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Quasi-platonic actions of some simple groups on Riemann surfaces and their dessins d'enfant

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Preliminary report

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Overview

- Quasi-platonic surfaces and actions.
- Dessins of QP actions and Galois action of $\mathrm{Gal}(\overline{\mathbb{Q}})$.
- Galois action on QP actions and the dessins of some simple groups.

Conformal actions

The finite group G *acts conformally* on the closed, orientable Riemann surface S if there is a monomorphism:

$$\epsilon : G \to \mathrm{Aut}(S),$$

the conformal automorphism group of S.

Example

Dihedral action on the sphere:

- $S = P^1(\mathbb{C})$,
- $G = D_n = \langle a, b : a^2 = b^n = 1, aba = b^{-1} \rangle$,
- G-action given by $a : z \to 1/z$ and $b : z \to \omega z$, $\omega = exp(\frac{2\pi i}{n})$.

Quasi-platonic actions - 1

Quasi-platonic actions satisfy:

- The quotient surface has genus zero: $S/G \simeq P^1(\mathbb{C})$.
- The quotient map $\beta : S \to S/G$ is a meromorphic function:

$$\beta : S \to P^1(\mathbb{C})$$

  which is ramified over at most three points, say $\{0, 1, \infty\}$.
- The map β is called a *regular Belyi function*, and S is called a *regular quasi-platonic surface*.

Example

Belyi function for the dihedral action:

$$\beta(z) = \frac{z^n + z^{-n} + 2}{4}.$$

Quasi-platonic actions - 2

- For all $w \in \beta^{-1}(0)$, $w \in \beta^{-1}(1)$, $w \in \beta^{-1}(\infty)$ the local degree of β at w has a common order l, m, n respectively.
- The stabilizer G_w at $w \in \beta^{-1}(0)$, $w \in \beta^{-1}(1)$, $w \in \beta^{-1}(\infty)$ is cyclic of order l, m, n respectively.
- If S has genus σ then Riemann-Hurwitz equation becomes:

$$\frac{2\sigma - 2}{|G|} = 1 - \frac{1}{l} - \frac{1}{m} - \frac{1}{n}.$$

- The triple (l, m, n) is called the *signature* of the action $\epsilon : G \to \mathrm{Aut}(S)$.

Covering by triangle groups - 1

- An (l, m, n) triangle group is a Fuchsian group with presentation

$$T_{l,m,n} = \langle A, B, C | A^l = B^m = C^n = ABC = 1 \rangle$$

- A, B, C are clockwise hyperbolic rotations through angles of $\frac{2\pi}{l}, \frac{2\pi}{m}, \frac{2\pi}{n}$ respectively, at the vertices of a hyperbolic triangle with angles $\frac{2\pi}{l}, \frac{2\pi}{m}, \frac{2\pi}{n}$.
- We look at a spherical icosahedral picture in the next section.

Covering by triangle groups - 2

Given a quasi-platonic action of G on S, of genus 2 or greater, there is a triangle group Δ, containing a torsion free Fuchsian group Π, such that:

- $\Pi \trianglelefteq \Delta$, and $S \simeq \mathbb{H}/\Pi$ and
- G acts on $S \simeq \mathbb{H}/\Pi$ via an epimorphism

$$\Pi \hookrightarrow \Delta \overset{\eta}{\twoheadrightarrow} G \tag{1}$$

  such that $\overline{\eta} : \Delta/\Pi \leftrightarrow G$ is the inverse of $\epsilon : G \to \mathrm{Aut}(S)$, upon identifying $S \simeq \mathbb{H}/\Pi$.
- η is called a surface kernel epimorphism.
- As we vary η and hence ϵ we get various surfaces $S \simeq \mathbb{H}/\Pi$ with QP G-action. We can transfer our efforts to the structure of G.

