Balance in generalized Tate cohomology

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BALANCE IN GENERALIZED TATE COHOMOLOGY#

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We consider two preenveloping classes of left $R$-modules $\mathcal{F}, \mathcal{C}$ such that $\text{Inj} \subset \mathcal{F} \subset \mathcal{C}$, and a left $R$-module $N$. For any left $R$-module $M$ and $n \geq 1$ we define the relative extension modules $\widehat{\text{Ext}}_{\mathcal{F}, \mathcal{C}}^n(M, N)$ and prove the existence of an exact sequence connecting these modules and the modules $\text{Ext}_{\mathcal{F}}^n(M, N)$ and $\text{Ext}_{\mathcal{C}}^n(M, N)$. We show that there is a long exact sequence of $\widehat{\text{Ext}}_{\mathcal{F}, \mathcal{C}}^n(M, -)$ associated with a $\text{Hom}( -, \mathcal{C})$ exact sequence $0 \to N' \to N \to N'' \to 0$ and a long exact sequence of $\text{Ext}_{\mathcal{F}}^n(-, N)$ associated with a $\text{Hom}(-, N)$ exact sequence $0 \to M' \to M \to M'' \to 0$. Using these properties we prove that for two complete hereditary cotorsion theories $(\mathcal{C}, \mathcal{L}), (\mathcal{L}, \mathcal{C})$ we have $\widehat{\text{Ext}}_{\mathcal{C}, \mathcal{L}}^n(M, N) \simeq \text{Ext}_{\mathcal{L}}^n(M, N)$ for any left $R$ modules $M, N$ and for any $n \geq 1$, where $\widehat{\text{Ext}}_{\mathcal{C}, \mathcal{L}}^n(M, N)$ are the generalized Tate cohomology modules (see Section 1 for the definition). So in this case we have an occurrence of balance, i.e. the generalized Tate cohomology can be computed either using a left $\mathcal{C}$-resolution and a projective resolution of $M$ or using a right $\mathcal{C}$-resolution and an injective resolution of $N$.

Key Words: Balance, Cotorsion theories, Generalized Tate cohomology.

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1. INTRODUCTION

Let $\mathcal{D}, \mathcal{C}$ be two precovering classes (see Section 2 below for the definition) of left $R$ modules such that $\text{Proj} \subset \mathcal{D} \subset \mathcal{C}$ where $\text{Proj}$ is the class of projective left $R$-modules. Let $M$ be a left $R$-module. Let $\mathcal{P}$ be a deleted $\mathcal{D}$-resolution of $M$, $\mathcal{C}$ a deleted $\mathcal{C}$-resolution of $M$, $u : \mathcal{P} \to \mathcal{C}$ a map of complexes induced by $\text{Id}_M$ and $M(u)$ be the associated mapping cone. We defined (Iacob) the generalized Tate cohomology modules $\widehat{\text{Ext}}_{\mathcal{D}, \mathcal{C}}^n(M, N)$ by the equality $\widehat{\text{Ext}}_{\mathcal{D}, \mathcal{C}}^n(M, N) = H^{n+1}(\text{Hom}(M(u), N))$ for any $n \geq 1$ and any left $R$-module $N$. In Iacob we proved that $\widehat{\text{Ext}}_{\mathcal{D}, \mathcal{C}}^n(M, N)$ is well defined and that when $\mathcal{C}$ is the class of Gorenstein projective modules, $\mathcal{D}$ is the class of projective left $R$-modules over a left noetherian

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ring \( R \) and \( \text{Gor proj dim} M < \infty \), the modules \( \widehat{\text{Ext}}_n^R(M, N) \) are the usual Tate cohomology modules for any \( n \geq 1 \) (see Avramov and Martsinkovsky, 2002, Section 5 for the definition of the Tate cohomology). In Iacob we showed the existence of a long exact sequence connecting these modules and the modules \( \text{Ext}_n^R(M, N) \) and \( \text{Ext}_n^R(M, N) \)

\[
0 \to \text{Ext}_n^R(M, N) \to \text{Ext}_n^R(M, N) \to \widehat{\text{Ext}}_n^R(M, N) \to \cdots \tag{1}
\]

analogous to that in Theorem 7.1 of Avramov and Martsinkovsky (2002).

We consider now two preenveloping classes \( \mathcal{J}, \mathcal{E} \) such that \( \text{Inj} \subset \mathcal{J} \subset \mathcal{E} \) where \( \text{Inj} \) is the class of injective left \( R \)-modules. Let \( N \) be a left \( R \)-module. For any \( n \geq 1 \) and any \( _R M \) we define the modules \( \widehat{\text{Ext}}_n^{\mathcal{E}}(M, N) \) and we show that we get an exact sequence

\[
0 \to \text{Ext}_n^{\mathcal{E}}(M, N) \to \text{Ext}_n^{\mathcal{E}}(M, N) \to \widehat{\text{Ext}}_n^{\mathcal{E}}(M, N) \to \cdots \tag{2}
\]

The classical instance of balance says that for any ring \( R \) and modules \( M \) and \( N \), the modules \( \text{Ext}_n^R(M, N) \) (for \( n \geq 0 \)) can be computed using either a projective resolution of \( M \) or an injective resolution of \( N \). But there are many other instances of such balance in relative homological algebra (see Enochs and Jenda, 2000, Section 8.2 and Holm, 2004 for examples). The classical instance above and other important instances occur when we have a pair \((\mathcal{E}, \mathcal{L}), (\mathcal{L}, \mathcal{E})\) of complete hereditary cotorsion theories on the category of left \( R \) modules (see Section 3). In this situation the same right derived functor of \( \text{Hom}_R(\_ , \_) \) for a given \( M \) and \( N \) can be computed using a left \( \mathcal{E} \)-resolution of \( M \) or a right \( \mathcal{E} \)-resolution of \( N \) (see Section 2 for definitions).

So now if \((\mathcal{P}, \mathcal{M}),(\mathcal{M}, \mathcal{J}),(\mathcal{E}, \mathcal{L}),(\mathcal{L}, \mathcal{E})\) are two such pairs of complete hereditary cotorsion theories then \( \text{Hom}_R(\_ , \_) \) is right balanced by \( \mathcal{P} \times \mathcal{J} \) and right balanced by \( \mathcal{E} \times \mathcal{E} \) (see Enochs and Jenda, 2000, Definition 8.2.13). So in this case we have

\[
\text{Ext}_n^\mathcal{P}(\_ , \_ ) \simeq \text{Ext}_n^\mathcal{M}(\_ , \_ ) \text{ and } \text{Ext}_n^\mathcal{E}(\_ , \_ ) \simeq \text{Ext}_n^\mathcal{E}(\_ , \_ ) \tag{3}
\]

(see Section 2, Definition 2).

