Δ has q parabolic elements. Then there is a q-parameter family $\Gamma(\ell_1, \ldots, \ell_q) < \Delta(\ell_1, \ldots, \ell_q)$ such that each member of the family has

- *the same codimension $d(\Gamma, \Delta)$*
- *the same index $[\Delta : \Gamma]$*
- *the same monodromy $M(\Delta, \Gamma)$ and monodromy vector $\mathcal{P}$.*
- *the same word map*
- *The pair $\Gamma(\ell_1, \ldots, \ell_q) < \Delta(\ell_1, \ldots, \ell_q)$ is hyperbolic for almost every choice of the ℓ_i.*

Constrained and tight pairs - 3

Remark

Every Fuchsian group pair is constrained or belongs to a unique family as above. The tight pair defining the family is called the parent tight pair.

Example

The previous example and the pair $T(7,7,7) < T(2,3,7)$, are constrained pairs.

Example

The family of triangle group pairs $T(2,d,2d) < T(2,3,2d)$ comes from the tight pair $T(2,\infty,\infty) < T(2,3,\infty)$. The monodromy vector is $((1,2),(1,2,3),(1,3))$.

Classification via Fuchsian groups

Lifting actions - 1

- For S a cyclic n-gonal surface, we have a covering diagram

$$\begin{array}{ccccc} \Gamma_C & \hookrightarrow & \Gamma_N & \hookrightarrow & \Gamma_A \\ \downarrow \eta & & \downarrow \eta & & \downarrow \eta \\ C & \hookrightarrow & N & \hookrightarrow & A \end{array} \tag{2}$$

- for an exact sequence

$$\Pi \hookrightarrow \Gamma_A \overset{\eta}{\twoheadrightarrow} A$$

- such that Π is torsion free and $S = \mathbb{H}/\Pi$

Lifting actions - 2

Also

- $$M(\Gamma_A, \Gamma_N) = M(A, N)$$
  $$M(\Gamma_N, \Gamma_C) = M(N, C) = M(N/C, \langle 1 \rangle) \simeq K$$
  $$M(\Gamma_A, \Gamma_C) = M(A, C) \simeq A.$$
- The last holds because of the super-elliptic condition.

Overview of classification method - 1

- Each triple of groups $C < N < A$ gives a triple of Fuchsian groups $\Gamma_C < \Gamma_N < \Gamma_A$.
- Classify, using computational group theory methods on
  - Fuchsian group signatures
  - monodromy of Fuchsian group pairs $\Gamma_C < \Gamma_N$, $\Gamma_N < \Gamma_A$
  - “word maps” of Fuchsian group pairs $\Gamma_C < \Gamma_N$, $\Gamma_N < \Gamma_A$
- Using the monodromy and word maps, the monodromy of the pairs $C < N$ and $N < A$, may be fused together to produce A.

Overview of classification method -2

- The superelliptic condition limits the possible triples.
- There are finitely many cases of parametric families and finitely many exceptional cases to consider. The two types of cases need separate computational methods.
- The methods used are a specific application of methods developed to study pairs of Fuchsian groups. For more details, see [1] and [2] in the references.

Steps of classification - 1

- Using a computer search determine all signature pairs $\mathcal{S}(\Gamma_N)$ and $\mathcal{S}(\Gamma_A)$ for codimension 0,1,2,3, treating constrained and tight pairs separately.
- The group K and the signature $\mathcal{S}(\Gamma_C)$ is automatically determined.
- For each candidate signature pair, compute all the compatible monodromy vectors up to conjugacy. Use the classification of primitive permutation groups (Magma or GAP).
- Some extra work, using towers of groups, is required in using the primitive data base to calculate all the $M(\Gamma_A, \Gamma_N)$, since the monodromy group it is only a transitive group, not necessarily primitive.

Steps of classification - 2

- From monodromy vectors of $\Gamma_N < \Gamma_A$ and $\Gamma_N < \Gamma_C$ compute the word maps of $\Gamma_C \hookrightarrow \Gamma_N$ and $\Gamma_N \hookrightarrow \Gamma_A$
- Compute the word map of $\Gamma_C \hookrightarrow \Gamma_A$ by substitution.
- Compute the monodromy group $M(\Gamma_A, \Gamma_C)$ using the Todd-Coxeter algorithm.
- If the stabilizer of a point in $M(\Gamma_A, \Gamma_C) \simeq A$ is not cyclic then reject this case. Generally $C = \Gamma_C/\Pi$ is weakly malnormal, it is just not cyclic.
- There are 202 constrained pairs and 597 tight pairs $\Gamma_N < \Gamma_A$ that could potentially lead to cyclic n-gonal surfaces. Obviously this cannot be done by hand unless we are missing something clever.