Covering by triangle groups - 3

- Given $\Delta = T_{l,m,n} = \langle A, B, C\rangle$ $\eta : \Delta \to G$, let

$$a = \eta(A), b = \eta(B), c = \eta(C).$$

- The triple (a, b, c) is called a *generating* (l, m, n)*-triple* of G.
- The generating triple satisfies:

$$G = \langle a, b, c\rangle \tag{2}$$

$$o(a) = l,\ o(b) = m,\ o(c) = n \tag{3}$$

$$abc = 1 \tag{4}$$

Equivalent epimorphisms and actions

- The surface-kernel epimorphisms of $T_{l,m,n}$, and hence quasi-platonic G-actions, are in $1-1$ correspondence to the generating (l,m,n)-triples of G.
- Two G-actions $\epsilon_1, \epsilon_2 : G \to \mathrm{Aut}(S)$ are called *algebraically equivalent* if $\epsilon_2 = \epsilon_1 \circ \omega$ for some $\omega \in \mathrm{Aut}(G)$, and the associated generating triples satisfy

$$(a_2, b_2, c_2) = (\omega(a_1), \omega(b_1), \omega(c_1)).$$

- We call such triples *algebraically equivalent* and $\mathrm{Aut}(G)$ orbits of triples (almost) classify surfaces with QP G-action via the canonical covering construction (1).

Algebraic classes of actions - 1

- To construct better partitions of the unwieldly set of all generating triples, we use an "approximate automorphism group" L satisfying

$$A = \mathrm{Aut}(G) \supseteq L \supseteq \mathrm{Inn}(G) = K.$$

- Set $g^L = \{\omega(g) : \omega \in L\}$ and define

$$L_G(a,b,c) = \{(x,y,z) : x \in a^L, y \in b^L, x \in c^L) : xyz = 1\}$$
$$L_G^\circ(a,b,c) = \{(x,y,z) \in L_G(a,b,c) : G = \langle x,y,z\rangle\}$$

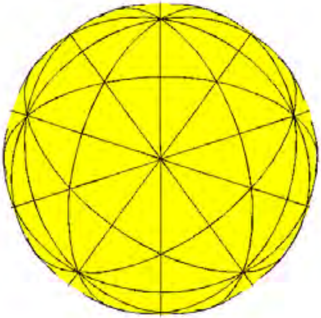
- If $L = K$ then g^L is a conjugacy class.

Algebraic classes of actions - 2

- The two sets on the previous slide can often be easily enumerated. Each $L_G^\circ(a,b,c)$ is a union a small number of L classes, upon each of which L acts freely.
- The sets $A_G^\circ(a,b,c)$ are useful in classifying QP-actions,
- the sets $K_G^\circ(a,b,c)$ work well with the action of $\mathrm{Gal}(\overline{\mathbb{Q}})$,
- and intermediate $L_G^\circ(a,b,c)$ are often useful for computation.
- To make these sets good approximations of G and $\mathrm{Aut}(G)$ we are going to assume that G has small center as in the case of a simple group or a cover of a simple group.

Example: Icosahedral triangular tiling and dessins

- Describe *G*-classes and quotient.
- Describe generating triple.
- Show the three dessins: dodecahedral, icosahedral, and rhombic.



The standard regular dessin d'enfant induced by G

- The dessin d'enfant (dessin) of a Belyi function is

$$\mathcal{D} = \beta^{-1}([0, 1]).$$

- G acts simply transitively on the edges of $\mathcal{D}$.
- $\mathcal{D}$ is a bipartite graph in S whose complement is a union of congruent dihedrally symmetric polygons.
- The tiling or map on S defined by the dessin is the origin of terminology quasi-platonic surfaces and actions.

Belyi's Theorem and Galois action

- Belyi's Theorem: **A quasi-platonic surface is defined over a number field.**
- An element $\theta \in \mathrm{Gal}(\overline{\mathbb{Q}})$ determines a new regular QP surface, and hence a new dessin, S^θ by acting on the coefficients of a defining equation of S.
- So, there is an action of $\mathrm{Gal}(\overline{\mathbb{Q}})$ on regular quasi-platonic surfaces and their dessins. The action is faithful on the set of all regular dessins.
- Defining equations are hard to find so we look at the action of $\mathrm{Gal}(\overline{\mathbb{Q}})$ on generating triples.