Now suppose that \( \mathcal{P} \subset \mathcal{E} \). Then (1) gives the exact sequence

\[
0 \to \text{Ext}_n^\mathcal{E}(M, N) \to \text{Ext}_n^\mathcal{E}(M, N) \to \widehat{\text{Ext}}_n^\mathcal{E}(M, N) \to \cdots .
\]

But \( \mathcal{P} \subset \mathcal{E} \) implies \( \mathcal{L} \subset \mathcal{M} \) and \( \mathcal{J} \subset \mathcal{E} \). So we get the exact sequence (2):

\[
0 \to \text{Ext}_n^\mathcal{E}(M, N) \to \text{Ext}_n^\mathcal{E}(M, N) \to \widehat{\text{Ext}}_n^\mathcal{E}(M, N) \to \cdots .
\]

So using the isomorphisms (3) above and comparing (1) and (2) it is natural to ask if \( \widehat{\text{Ext}}_n^{\mathcal{E}, \mathcal{P}}(M, N) \simeq \widehat{\text{Ext}}_n^{\mathcal{E}, \mathcal{J}}(M, N) \). We show that this is the case (Theorem 1) when \( \mathcal{P} \) is the class of projective modules (so \( \mathcal{J} \) is the class of injective modules). So in this case the generalized Tate cohomology can be computed two ways: using a left \( \mathcal{E} \)-resolution and a projective resolution of \( M \) or using a right \( \mathcal{E} \)-resolution and
an injective resolution of $N$. The proof uses the following properties of the functors $\overline{\operatorname{Ext}}_{\mathcal{E}, \mathcal{F}}(-, -)$:

- If $0 \to M' \to M \to M'' \to 0$ is a $\operatorname{Hom}(\mathcal{E}, -)$ exact sequence of left $R$-modules then for any $_RM$ there is an exact sequence $0 \to \overline{\operatorname{Ext}}^{1}_{\mathcal{E}, \mathcal{F}}(M'', N) \to \overline{\operatorname{Ext}}^{1}_{\mathcal{E}, \mathcal{F}}(M, N) \to \overline{\operatorname{Ext}}^{1}_{\mathcal{E}, \mathcal{F}}(M', N) \to \overline{\operatorname{Ext}}^{2}_{\mathcal{E}, \mathcal{F}}(M'', N) \to \cdots$ (Proposition 1)
- If $0 \to N' \to N \to N'' \to 0$ is a $\operatorname{Hom}(\mathcal{E}, -)$ exact sequence of left $R$-modules then for any $_RM$ we have an exact sequence: $0 \to \overline{\operatorname{Ext}}^{1}_{\mathcal{E}, \mathcal{F}}(M, N') \to \overline{\operatorname{Ext}}^{1}_{\mathcal{E}, \mathcal{F}}(M, N) \to \overline{\operatorname{Ext}}^{1}_{\mathcal{E}, \mathcal{F}}(M, N'') \to \overline{\operatorname{Ext}}^{2}_{\mathcal{E}, \mathcal{F}}(M, N') \to \cdots$ (Proposition 2)

as well as the dual results for $\overline{\operatorname{Ext}}^{*}_{\mathcal{E}, \mathcal{F}}(-, -)$ (Propositions 3, 4).

2. PRELIMINARIES

**Definition 1** (Enochs and Jenda, 2000, p. 167). Let $\mathcal{C}$ be a class of left $R$-modules. For a left $R$-module $M$ a morphism $\varphi : C \to M$ where $C \in \mathcal{C}$ is a $\mathcal{C}$-precover of $M$ if $\operatorname{Hom}(C', C) \to \operatorname{Hom}(C', M) \to 0$ is exact for any $C' \in \mathcal{C}$. $\mathcal{C}$ is said to be precovering if any left $R$-module $M$ has a $\mathcal{C}$ precover.

By a left $\mathcal{C}$-resolution of $M$ we mean a $\operatorname{Hom}(\mathcal{C}, -)$ exact complex $\mathcal{C} : \cdots \to C_{1} \to C_{0} \to M \to 0$ (not necessarily exact) with each $C_{i} \in \mathcal{C}$.

If $\mathcal{C}$ contains all the projective left $R$-modules, then any $\mathcal{C}$-precover is a surjection. We note that a complex $\mathcal{C}$ as above is a $\mathcal{C}$-resolution of $M$ if and only if $C_{0} \to M, C_{1} \to \operatorname{Ker}(C_{0} \to M), C_{i} \to \operatorname{Ker}(C_{i-1} \to C_{i-2}), i \geq 2$ are $\mathcal{C}$-precovers. So, if $\mathcal{C}$ contains all the projectives then a left $\mathcal{C}$-resolution of $M$ is an exact complex.

**Definition 2** (Enochs and Jenda, 2000, p. 170). Let $M$ be a left $R$-module that has a left $\mathcal{C}$-resolution $\mathcal{C} : \cdots \to C_{1} \to C_{0} \to M \to 0$. Then $\overline{\operatorname{Ext}}^{n}_{\mathcal{E}}(M, N) = H^{n}(\operatorname{Hom}(\mathcal{C}, N))$ for any left $R$-module $N$ and any $n \geq 0$, where $\mathcal{C}$ is the deleted resolution.

Dually, an $\mathcal{F}$-preenvelope of a left $R$-module $N$ is a morphism $\psi : N \to F$ where $F \in \mathcal{F}$ and $\operatorname{Hom}(F, F') \to \operatorname{Hom}(N, F') \to 0$ is exact for any $F' \in \mathcal{F}$. $\mathcal{F}$ is preenveloping if any $_RN$ has an $\mathcal{F}$-preenvelope.

If $\mathcal{F}$ contains all the injective left $R$-modules then any $\mathcal{F}$-preenvelope is an injection.

A right $\mathcal{F}$-resolution of $N$ is a $\operatorname{Hom}(-, \mathcal{F})$ exact complex (not necessarily exact) $0 \to N \to F^{0} \to F^{1} \to \cdots$ with each $F^{i} \in \mathcal{F}$. If $\mathcal{F}$ contains all the injectives then a right resolution of $N$ is an exact complex.

If $\mathcal{F} : 0 \to N \to F^{0} \to F^{1} \to \cdots$ is a right $\mathcal{F}$ resolution of $N$ then $\overline{\operatorname{Ext}}^{n}_{\mathcal{E}}(M, N) = H^{n}(\operatorname{Hom}(M, F_{n}))$ for any $_RM$, for any $n \geq 0$ (where $F_{n}$ is the deleted resolution).

We show the existence of the exact sequence (2).

Let $\mathcal{C}$ and $\mathcal{F}$ be preenveloping classes that contain all the injective left $R$-modules and such that $\mathcal{F} \subset \mathcal{C}$.

Then any left $R$ module $N$ has an exact $\mathcal{C}$-resolution

$$E : 0 \to N \to E^{0} \to E^{1} \to \cdots$$
and an exact \( J \)-resolution \( I \):

\[
0 \to N \to I^0 \to I^1 \to \cdots
\]

Let \( v : E \to I \) be a chain map induced by \( \text{Id}_N \). Then \( 0 \to I \to M(v) \to E[1] \to 0 \) is an exact sequence of complexes, so the exactness of the mapping cone \( M(v) \) follows.

\[
M(v) : 0 \to N \to N \oplus E^0 \to I^0 \oplus E^1 \to I^1 \oplus E^2 \to \cdots
\]

has the exact subcomplex \( 0 \to N \cong N \to 0 \). Forming the quotient we get an exact complex \( M(v) \) that is the mapping cone of the map of complexes \( v : E \to I \) (where \( E \) and \( I \) are the deleted resolutions). The sequence \( 0 \to I \to M(v) \to E[1] \to 0 \) is split exact in each degree, so for each left \( R \)-module \( M \) the sequence

\[
0 \to \text{Hom}(M, I) \to \text{Hom}(M, M(v)) \to \text{Hom}(M, E[1]) \to 0
\]

is still exact. Therefore we have an associated long exact sequence

\[
\cdots \to H^n(\text{Hom}(M, I)) \to H^n(\text{Hom}(M, M(v))) \to H^n(\text{Hom}(M, E[1]))
\]

\[
\to H^{n+1}(\text{Hom}(M, I)) \to H^{n+1}(\text{Hom}(M, M(v))) \to \cdots
\]

But

\[
M(v) : 0 \to E^0 \to I^0 \oplus E^1 \to I^1 \oplus E^2 \to I^2 \oplus E^3 \to \cdots
\]

(with \( I^0 \oplus E^1 \) in the 0th place) is an exact complex and the functor \( \text{Hom}(M, -) \) is left exact, so

\[
H^{-1}(\text{Hom}(M, M(v))) = H^0(\text{Hom}(M, M(v))) = 0.
\]