Restrictions on signatures and structure

Super-elliptic restriction on signatures - 1

Theorem

If S is super-elliptic then Γ_N has at most 3 more periods than Γ_A. If Γ_A and Γ_N have the same number of canonical generators, then they appear in Singerman's list [5]. The signatures for Γ_A and Γ_N appear as a pair in following table. In the table

- (a_1, a_2, a_3) *or* (k, k) *is the signature of* $K = \Gamma_N/\Gamma_C$,
- *the* m_i *equal either* 1 *or* n,
- *the number of periods denoted by* n *is the same for each, and could be zero.*

The signature for Γ_C is automatically determined from Γ_N.

Super-elliptic restriction on signatures - 2

Case	Signature of Γ_N	Signature of Γ_A
$0A$	$(0; a_1 m_1, a_2 m_2, a_3 m_3, n, \ldots, n)$	$(0; b_1, b_2, b_3, n, \ldots, n)$
$0B$	$(0; km_1, km_2, n, \ldots, n)$	$(0; b_1, b_2, n, \ldots, n)$
$1A$	$(0; a_1 m_1, a_2 m_2, a_3 m_3, n, \ldots, n)$	$(0; b_1, b_2, n, \ldots, n)$
$1B$	$(0; km_1, km_2, n, \ldots, n)$	$(0; b_1, n, \ldots, n)$
$2A$	$(0; a_1 m_1, a_2 m_2, a_3 m_3, n, \ldots, n)$	$(0; b_1, n, \ldots, n)$
$2B$	$(0; km_1, km_2, n, \ldots, n)$	$(0; n, \ldots, n)$
$3A$	$(0; a_1 m_1, a_2 m_2, a_3 m_3, n, \ldots, n)$	$(0; n, \ldots, n)$

Table : Signatures for Γ_A and Γ_N

Non self-normalizing case.

Theorem

If N is not self normalizing in A then N contains a copy of $\mathbb{Z}_n \times \mathbb{Z}_n$ and there are three possibilities given in the table below.

In this table $m = |\Gamma_A/\Gamma_N|$.

$\mathcal{S}(\Gamma_A)$	$\mathcal{S}(\Gamma_N)$	m	$\|C\|$	genus	Group
$(2,3,2n)$	$(2,n,2n)$	3	$n \geq 5$	$\frac{(n-1)(n-2)}{2}$	$\Sigma_3 \times (\mathbb{Z}_n \ltimes \mathbb{Z}_n)$
$(2,2,2,n)$	$(2,2,n,n)$	2	$n \geq 3$	$(n-1)^2$	$V_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$
$(2,4,2n)$	$(2,2n,2n)$	2	$n \geq 3$	$(n-1)^2$	$D_4 \ltimes (\mathbb{Z}_n \ltimes \mathbb{Z}_n)$

Examples and results

cyclic n-gonal groups from constrained pairs

We think these are the only possibilities.

$\mathcal{S}(\Gamma_A)$	$\mathcal{S}(\Gamma_N)$	$\mathcal{S}(K)$	$\lvert\Gamma_A/\Gamma_N\rvert$	$\lvert C\rvert$	genus	Group
$(2,4,5)$	$(4,4,5)$	$(4,4)$	6	5	4	Σ_6
$(2,3,7)$	$(3,3,7)$	$(3,3)$	8	7	3	$PSL_2(7)$

cyclic n-gonal groups from tight pairs

- Here are some examples, admittedly calculated by hand.
- Some tight pairs admit only a finite number of n-gonal surfaces, see lines 1 and 2
- The authors are currently working out a uniform method to deal with the family of cases arising from a single tight pair.
- In this table $m = |\Gamma_A/\Gamma_N|$.

$\mathcal{S}(\Gamma_A)$	$\mathcal{S}(\Gamma_N)$	m	$\|C\|$	genus	Group
$(2,3,4n)$	$(2,2,3,n)$	4	$n=2$	2	$GL(2,3)$
$(2,3,3n)$	$(3,n,3n)$	4	$n=4$	3	$SL(2,3)/CD$
$(2,3,2n)$	$(2,n,2n)$	3	$n\geq 5$	$\frac{(n-1)(n-2)}{2}$	$\Sigma_3 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$
$(2,2,2,n)$	$(2,2,n,n)$	2	$n\geq 3$	$(n-1)^2$	$V_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$
$(2,4,2n)$	$(2,2n,2n)$	2	$n\geq 3$	$(n-1)^2$	$D_4 \ltimes (\mathbb{Z}_n \ltimes \mathbb{Z}_n)$

References

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2 S. Allen Broughton, talk at UNED *http://www.rose-hulman.edu/˜brought/Epubs/UNED/UNED.html*.

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4 Bujalance, E., Gamboa, J. M., Gromadzki, G. *The full automorphism groups of hyperelliptic Riemann surfaces.* Manuscripta Math. 79 (1993), no. 3-4, 267-282.

5 D. Singerman, *Finitely Maximal Fuchsian Groups*, J. London Math. Society(2) 6, (1972),17-32

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Questions

- Any questions?
- The slides of this talk will be available at *http://www.rose-hulman.edu/˜brought/Epubs/Oslo/Oslo.html*