Galois action - 1

- Applying θ (extended to $\mathbb{C}$) pointwise, induces a bijection

$$\theta_S : S \to S^\theta$$

which in turn transfers a given G action $\epsilon : G \to \mathrm{Aut}(S)$, to an action on S^θ via θ_S:

$$\epsilon^\theta(g) = \theta_S \circ \epsilon(g) \circ \theta_S^{-1}.$$

- The quotient maps $\beta : S \to S/G = P^1$ and $\beta^\theta : S^\theta \to S^\theta/G = P^1$ satisfy:

$$\begin{array}{ccc} S & \xrightarrow{\theta_S} & S^\theta \\ \downarrow \beta & & \downarrow \beta^\theta \\ P^1(\mathbb{C}) & \xrightarrow{\theta_{P^1}} & P^1(\mathbb{C}) \end{array}$$

Galois action - 2

- The ϵ^θ action is quasi-platonic and has the same signature as the ϵ action.
- The bijection $S \xrightarrow{\theta_S} S^\theta$ maps $\beta^{-1}(0, 1, \infty)$ = vertices of tiling on S to $(\beta^\theta)^{-1}(0, 1, \infty)$ = vertices of tiling on S^θ.
- However, θ_S does not map the edges and faces of the dessin on S to those on S^θ.

Galois action - 3

- If $g \in G$ fixes $w \in \beta^{-1}(0)$ by $\epsilon(g)$ then g acts at w as a local rotation by ζ an l'th root of unity. Call ζ the rotation number.
- But g also fixes $\theta(w) \in (\beta^{\theta})^{-1}(0)$ via $\epsilon^{\theta}(g)$ and g acts at $\theta(w)$ as a local rotation by $\theta(\zeta)$.
- Similar remarks apply to $w \in \beta^{-1}(1)$ and $w \in \beta^{-1}(\infty)$.
- Let $N = \mathrm{lcm}(l, m, n)$ then θ acts on the Nth roots of unity, and hence the rotation numbers, by $\zeta \to \zeta^s$ for an s relatively prime to N.

Galois action on generating triples- 1

Theorem (Branch cycle argument)

Let notation be as above and let (a, b, c) be a generating (l, m, n)-triple for the $\epsilon(G)$ action on S, and select t so that $st = 1 \; mod \; N$. Then there are $x, y, z \in G$ such that a generating triple for the $\epsilon^\theta(G)$ action on S^θ is

$$(a', b', c') = (xa^t x^{-1}, yb^t y^{-1}, zc^t z^{-1}).$$

Theorem (Gonzales Diez & Jaikin-Zapirain)

The absolute Galois group acts faithfully on regular dessins.

Galois action on generating triples - 2

Remark

- *The Galois action on generating triples maps the set* $K_G^\circ(a, b, c)$ *to* $K_G^\circ(a^t, b^t, c^t)$. *If* $K_G^\circ(a, b, c)$ *consists of several* K*-orbits (called companion classes or actions) then the Galois action is ambiguous.*
- *In addition we need to resolve the mapping of equivalence classes for the inclusion* $K_G^\circ(a^t, b^t, c^t) \to A_G^\circ(a^t, b^t, c^t)$.

Splitting the Galois action

Construct a "cyclotomic" splitting of $\mathrm{Gal}(\overline{\mathbb{Q}})$ as follows:

- Let

$$\mathbb{Q}_n = \mathbb{Q}[\exp(2\pi i/n)], \mathbb{Q}_\infty = \bigcup_n \mathbb{Q}_n$$

$$\mathcal{K}_n = \mathcal{K}_n(\mathbb{Q}) = \{\theta \in \mathrm{Gal}(\overline{\mathbb{Q}}) : \theta|\mathbb{Q}_n = id\} \lhd \mathrm{Gal}(\overline{\mathbb{Q}})$$

$$\mathcal{Q}_n = \mathcal{Q}_n(\mathbb{Q}) = \mathrm{Gal}(\overline{\mathbb{Q}})/\mathcal{K}_n(\mathbb{Q})$$

- and

$$\mathcal{K} = \mathcal{K}_\infty(\mathbb{Q}), \mathcal{Q} = \mathcal{Q}_\infty(\mathbb{Q}).$$

- We have exact sequences:

$$\mathcal{K}_n \hookrightarrow \mathrm{Gal}(\overline{\mathbb{Q}}) \twoheadrightarrow \mathcal{Q}_n$$

$$\mathcal{K} \hookrightarrow \mathrm{Gal}(\overline{\mathbb{Q}}) \twoheadrightarrow \mathcal{Q}$$

- Since $\mathcal{Q}_n \backsimeq \mathrm{Gal}(\mathbb{Q}_n)$ is cyclic, the first sequence is split.