Since

\[
H^i(\text{Hom}(M, I)) = \text{Ext}_J^i(M, N) \quad \text{for any } i \geq 0,
\]

\[
H^i(\text{Hom}(M, E[1])) = \text{Ext}_{E}^{i+1}(M, N) \quad \text{for any } i \geq -1 \quad \text{and}
\]

\[
\text{Ext}_{J}^0(M, N) \cong \text{Ext}_{E}^0(M, N) \cong \text{Hom}_{R}(M, N)
\]

we have an exact sequence

\[
0 \to \text{Hom}(M, N) \to \text{Hom}(M, N) \to 0 \to \text{Ext}_{E}^{1}(M, N)
\]

\[
\to \text{Ext}_{J}^{1}(M, N) \to \overline{\text{Ext}}_{E,J}^{-1}(M, N) \to \text{Ext}_{E}^{2}(M, N) \to \cdots
\]

with \( \overline{\text{Ext}}_{E,J}(M, N) \) defined by the equality

\[
\overline{\text{Ext}}_{E,J}(M, N) = H^i(\text{Hom}(M, M(v))), \quad \text{for } i \geq 1.
\]
After factoring out the exact sequence

\[ 0 \to \text{Hom}(M, N) \to \text{Hom}(M, N) \to 0 \]

we obtain the exact sequence

\[ 0 \to \text{Ext}^1_{\mathcal{I}}(M, N) \to \text{Ext}^1_{\mathcal{I}}(M, N) \to \overline{\text{Ext}}^1_{\mathcal{I}}(M, N) \to \cdots. \]

We prove first that \( \overline{\text{Ext}}^1_{\mathcal{I}}(M, N) \) does not depend on the \( \mathcal{I} \) and \( \mathcal{E} \) resolutions that we use.

Let \( \mathcal{I}, \mathcal{I}' \) be two \( \mathcal{I} \) resolutions and let \( \mathcal{E}, \mathcal{E}' \) be two \( \mathcal{E} \) resolutions of \( N \).

\[ \mathcal{I} : 0 \to N \to I^0 \to I^1 \to \cdots, \quad \mathcal{I}' : 0 \to N \to I'^0 \to I'^1 \to \cdots \]

\[ \mathcal{E} : 0 \to N \to E^0 \to E^1 \to \cdots, \quad \mathcal{E}' : 0 \to N \to E'^0 \to E'^1 \to \cdots \]

There exist maps of complexes \( \mathcal{E} \to \mathcal{I} \) and \( \mathcal{E}' \to \mathcal{I}' \), induced by \( \text{Id}_N \).

\[ M(v) : 0 \to N \xrightarrow{\delta} N \oplus E^0 \xrightarrow{\delta_0} I^0 \oplus E^1 \xrightarrow{\delta_1} I^1 \oplus E^2 \to \cdots, \quad \text{with} \]

\[ \delta(x) = (x, -j(x)), \]

\[ \delta_0(x, y) = (i(x) + v_0(y), -g_0(y)), \]

\[ \delta_k(x, y) = (f_{k-1}(x) + v_k(y), -g_k(y)) \quad \text{for } k \geq 1 \]

is the mapping cone of \( v \) and

\[ M(w) : 0 \to N \xrightarrow{\delta'} N \oplus E^0 \xrightarrow{\delta'_0} I'^0 \oplus E'^1 \xrightarrow{\delta'_1} I'^1 \oplus E'^2 \to \cdots, \quad \text{with} \]

\[ \delta'(x) = (x, -j'(x)), \]

\[ \delta'_0(x, y) = (i'(x) + w_0(y), -g'_0(y)), \]

\[ \delta'_k(x, y) = (f'_{k-1}(x) + w_k(y), -g'_k(y)) \quad \text{for } k \geq 1 \]

is the mapping cone of \( w \).

For the maps of complexes \( \mathcal{E} \to \mathcal{I}, \mathcal{E}' \to \mathcal{I}' \) (\( \mathcal{E}, \mathcal{I}, \mathcal{E}', \mathcal{I}' \) are the deleted resolutions) the associated mapping cones are

\[ M(v_\mu) : 0 \to E^0 \xrightarrow{\delta_0} I^0 \oplus E^1 \xrightarrow{\delta_1} I^1 \oplus E^2 \to \cdots \quad \text{with} \]

\[ \overline{\delta}_0(y) = (v_0(y), -g_0(y)) \]

and

\[ M(w_\mu) : 0 \to E^0 \xrightarrow{\delta'_0} I'^0 \oplus E'^1 \xrightarrow{\delta'_1} I'^1 \oplus E'^2 \to \cdots \quad \text{with} \]

\[ \overline{\delta}'_0(y) = (w_0(y), -g'_0(y)). \]

\( M(v_\mu) \) and \( M(w_\mu) \) are exact complexes (since \( M(v) \) and \( M(w) \) are exact).
The exact sequence of complexes $0 \rightarrow I \rightarrow M(v) \rightarrow E[1] \rightarrow 0$ is split exact in each degree, so for each $I^j$ we have an exact sequence

$$0 \rightarrow \text{Hom}(E[1], I^j) \rightarrow \text{Hom}(M(v), I^j) \rightarrow \text{Hom}(I, I^j) \rightarrow 0.$$ 

Since $I^j \in \mathcal{F} \subseteq \mathcal{E}$ both complexes $\text{Hom}(E[1], I^j)$ and $\text{Hom}(I, I^j)$ are exact, so the exactness of $\text{Hom}(M(v), I^j)$ follows.

Let $\overline{N}$ be the complex $0 \rightarrow N \xrightarrow{\sim} N \rightarrow 0$.

The sequence $0 \rightarrow \overline{N} \rightarrow M(v) \rightarrow M(v) \rightarrow 0$ is split exact in each degree, so

$$0 \rightarrow \text{Hom}(M(v_1), I^j) \rightarrow \text{Hom}(M(v), I^j) \rightarrow \text{Hom}(\overline{N}, I^j) \rightarrow 0$$

is exact $\forall j \geq 0$. Since both $\text{Hom}(M(v), I^j)$ and $\text{Hom}(\overline{N}, I^j)$ are exact complexes it follows that

$$\text{Hom}(M(v), I^j) \text{ is exact } \forall j \geq 0 \quad (4)$$

$\text{Id}_N$ induces maps of complexes $h : E. \rightarrow E'_E$ and $k : I. \rightarrow I'_I$. Since $w \circ h$ and $k \circ v : E. \rightarrow I'_I$ are both maps of complexes induced by $\text{Id}_{E'}$, it follows that $k \circ v$ and $w \circ h$ are homotopic. So there exists $s_i \in \text{Hom}(E^i, I^{i-1})$, $i \geq 1$ such that

$$w_0 \circ h_0 - k_0 \circ v_0 = s_1 \circ g_0 \text{ and }$$

$$w_n \circ h_n - k_n \circ v_n = f_{n-1}' \circ s_n + s_{n+1} \circ g_n \text{ for } n \geq 1.$$ 

Then $\gamma : M(v_i) \rightarrow M(v_i)$ defined by

$$\gamma : E^0 \rightarrow E'^0, \gamma = h_0,$$

$$\gamma_n : I^n \oplus E^{n+1} \rightarrow I'^n \oplus E'^{n+1},$$

$$\gamma_n(x, y) = (k_n(x) - s_{n+1}(y), h_{n+1}(y)), \text{ for } n \geq 0,$$

is a map of complexes.