Splitting the Galois action on triples

- The “splitting” on the previous slide descends to a splitting of the action on triples.
- The elements of $\mathcal{K}_\infty$ acts trivially on rotation numbers at the fixed points of G.
- Therefore, in the branch cycle action, elements of $\mathcal{K}_\infty$ yeild

$$(a, b, c) \to (xax^{-1}, yby^{-1}, zcz^{-1}).$$

- We make the distinction because the action of $\mathcal{Q}_\infty$ is easier to determine, whereas the action of $\mathcal{K}_\infty$ requires a defining field of the surface to get anywhere.
- Elements of $\mathcal{Q}_\infty$ can induce transforms as above if a^t, b^t, c^t are conjugate to a, b, c respectively.

$PSL_2(q)$ Examples - 1

- Set $q = p^e$, $G = PsL_2(q)$, $L = PGL_2(q)$.
- Call L-equivalent actions geometrically equivalent.
    - For $q = p > 2$, $K < L = A$, and $|L/K| = 2$.
    - For $q = p^e$, $K < L < A$ and $A/L \simeq \mathrm{Gal}(\mathbb{F}_q)$.
    - $p = 2$ is a separate case.

- Tables of Galois orbits for $PSL_2(q), q = 7, 8$
  `PSL-QPGalActTables.pdf`
- $PSL_2(47)$: there are 2431 $(23, 23, 23)$ actions consisting of 121 Galois orbits each of size 11.
- $PSL_2(32)$: there are 2940 geometric classes of $(31, 31, 31)$ triples in 196 Galois orbits of size 15 each. Each Galois orbit provides $3 = 15/5$ inequivalent actions. Note: $|\mathrm{Gal}(\mathbb{F}_{32})| = 5$.

$PSL_2(q)$ Examples - 2

Theorem

For $G = PSL_2(q)$ the action of $\mathcal{K}_\infty$ is trivial.

Proof Sketch

- Use Macbeath's results on generating triples to show that $L_G^\circ(a, b, c)$ has one or two L orbits.
- If $L_G^\circ(a, b, c)$ is a single L-orbit, then $\mathcal{K}_\infty$ acts trivially
- Lift triples (a, b, c) to covering triples $(\widetilde{a}, \widetilde{b}, \widetilde{c})$ in the Schur cover $\widetilde{G} = SL_2(q)$. Classify with corresponding triple of traces (α, β, γ).
- Lifting to the Schur cover separates L-orbits and the Galois action is no longer ambiguous in $\widetilde{G}$.
- In the case of composite q we have to work further with $A/L \simeq \mathrm{Gal}(\mathbb{F}_q)$

Simple groups - 1

- Let G be a simple group and $\widetilde{G}$ its Schur cover.
- Work with $K^\circ_G(a, b, c)$ triple sets:
  - a, b, c range over representatives of conjugacy classes
  - convenient for MAGMA computations
  - equivalence by $A = \mathrm{Aut}(G)$
  - or L = geometric automorphisms
- Non-trivial $Z(\widetilde{G})$ can produce companion orbits in $K^\circ_G(a, b, c)$.
- The action of $\mathrm{Aut}(G)$ on the classes of powers $(a^t)^G$, etc., (i.e., $N_A(\langle a \rangle)$, etc.) needs to be worked out.

Simple groups - 2

Not much can be said at this point. Here are some examples.

- Table for alternating groups
  `PSL-QPSimpleDataTables.pdf`
- Table for linear groups
  same file `PSL-QPSimpleDataTables.pdf`

References

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- G. Gonzales Diez & A. Jaikin-Zapirain, The absolute Galois group acts faithfully on regular dessins and on Beauville surfaces, preprint (2013).
- P.L. Clark and J. Voight, Algebraic Curves Uniformized by Congruence Subgroups of Triangle Groups, preprint (2015).

Any Questions?

Sample hyperbolic quasi-platonic actions of simple groups - tables

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Group Pair (G,A)	$\left\|Z(\widetilde{G})\right\|$	Signature (l,m,n)	K_G° tag (a^G,b^G,c^G)	Genus	Actions
(Alt_7,Sym_7)	6	$(3,3,5)$	(K_4,K_4,K_6)	169	1
(Alt_7,Sym_7)	6	$(3,3,7)$	(K_3,K_4,K_8)	241	1
(Alt_7,Sym_7)	6	$(4,4,4)$	(K_5,K_5,K_5)	316	12
(Alt_7,Sym_7)	6	$(4,4,5)$	(K_5,K_5,K_6)	379	11
(Alt_7,Sym_7)	6	$(4,5,6)$	(K_5,K_6,K_7)	484	4
(Alt_7,Sym_7)	6	$(5,5,5)$	(K_6,K_6,K_6)	505	1
(Alt_7,Sym_7)	6	$(5,5,7)$	(K_6,K_6,K_7)	577	12
(Alt_7,Sym_7)	6	$(5,6,7)$	(K_6,K_7,K_8)	619	7
(Alt_7,Sym_7)	6	$(6,6,6)$	(K_7,K_7,K_7)	631	1
(Alt_7,Sym_7)	6	$(7,7,7)$	(K_8,K_8,K_8)	721	23
(Alt_8,Sym_8)	2	$(4,4,15)$	(K_6,K_6,K_{13})	4369	3
(Alt_8,Sym_8)	2	$(4,4,15)$	(K_6,K_7,K_{13})	4369	12
(Alt_8,Sym_8)	2	$(4,4,15)$	(K_6,K_6,K_{13})	4369	15
(Alt_8,Sym_8)	2	$(7,7,7)$	(K_{11},K_{11},K_{11})	5761	177
(Alt_8,Sym_8)	2	$(15,15,15)$	(K_{13},K_{13},K_{13})	8065	20

Sample actions for alternating groups
Braid automorphisms not accounted for.

Group Pair (G, L)	$\lvert A/L \rvert$	$\left\lvert Z(\widetilde{G})\right\rvert$	Signature (l, m, n)	K_G° tag (a^G, b^G, c^G)	Genus	Actions
$(PSL_3(3), PGL_3(3))$	1	1	$(2, 3, 13)$	(K_2, K_4, K_9)	253	1
$(PSL_3(3), PGL_3(3))$	1	1	$(3, 3, 13)$	(K_3, K_4, K_{10})	721	1
$(PSL_3(3), PGL_3(3))$	1	1	$(3, 3, 13)$	(K_3, K_4, K_9)	721	3
$(PSL_3(3), PGL_3(3))$	1	1	$(13, 13, 13)$	(K_9, K_9, K_9)	2161	5
$(PSL_3(3), PGL_3(3))$	1	1	$(13, 13, 13)$	(K_9, K_9, K_{10})	2161	1
$(PSL_3(3), PGL_3(3))$	1	1	$(13, 13, 13)$	(K_9, K_9, K_{12})	2161	3
$(PSL_3(4), PGL_3(4))$	2	48	$(5, 5, 5)$	(K_7, K_7, K_7)	4033	44
$(PSL_3(4), PGL_3(4))$	2	48	$(5, 5, 5)$	(K_7, K_7, K_8)	4033	48
$(PSL_3(4), PGL_3(4))$	2	48	$(7, 7, 7)$	(K_9, K_9, K_9)	5761	20
$(PSL_3(4), PGL_3(4))$	2	48	$(7, 7, 7)$	(K_9, K_9, K_{10})	5761	18
$(PSL_3(5), PGL_3(5))$	1	1	$(2, 3, 13)$	(K_2, K_3, K_{21})	25001	44
$(PSL_3(5), PGL_3(5)$	1	1	$(3, 3, 13)$	(K_3, K_3, K_{21})	56001	19
$(PSL_3(5), PGL_3(5))$	1	1	$(5, 5, 5)$	(K_8, K_8, K_8)	74401	24
$(PSL_3(5), PGL_3(5))$	1	1	$(31, 31, 31)$	(K_{21}, K_{21}, K_{21})	168001	19
$(PSL_3(5), PGL_3(5))$	1	1	$(31, 31, 31)$	(K_{21}, K_{21}, K_{22})	168001	11
$(PSL_3(5), PGL_3(5))$	1	1	$(31, 31, 31)$	(K_{21}, K_{21}, K_{28})	168001	15