$\text{Id}_N$ also induces maps of complexes $l : I'_I \rightarrow I_I$ and $t : E'_E \rightarrow E_E$. Then $v \circ t \sim l \circ w$. So there exists $\overline{s}_i \in \text{Hom}(E'^i, I'^{i-1})$ so that

$$v_0 \circ t_0 - l_0 \circ w_0 = \overline{s}_1 \circ g_0' \text{ and }$$

$$v_k \circ t_k - l_k \circ w_k = f_{k-1}' \circ \overline{s}_k + \overline{s}_{k+1} \circ g_k' \text{ for } k \geq 1.$$ 

Then $\psi : M(v_i) \rightarrow M(v_i)$ defined by

$$\overline{\psi} : E'^0 \rightarrow E^0, \overline{\psi} = t_0 \text{ and }$$

$$\psi_n : I'^n \oplus E'^{n+1} \rightarrow I^n \oplus E^{n+1},$$

$$\psi_n(x, y) = (l_n(x) - \overline{s}_{n+1}(y), t_{n+1}(y)) \text{ for } n \geq 0$$

is a map of complexes.
We prove that $\psi \circ \gamma \sim \text{Id}_{M(a)}$

\[
\begin{array}{c}
M(v_0) : 0 \to E^0 \xrightarrow{\bar{\gamma}} I^0 \oplus E^1 \xrightarrow{\delta_1} I^1 \oplus E^2 \xrightarrow{\delta_2} \cdots \\
M(v_1) : 0 \to E^0 \xrightarrow{\bar{\gamma}} I^0 \oplus E^1 \xrightarrow{\delta_1} I^1 \oplus E^2 \xrightarrow{\delta_2} \cdots \\
M(v_2) : 0 \to E^0 \xrightarrow{\bar{\gamma}} I^0 \oplus E^1 \xrightarrow{\delta_1} I^1 \oplus E^2 \xrightarrow{\delta_2} \cdots 
\end{array}
\]

Since $t \circ h : E \to E$ is induced by $\text{Id}_N$ we have $t \circ h \sim \text{Id}_E$. So there exists $\beta_i \in \text{Hom}(E^i, E^{i-1})$, $i \geq 1$, such that

\[
\text{Id} - t_{0} \circ h_{0} = \beta_1 \circ g_0, \quad \text{Id} - t_{i} \circ h_{i} = g_{i-1} \circ \beta_i + \beta_{i+1} \circ g_i, \quad i \geq 1.
\]

Let $\chi_1 : I^0 \oplus E^1 \to E^0$, $\chi_1(x, y) = \beta_1(y)$. We have

\[
\begin{align*}
\bar{\psi} \circ \bar{\gamma}(y) - \text{Id}(y) &= t_0 \circ h_0(y) - \text{Id}(y) = -\beta_1(g_0(y)) \quad \text{and} \\
\chi_1 \circ \delta_0(y) &= \chi_1(v_0(y), -g_0(y)) = \beta_1(-g_0(y)) = -\beta_1(g_0(y)).
\end{align*}
\]

So

\[
\bar{\psi} \circ \bar{\gamma} = \chi_1 \circ \delta_0 + \text{Id}.
\]

By (4) the complex

\[
\cdots \to \text{Hom}(I^2 \oplus E^3, I^0) \xrightarrow{\text{Hom}(\delta_2, \rho)} \text{Hom}(I^1 \oplus E^2, I^0) \xrightarrow{\text{Hom}(\delta_1, \rho)} \text{Hom}(I^0 \oplus E^1, I^0) \xrightarrow{\text{Hom}(\delta_0, \rho)} \text{Hom}(E^0, I^0) \to 0.
\]

is exact.

Let $\pi_0 : I^0 \oplus E^1 \to I^0$ be defined by $\pi_0(a, b) = a$. We have

\[
\begin{align*}
\pi_0 \circ (-\delta_0 \circ \chi_1 + \psi_0 \circ \gamma_0 - \text{Id}) \circ \delta_0 &= \pi_0 \circ (-\delta_0 \circ (\psi_0 \circ \bar{\gamma} - \text{Id}) + \psi_0 \circ \gamma_0 \circ \delta_0 - \delta_0) = 0.
\end{align*}
\]

Since

\[
\pi_0 \circ (-\delta_0 \circ \chi_1 + \psi_0 \circ \gamma_0 - \text{Id}) \in \text{Ker} \text{Hom}(\delta_0, I^0) = \text{Im} \text{Hom}(\delta_1, I^0),
\]

there exists $x_1 \in \text{Hom}(I^1 \oplus E^2, I^0)$ such that

\[
\pi_0 \circ (-\delta_0 \circ \chi_1 + \psi_0 \circ \gamma_0 - \text{Id}) = x_1 \circ \delta_1
\] (5)
We have
\[(\psi_0 \circ \gamma_0 - \text{Id} - \delta_0 \circ \chi_1)(x, y)\]
\[= \psi_0(k_0(x) - s_1(y), h_1(y)) - (x, y) - \delta_0(\beta_1(y))\]
\[= ((l_0 \circ k_0 - \text{Id})(x) - (l_0 \circ s_1 + \overline{s_1} \circ h_1\]
\[+ v_0 \circ \beta_1)(y), (t_1 \circ h_1 - \text{Id} + g_0 \circ \beta_1)(y))\]
\[= (a, b) \in I^0 \oplus E^1.\]

So
\[b = (t_1 \circ h_1 - \text{Id} + g_0 \circ \beta_1)(y) = -\beta_2 \circ g_1(y) = \beta_2 \circ p_2 \circ \delta_1(x, y) \quad \text{with}\]
\[p_2 : I^1 \oplus E^2 \rightarrow E^2, \quad p_2(c, d) = d.\]

By (5) \(a = \alpha_1 \circ \delta_1(x, y)\) for some \(\alpha_1 \in \text{Hom}(I^1 \oplus E^2, I^0).\)

Let
\[\chi_2 : I^1 \oplus E^2 \rightarrow I^0 \oplus E^1, \quad \chi_2(x, y) = (\alpha_1(x, y), \beta_2(p_2(x, y))).\]

By the above
\[(\psi_0 \circ \gamma_0 - \text{Id} - \delta_0 \circ \chi_1)(x, y) = (a, b)\]
\[= (\alpha_1(\delta_1(x, y)), \beta_2(p_2(\delta_1(x, y)))) = \chi_2(\delta_1(x, y)).\]

So
\[\psi_0 \circ \gamma_0 - \text{Id} = \delta_0 \circ \chi_1 + \chi_2 \circ \delta_1.\]

Similarly there exists \(\chi_j \in \text{Hom}(I^{j-1} \oplus E^{j-1}, I^{j-2} \oplus E^{j-1})\) such that
\[\chi_j \circ \delta_{j-2} + \delta_{j-1} \circ \chi_{j-1} = \psi_{j-2} \circ \gamma_{j-2} - \text{Id}, \quad \text{for all } j \geq 2.\]

Thus \(\psi \circ \gamma \sim \text{Id}_{M(v)}.\)

Similarly \(\gamma \circ \psi \sim \text{Id}_{M(w)}.\)

Then
\[H^n(\text{Hom}(M, M(v))) \simeq H^n(\text{Hom}(M, M(w))).\]

for any \(R M\), for any \(n \geq 0.\)

We consider now two precovering classes of left \(R\)-modules \(\mathcal{P}, \mathcal{C}\), both closed under finite direct sums, such that \(\text{Proj} \subset \mathcal{P} \subset \mathcal{C}\). We check the following properties of \(\hat{\text{Ext}}^n_{E, \mathcal{P}}(M, N)\):

**Proposition 1.** If \(0 \rightarrow M' \xrightarrow{1} M \xrightarrow{h} M'' \rightarrow 0\) is a \(\text{Hom}(\mathcal{C}, -)\) exact sequence of left \(R\)-modules then for any \(\mathcal{P} N\) there is an exact sequence:

\[0 \rightarrow \hat{\text{Ext}}^1_{E, \mathcal{P}}(M'', N) \rightarrow \hat{\text{Ext}}^1_{E, \mathcal{P}}(M, N) \rightarrow \hat{\text{Ext}}^1_{E, \mathcal{P}}(M', N) \rightarrow \cdots\]
Proof. Let
\[ C' : \cdots \to C_2^i \xrightarrow{g_2'} C_1^i \xrightarrow{g_1'} C_0^i \xrightarrow{g_0'} M' \to 0 \]
be a $\mathcal{C}$ resolution of $M'$ and let
\[ C'' : \cdots \to C_2^{i''} \xrightarrow{g_2''} C_1^{i''} \xrightarrow{g_1''} C_0^{i''} \xrightarrow{g_0''} M'' \to 0 \]
be a $\mathcal{C}$-resolution of $M''$. Since $\mathcal{C}$ is a precovering class closed under finite direct sums and $0 \to M' \to M \to M'' \to 0$ is a $\text{Hom}(\mathcal{C}, -)$ exact sequence, we can construct a commutative diagram:

```
\[
\begin{array}{cccc}
\vdots & \vdots & \vdots & \\
\downarrow & \downarrow & \downarrow & \\
0 & \xrightarrow{i} & C_1^i & \xrightarrow{p_1} C_1' & \xrightarrow{g_1} 0 \\
\downarrow & \downarrow & \downarrow & \\
0 & \xrightarrow{i_0} & C_0 & \xrightarrow{p_0} C_0' & \xrightarrow{g_0} 0 \\
\downarrow & \downarrow & \downarrow & \\
0 & \xrightarrow{l} & M' & \xrightarrow{h} M'' & \xrightarrow{0} 0 \\
\end{array}
\]
```

with $C : \cdots \to C_1^i \oplus C_1'' \xrightarrow{g_1} C_0 \oplus C_0'' \xrightarrow{g_0} M \to 0$ being a $\mathcal{C}$-resolution of $M$ (Horseshoe Lemma).

Similarly, there is a commutative diagram:

```
\[
\begin{array}{cccc}
\vdots & \vdots & \vdots & \\
\downarrow & \downarrow & \downarrow & \\
0 & \xrightarrow{j} P_1 & \xrightarrow{\eta_1} P_1' & \xrightarrow{f_1} 0 \\
\downarrow & \downarrow & \downarrow & \\
0 & \xrightarrow{j_0} P_0 & \xrightarrow{\eta_0} P_0' & \xrightarrow{f_0} 0 \\
\downarrow & \downarrow & \downarrow & \\
0 & \xrightarrow{l} M' & \xrightarrow{h} M'' & \xrightarrow{0} 0 \\
\end{array}
\]
```
with

\[ \mathbf{P'}: \cdots \rightarrow P'_2 \xrightarrow{f'_2} P'_1 \xrightarrow{f'_1} P'_0 \xrightarrow{f'_0} M' \rightarrow 0, \]

\[ \mathbf{P'':} \cdots \rightarrow P''_2 \xrightarrow{f''_2} P''_1 \xrightarrow{f''_1} P''_0 \xrightarrow{f''_0} M'' \rightarrow 0 \]

\( \mathcal{P} \)-resolutions of \( M', M'' \) and

\[ \mathbf{P}: \cdots \rightarrow P'_1 \oplus P''_1 \xrightarrow{f'_1} P'_0 \oplus P''_0 \xrightarrow{f_0} M \rightarrow 0 \]

a \( \mathcal{P} \)-resolution of \( M \).

Let \( u: \mathbf{P'} \rightarrow \mathbf{C'} \) be a map of complexes induced by \( \text{Id}_{M'} \).

\[ \begin{array}{c}
\mathbf{P'}: \cdots \rightarrow P'_2 \xrightarrow{f'_2} P'_1 \xrightarrow{f'_1} P'_0 \xrightarrow{f'_0} M' \rightarrow 0 \\
\downarrow{u} \downarrow{u_2} \downarrow{u_1} \downarrow{u_0} \\
\mathbf{C'}: \cdots \rightarrow C'_2 \xrightarrow{\delta'_2} C'_1 \xrightarrow{\delta'_1} C'_0 \xrightarrow{\delta'_0} M' \rightarrow 0
\end{array} \]

Let

\[ \alpha_0: P'_0 \rightarrow M, \alpha_0 = f_0 \circ e_0 \quad \text{with} \quad e_0: P''_0 \rightarrow P'_0 \oplus P''_0, e_0(y) = (0, y). \]

Since \( P''_0 \in \mathcal{P} \subset \mathcal{C}, C'_0 \oplus C''_0 \xrightarrow{\beta_0} M \) is a \( \mathcal{C} \)-precover and \( \alpha_0 \in \text{Hom}(P''_0, M) \) there is \( \beta_0 \in \text{Hom}(P''_0, C'_0 \oplus C''_0) \) so that \( g_0 \circ \beta_0 = \alpha_0 \).

Let

\[ \omega_0: P'_0 \oplus P''_0 \rightarrow C'_0 \oplus C''_0, \omega_0(x, y) = (u_0(x), 0) + \beta_0(y). \]

Then

\[ g_0 \circ \omega_0(x, 0) = g_0((u_0(x), 0)) = g_0 \circ i_0 \circ u_0(x) = l \circ g_0 \circ u_0(x) = l \circ f''_0(x) = f_0 \circ j_0(x) = f_0(x, 0) \quad \text{and} \]
\[ g_0 \circ \omega_0(0, y) = g_0 \circ \beta_0(y) = \alpha_0(y) = f_0(0, y). \]

So

\[ g_0 \circ \omega_0(x, y) = f_0(x, y) \quad \text{for any} \quad (x, y) \in P'_0 \oplus P''_0. \]

\[ \begin{array}{c}
\mathbf{P}: \cdots \rightarrow P'_2 \oplus P''_2 \xrightarrow{f'_2} P'_1 \oplus P''_1 \xrightarrow{f'_1} P'_0 \oplus P''_0 \xrightarrow{f_0} M \rightarrow 0 \\
\downarrow{e_2} \downarrow{e_1} \downarrow{e_0} \\
\mathbf{C}: \cdots \rightarrow C'_2 \oplus C''_2 \xrightarrow{\delta'_2} C'_1 \oplus C''_1 \xrightarrow{\delta'_1} C'_0 \oplus C''_0 \xrightarrow{\delta'_0} M \rightarrow 0
\end{array} \]

Let \( e_1: P''_1 \rightarrow P'_1 \oplus P''_1, e_1(y) = (0, y) \). We have \( g_0 \circ \omega_0 \circ f_1 \circ e_1 = f_0 \circ f_1 \circ e_1 = 0. \)
Since $P''_1 \in \mathcal{E}$ and $C$ is a $\mathcal{E}$ resolution of $M$, the complex $\cdots \to \text{Hom}(P''_1, C'_1 \oplus C''_1) \to \text{Hom}(P''_1, C'_0 \oplus C''_0) \to \text{Hom}(P''_1, M) \to 0$ is exact. We have $\omega_0 \circ f_1 \circ e_1 \in \text{Ker} \text{Hom}(P''_1, g_0) = \text{Im} \text{Hom}(P''_1, g_1)$ so there is $\beta_1 \in \text{Hom}(P''_1, C'_1 \oplus C''_1)$ such that $\omega_0 \circ f_1 \circ e_1 = g_1 \circ \beta_1$.