Sample actions for linear groups
Braid automorphisms not accounted for.

Galois action on hyperbolic quasi-platonic $PSL_2(q)$-actions - tables

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Signature (l, m, n)	Covering trace triple (α, β, γ)	Genus	Galois Orbit Sizes
$(2, 3, 7)$	$(0, 1, 2)$	3	$\{1\}$
$(2, 4, 7)$	$(0, 3, 2)$	10	$\{1\}$
$(2, 7, 7)$	$(0, 2, 2)$	19	$\{1\}$
$(3, 3, 4)$	$(1, 1, 3), (1, 1, 4)$	8	$\{2\}$
$(3, 3, 7)$	$(1, 1, -2)$	17	$\{1\}$
$(3, 4, 4)$	$(-1, 3, 3)$	15	$\{1\}$
$(3, 4, 7)$	$(1, 3, 2), (1, 3, -2)$	24	$\{1\}$
$(3, 7, 7)$	$(1, 2, 2), (-1, 2, 2)$	33	$\{1, 1\}$
$(4, 4, 4)$	$(3, 3, 3), (3, 3, 4)$	22	$\{2\}$
$(4, 4, 7)$	$(3, 3, -2)$	31	$\{1\}$
$(4, 7, 7)$	$(3, 2, 2), (-3, 2, 2)$	40	$\{2\}$
$(7, 7, 7)$	$(2, 2, -2)$	49	$\{1\}$

Galois action on hyperbolic $PSL_2(7)$ actions

Signature (l,m,n)	Covering trace triple (α,β,γ)	Genus	Galois Orbit Sizes
$(2,3,7)$	$(0,1,w^3)$	7	$\{1\}$
$(2,3,9)$	$(0,1,w^2)$	15	$\{1\}$
$(2,7,7)$	$(0,w^3,w^5),(0,w^3,w^6)$	55	$\{1,1\}$
$(2,7,9)$	$(0,w^3,w),(0,w^3,w^2),(0,w^3,w^4)$	63	$\{3\}$
$(2,9,9)$	$(0,w,w^2),(0,w,w^4)$	71	$\{1,1\}$
$(3,3,7)$	$(1,1,w^3)$	41	$\{1\}$
$(3,3,9)$	$(1,1,w)$	57	$\{1\}$
$(3,7,7)$	$(1,w^3,w^3),(1,w^3,w^5),(1,w^3,w^6)$	97	$\{3\}$
$(3,7,9)$	$(1,w^3,w),(1,w^3,w^2),(1,w^3,w^4)$	105	$\{3\}$
$(3,9,9)$	$(1,w,w),(1,w^2,w^2),(1,w^4,w^4)$	113	$\{1\}$
$(7,7,7)$	$(w^3,w^3,w^5),(w^3,w^3,w^6),$ $(w^3,w^5,w^3),(w^3,w^6,w^3)$	145	$\{1,1,1,1\}$
$(7,7,9)$	$(w^a,w^b,w),a,b=3,5,6$	153	$\{3,3,3\}$
$(7,9,9)$	$(w^3,w^a,w^b),a,b=1,2,4$	161	$\{3,3,3\}$
$(9,9,9)$	$(w,w,w),(w,w,w^4),$ (w,w^2,w^2) $(w,w^2,w^4),(w,w^4,w),$ (w,w^4,w^2)	169	$\{1,1,1,1,1,1\}$

Galois action on hyperbolic $PSL_2(8)$ actions