Let

$$\omega_1 : P'_1 \oplus P''_1 \to C'_0 \oplus C''_0, \omega_1(x, y) = (u_1(x), 0) + \beta_1(y).$$

Then

$$g_1 \circ \omega_1(x, 0) = g_1 \circ i_1 \circ u_1(x) = i_0 \circ g'_1 \circ u_1(x) = i_0 \circ u_0 \circ f'_1(x)$$

$$= \omega_0 \circ j_0 \circ f'_1(x) = \omega_0 \circ f_1 \circ j_1(x) = \omega_0 \circ f_1(x, 0))$$

(we used $\omega_0 \circ j_0(a) = \omega_0(a, 0) = (u_0(a), 0) = i_0 \circ u_0(a), \forall a \in P'_0$) and

$$g_1 \circ \omega_1(0, y) = g_1 \circ \beta_1(y) = \omega_0 \circ f_1 \circ e_1(y) = \omega_0 \circ f_1(0, y).$$

So

$$(g_1 \circ \omega_1)(x, y) = (\omega_0 \circ f_1)(x, 0) + (\omega_0 \circ f_1)(0, y)$$

$$= (\omega_0 \circ f_1)(x, y), \quad \forall (x, y) \in P'_1 \oplus P''_1.$$

Similarly there exists $\omega_k \in \text{Hom}(P'_k, P''_k, C'_k \oplus C''_k)$ such that $g_k \circ \omega_k = \omega_{k-1} \circ f_k \forall k \geq 2$. So $\omega : P \to C$ is a map of complexes induced by $\text{Id}_M$ and therefore $M(\omega_*)$ can be used to compute $\text{Ext}^n_{\mathcal{E}, \mathcal{C}}(M, N)$ (where $M(\omega_*)$ is the mapping cone of the chain map $\omega : P_r \to C, P_s, C, C$, being the deleted resolutions).

We show that $p \circ \beta : P'' \to C''$ is a map of complexes (where $\beta_k \in \text{Hom}(P''_k, C'_k \oplus C''_k)$) such that $g_0 \circ \beta_0 = f_0 \circ e_0$ and $g_k \circ \beta_k = \omega_{k-1} \circ f_k \circ e_k, \forall k \geq 1$.

$$g''_0 \circ p_0 \circ \beta_0(y) = h \circ g_0 \circ \beta_0(y) = h \circ f_0 \circ e_0(y)$$

$$= f''_0 \circ p_0(0, y) = f''_0(y), \quad \forall y \in P''_0.$$

If $f_k(0, y) = (a, b)$ then $\omega_{k-1}(a, b) = (u_{k-1}(a), 0) + \beta_{k-1}(b)$.

We have

$$g''_k \circ p_k \circ \beta_k(y) = p_{k-1} \circ g_k \circ \beta_k(y) = p_{k-1} \circ \omega_{k-1} \circ f_k(0, y)$$

$$= p_{k-1}((u_{k-1}(a), 0) + \beta_{k-1}(b)) = p_{k-1}(\beta_{k-1}(b)).$$

Since $\pi_{k-1} \circ f_k = f''_k \circ \pi_k$ we have

$$\pi_{k-1}(a, b) = f''_k(\pi_k(0, y)) \iff b = f''_k(y).$$

So

$$g''_k \circ p_k \circ \beta_k(y) = p_{k-1} \circ \beta_{k-1} \circ f''_k(y) \quad \text{for any } y \in P''_k.$$
Thus
\[ g_k'' \circ p_k \circ \beta_k = p_{k-1} \circ \beta_{k-1} \circ f_k'' \text{ for all } k \geq 1. \]

So \( v = p \circ \beta : \mathbf{P}' \to \mathbf{C}' \) is a map of complexes induced by \( \text{Id}_{\mathbf{M}'} \) and therefore \( \hat{\text{Ext}}^n_{\mathcal{C},\mathcal{P}}(\mathbf{M}', N) = H^{n+1}(\text{Hom}(\mathbf{M}(v), N)) \) for any \( \mathcal{R}N \) and any \( n \geq 0 \) (where \( \mathbf{M}(v) \) is the mapping cone of \( v : \mathbf{P}' \to \mathbf{C}' \), \( \mathbf{P}', \mathbf{C}' \) being the deleted resolutions).

We have an exact sequence of complexes \( 0 \to \mathbf{M}(u) \to \mathbf{M}(\omega) \to \mathbf{M}(v) \to 0 \) (with \( \mathbf{P}', \mathbf{C}' \) the deleted resolutions, and \( \mathbf{M}(u) \) the mapping cone of \( u : \mathbf{P}' \to \mathbf{C}' \)).

\[
\begin{array}{cccccccc}
M(u) : & \cdots & \rightarrow & C_3' \oplus P_2' & \rightarrow & C_2' \oplus P_1' & \rightarrow & C_1' \oplus P_0' & \rightarrow & C_0' & \rightarrow & 0 \\
& \downarrow(i_1,j_2) & & \downarrow(i_2,j_1) & & \downarrow(i_1,j_0) & & \downarrow i_0 & & \\
M(u) : & \cdots & \rightarrow & C_3 \oplus P_2 & \rightarrow & C_2 \oplus P_1 & \rightarrow & C_1 \oplus P_0 & \rightarrow & C_0 & \rightarrow & 0 \\
& \downarrow(p_3,n_2) & & \downarrow(p_2,n_1) & & \downarrow(p_1,n_0) & & \downarrow n_0 & & \\
M(v) : & \cdots & \rightarrow & C_3'' \oplus P_2'' & \rightarrow & C_2'' \oplus P_1'' & \rightarrow & C_1'' \oplus P_0'' & \rightarrow & C_0'' & \rightarrow & 0 \\
\end{array}
\]

The sequence \( 0 \to \mathbf{M}(u) \to \mathbf{M}(\omega) \to \mathbf{M}(v) \to 0 \) is split exact in each degree. Therefore \( 0 \to \text{Hom}(\mathbf{M}(v), N) \to \text{Hom}(\mathbf{M}(\omega), N) \to \text{Hom}(\mathbf{M}(u), N) \to 0 \) is an exact sequence of complexes, for any \( \mathcal{R}N \). We have an associated long exact sequence

\[
\cdots \to H^n(\text{Hom}(\mathbf{M}(v), N)) \to H^n(\text{Hom}(\mathbf{M}(\omega), N)) \to \cdots 
\]

(6)

Since \( 0 \to \mathbf{C}' \to \mathbf{M}(u) \to \mathbf{P}'[1] \to 0 \) is an exact sequence of complexes and both \( \mathbf{C}' \) and \( \mathbf{P}'[1] \) are exact complexes, it follows that \( \mathbf{M}(u) : \cdots \rightarrow C_3' \oplus P_2' \rightarrow C_2' \oplus P_1' \rightarrow C_1' \oplus P_0' \rightarrow C_0' \rightarrow 0 \) is also exact. After factoring out the exact subcomplex \( 0 \to M' \simeq M' \to 0 \) we obtain the exact complex \( \cdots \rightarrow C_2' \oplus P_1' \rightarrow C_1' \oplus P_0' \rightarrow C_0' \rightarrow 0 \) which is \( \mathbf{M}(u) \). Similarly \( \mathbf{M}(u) \) and \( \mathbf{M}(w) \) are exact complexes.

Since \( \text{Hom}(\mathbf{P}, N) \) is a left exact functor it follows that \( H^i((\text{Hom}(\mathbf{M}(u), N)) = H^i((\text{Hom}(\mathbf{M}(u), N)) = H^i((\text{Hom}(\mathbf{M}(v), N)) = 0, i = 0, 1, 2, \ldots \)

By definition

\[
H^{n+1}(\text{Hom}(\mathbf{M}(u), N)) = \hat{\text{Ext}}^n_{\mathcal{C},\mathcal{P}}(M', N), H^{n+1}(\text{Hom}(\mathbf{M}(\omega), N)) = \hat{\text{Ext}}^n_{\mathcal{C},\mathcal{P}}(M'', N), \forall n \geq 1,
\]

so the long exact sequence (6) is

\[
0 \to \hat{\text{Ext}}^1_{\mathcal{C},\mathcal{P}}(M'', N) \to \hat{\text{Ext}}^1_{\mathcal{C},\mathcal{P}}(M, N) \to \hat{\text{Ext}}^1_{\mathcal{C},\mathcal{P}}(M', N) \to \hat{\text{Ext}}^2_{\mathcal{C},\mathcal{P}}(M'', N) \to \cdots
\]

\[ \square \]

**Proposition 2.** If \( 0 \to N' \to N \to N'' \to 0 \) is a complex of left \( \mathcal{R}M \)-modules such that \( 0 \to \text{Hom}(\mathbf{C}, N') \to \text{Hom}(\mathbf{C}, N) \to \text{Hom}(\mathbf{C}, N'') \to 0 \) is exact for any \( \mathbf{C} \in \mathcal{C} \), then for any \( \mathcal{R}M \) we have an exact sequence:

\[
0 \to \hat{\text{Ext}}^1_{\mathcal{C},\mathcal{P}}(M, N') \to \hat{\text{Ext}}^1_{\mathcal{C},\mathcal{P}}(M, N) \to \hat{\text{Ext}}^1_{\mathcal{C},\mathcal{P}}(M, N'') \to \hat{\text{Ext}}^2_{\mathcal{C},\mathcal{P}}(M, N') \to \cdots
\]
Proof. Let $P_0 : \cdots \to P_2 \to P_1 \to P_0 \to 0$ be a deleted left $\mathcal{P}$-resolution of $M$ and let $C_0 : \cdots \to C_2 \to C_1 \to C_0 \to 0$ be a deleted left $\mathcal{C}$-resolution of $M$. If $u : P_0 \to C_0$ is a map of complexes induced by $\text{Id}_M$ then $M(u) : \cdots \to C_2 \oplus P_1 \to C_1 \oplus P_0 \to C_0 \to 0$ is an exact complex (same argument as in the proof of Proposition 1).

Since $C_0, C_{i+1} \oplus P_i \in \mathcal{C}, \forall i \geq 0$, and $0 \to N' \to N \to N'' \to 0$ is a $\text{Hom}(\mathcal{C}, -)$ exact sequence it follows that

$$0 \to \text{Hom}(M(u), N') \to \text{Hom}(M(u), N) \to \text{Hom}(M(u), N'') \to 0$$

is an exact sequence of complexes. Therefore we have an associated long exact sequence:

$$\cdots \to H^n(\text{Hom}(M(u), N')) \to H^n(\text{Hom}(M(u), N)) \to H^n(\text{Hom}(M(u), N'')) \to H^{n+1}(\text{Hom}(M(u), N')) \to \cdots.$$

Since $M(u)$ is exact, $H^0(\text{Hom}(M(u), T)) = H^1(\text{Hom}(M(u), T)) = 0$ for any $\mathfrak{R} \mathfrak{T}$. So the exact sequence above is:

$$0 \to H^2(\text{Hom}(M(u), N')) \to H^2(\text{Hom}(M(u), N)) \to H^2(\text{Hom}(M(u), N'')) \to \cdots \ i.e.,$$

$$0 \to \widehat{\text{Ext}}_{\mathcal{E},\mathcal{P}}^1(M, N') \to \widehat{\text{Ext}}_{\mathcal{E},\mathcal{P}}^1(M, N) \to \widehat{\text{Ext}}_{\mathcal{E},\mathcal{P}}^1(M, N'') \to \widehat{\text{Ext}}_{\mathcal{E},\mathcal{P}}^2(M, N') \to \cdots$$

is an exact sequence.

Let $\mathcal{E}, \mathcal{F}$ be two preenveloping classes of left $\mathcal{R}$-modules, both closed under finite direct sums, such that $\text{Inj} \subset \mathcal{F} \subset \mathcal{E}$. A dual argument of the proof of Proposition 1 gives us:

**Proposition 3.** If $0 \to N' \to N \to N'' \to 0$ is a $\text{Hom}(-, \mathcal{E})$ exact complex then for each $\mathcal{R}M$ there is an exact sequence: $0 \to \widehat{\text{Ext}}_{\mathcal{E},\mathcal{F}}^1(M, N') \to \widehat{\text{Ext}}_{\mathcal{E},\mathcal{F}}^1(M, N) \to \widehat{\text{Ext}}_{\mathcal{E},\mathcal{F}}^2(M, N') \to \widehat{\text{Ext}}_{\mathcal{E},\mathcal{F}}^2(M, N) \to \cdots$.

The dual result of Proposition 2 is:

**Proposition 4.** If $0 \to M' \to M \to M'' \to 0$ is a $\text{Hom}(-, \mathcal{E})$ exact sequence then for any $\mathcal{R}N$ we have an exact sequence $0 \to \widehat{\text{Ext}}_{\mathcal{E},\mathcal{F}}^1(M', N) \to \widehat{\text{Ext}}_{\mathcal{E},\mathcal{F}}^1(M, N) \to \widehat{\text{Ext}}_{\mathcal{E},\mathcal{F}}^2(M', N) \to \widehat{\text{Ext}}_{\mathcal{E},\mathcal{F}}^2(M'', N) \to \cdots$.

3. **MAIN RESULT**

We recall first a few things about cotorsion theories.

For a class $\mathcal{T}$ of $\mathcal{R}$-modules $\perp \mathcal{T}$ denotes the class of $\mathcal{R}$-modules $M$ such that $\text{Ext}_{\mathcal{R}}^1(M, F) = 0$ for all $F \in \mathcal{T}$ and $\mathcal{T}^\perp$ denotes the class of $\mathcal{R}$-modules $N$ such that $\text{Ext}_{\mathcal{R}}^1(F, N) = 0$ for all $F \in \mathcal{T}$.

We note that $\mathcal{P} \subset \mathcal{T}$ implies $\mathcal{P}^\perp \subset \mathcal{T}^\perp$ and $\perp \mathcal{T} \subset \perp \mathcal{P}$.
**Definition 3** (Enochs and Jenda, 2000, p. 152). A pair \((\mathcal{F}, \mathcal{C})\) of classes of R-modules is called a cotorsion theory if \(\mathcal{F}^\perp = \mathcal{C}\) and \(\mathcal{C}^\perp = \mathcal{F}\).

**Definition 4** (Enochs and Jenda, 2000, p. 153). A cotorsion theory \((\mathcal{F}, \mathcal{C})\) is said to have enough injectives if for every module \(M\) there is an exact sequence \(0 \rightarrow M \rightarrow C \rightarrow F \rightarrow 0\) with \(C \in \mathcal{C}\) and \(F \in \mathcal{F}\).

We note that if \(0 \rightarrow M \rightarrow C \rightarrow F \rightarrow 0\) is such an exact sequence then \(M \rightarrow C\) is a \(\mathcal{C}\)-preenvelope of \(M\) (called a special \(\mathcal{C}\)-preenvelope of \(M\)).

Dually, \((\mathcal{F}, \mathcal{C})\) has enough projectives if for every \(M\) there is an exact sequence \(0 \rightarrow C \rightarrow F \rightarrow M \rightarrow 0\) with \(C \in \mathcal{C}\) and \(F \in \mathcal{F}\). If \(0 \rightarrow C \rightarrow F \rightarrow M \rightarrow 0\) is such an exact sequence then \(F \rightarrow M\) is an \(\mathcal{F}\)-precover of \(M\) (called a special \(\mathcal{F}\)-precover of \(M\)).

By Enochs and Jenda (2000, Proposition 7.1.7), if \((\mathcal{F}, \mathcal{C})\) is a cotorsion theory on the category of \(R\)-modules having enough injectives (projectives) then it also has enough projectives (injectives).

A cotorsion theory is said to be **complete** if it has enough projectives.

**Definition 5** (Enochs et al., 1998, 3.5). A cotorsion theory \((\mathcal{F}, \mathcal{C})\) is hereditary if \(\text{Ext}^i_R(F, C) = 0\) for all \(i \geq 1\) and all \(F \in \mathcal{F}\) and \(C \in \mathcal{C}\).

By Enochs et al. (1998, Theorem 4.1), if \((\mathcal{F}, \mathcal{C}), (\mathcal{C}, \mathcal{L})\) are hereditary cotorsion theories on the category of \(R\)-modules then \(\text{Hom}(\cdot, \cdot)\) is right balanced by \(\mathcal{F} \times \mathcal{L}\). So we can compute \(\text{Ext}^i_{\mathcal{L}}(M, N)\) using either a left \(\mathcal{F}\)-resolution of \(M\) or a right \(\mathcal{L}\)-resolution of \(N\).

We can prove now the main result:

**Theorem 1.** If \((\mathcal{C}, \mathcal{L}), (\mathcal{L}, \mathcal{C})\) are complete hereditary cotorsion theories, then \(\text{Ext}^i_{\mathcal{L}}(M, N) \cong \text{Ext}^i_{\mathcal{C}}(M, N)\) for all \(i \geq 1\), for any \(R\)-modules \(M, N\), where \(\mathcal{P} = \text{Proj}, \mathcal{I} = \text{Inj}\).

**Proof.** \(\mathcal{C} = \mathcal{L} = \{C|\text{Ext}^1_R(C, L) = 0\text{ for all }L \in \mathcal{L}\}\). So \(\text{Proj} \subset \mathcal{C}\).

Consequently, for any left \(R\)-modules \(M, N\) we have an exact sequence \(0 \rightarrow \text{Ext}^1_{\mathcal{C}}(M, N) \rightarrow \text{Ext}^1_R(M, N) \rightarrow \text{Ext}^1_{\mathcal{L}}(M, N) \rightarrow \cdots\). Also \(\mathcal{I} = \text{Inj} \subset \mathcal{C}\), so for any \(M, N\) we have an exact sequence

\[
0 \rightarrow \text{Ext}^1_{\mathcal{C}}(M, N) \rightarrow \text{Ext}^1_R(M, N) \rightarrow \text{Ext}^1_{\mathcal{L}}(M, N) \rightarrow \cdots.
\]

Let \(L \in \mathcal{L}\). \((\mathcal{C}, \mathcal{L})\) is hereditary, so \(\text{Ext}^i_R(C, L) = 0\) for all \(i \geq 1\), for any \(C \in \mathcal{C}\). Since the sequence

\[
0 \rightarrow \text{Ext}^1_{\mathcal{C}}(C, L) \rightarrow \text{Ext}^1_R(C, L) \rightarrow \text{Ext}^1_{\mathcal{L}}(C, L) \rightarrow \text{Ext}^2_{\mathcal{C}}(C, L) \rightarrow \cdots
\]

is exact, \(\text{Ext}^i_{\mathcal{C}}(C, \cdot) = 0\) and \(\text{Ext}^i_R(C, L) = 0\) for all \(i \geq 1\), for \(C \in \mathcal{C}\), it follows that \(\text{Ext}^i_{\mathcal{L}}(C, L) = 0\) for all \(i \geq 1\), for any \(C \in \mathcal{C}\).

A dual argument gives us \(\text{Ext}^i_{\mathcal{C}}(C, L) = 0\) for any \(C \in \mathcal{C}\) and any \(L \in \mathcal{L}\). Let \(\mathcal{P}M\) be any left \(R\)-module.
\((\mathcal{E}, \mathcal{L}), (\mathcal{L}, \mathcal{E})\) are complete hereditary cotorsion theories, so there is an exact sequence \(0 \to K \to C \to M \to 0\) with \(C \in \mathcal{E}, K \in \mathcal{L}\) and so that \(\text{Hom}(-, E)\) leaves it exact for any \(E \in \mathcal{E}\) (Enochs et al., 1998, p. 23).

The sequence is \(\text{Hom}(\mathcal{E}, -)\) exact (since \(K \in \mathcal{L} = \mathcal{E}^\perp\)), so by Proposition 1, for any \(_R T\) we have an exact sequence \(0 \to \text{Ext}^{-1}_{\mathcal{E}, \mathcal{S}}(M, T) \to \text{Ext}^{-1}_{\mathcal{E}, \mathcal{S}}(C, T) \to \text{Ext}^{-1}_{\mathcal{E}, \mathcal{S}}(K, T) \to \text{Ext}^{-2}_{\mathcal{E}, \mathcal{S}}(M, T) \to \cdots\). In particular, for \(T = L \in \mathcal{L}\) we have an exact sequence \(0 \to \text{Ext}^{-1}_{\mathcal{E}, \mathcal{S}}(M, L) \to \text{Ext}^{-1}_{\mathcal{E}, \mathcal{S}}(C, L) = 0\). So \(\text{Ext}^{-1}_{\mathcal{S}, \mathcal{S}}(M, L) = 0\) for any \(_R M\) for all \(L \in \mathcal{L}\).

A dual argument shows that \(\text{Ext}^{-1}_{\mathcal{E}, \mathcal{S}}(M, L) = 0\) for any \(_R M\), for all \(L \in \mathcal{L}\).

Let \(_R N\) be any left \(R\)-module.

Since \((\mathcal{E}, \mathcal{L}), (\mathcal{L}, \mathcal{E})\) are complete hereditary cotorsion theories there exists an exact sequence \(0 \to N \to E \to L \to 0\) with \(E \in \mathcal{E}, L \in \mathcal{L}\) and so that \(\text{Hom}(C, -)\) leaves it exact for any \(C \in \mathcal{E}\) (Enochs et al., 1998, p. 23). Therefore by Proposition 2 for each \(_R M\) we have an exact sequence \(0 \to \text{Ext}^{-1}_{\mathcal{E}, \mathcal{S}}(M, N) \to \text{Ext}^{-1}_{\mathcal{E}, \mathcal{S}}(M, E) \to \text{Ext}^{-1}_{\mathcal{E}, \mathcal{S}}(M, L) = 0\). So we have

\[
\text{Ext}^{-1}_{\mathcal{E}, \mathcal{S}}(M, N) \cong \text{Ext}^{-1}_{\mathcal{E}, \mathcal{S}}(M, E) \tag{7}
\]

Since \(L \in \mathcal{L} = \mathcal{E}^\perp\) the sequence \(0 \to N \to E \to L \to 0\) is also \(\text{Hom}(-, \mathcal{E})\) exact and therefore (Proposition 3) for each \(_R M\) we have an exact sequence \(0 \to \text{Ext}^{-1}_{\mathcal{E}, \mathcal{S}}(M, N) \to \text{Ext}^{-1}_{\mathcal{E}, \mathcal{S}}(M, E) \to \text{Ext}^{-1}_{\mathcal{E}, \mathcal{S}}(M, L) = 0\). So

\[
\text{Ext}^{-1}_{\mathcal{E}, \mathcal{S}}(M, N) \cong \text{Ext}^{-1}_{\mathcal{E}, \mathcal{S}}(M, E) \tag{8}
\]

We have an exact sequence \(0 \to \text{Ext}^{i}_{\mathcal{E}}(M, E) \to \text{Ext}^{i}_{\mathcal{S}}(M, E) \to \text{Ext}^{i}_{\mathcal{S}}(M, L) \to \text{Ext}^{i}_{\mathcal{S}}(C, L) \to \text{Ext}^{i}_{\mathcal{S}}(K, L) \to \text{Ext}^{i+1}_{\mathcal{S}}(M, L) \to \cdots\). Since \(\text{Ext}^{i}_{\mathcal{E}}(-, E)\) is exact for all \(i \geq 1\) it follows that \(\text{Ext}^{i}_{\mathcal{E}}(M, E) \cong \text{Ext}^{i}_{\mathcal{S}}(M, E)\) for all \(i \geq 1\).

Since \(0 \to \text{Ext}^{i}_{\mathcal{E}}(M, E) \to \text{Ext}^{i}_{\mathcal{S}}(M, E) \to \text{Ext}^{i}_{\mathcal{S}}(M, L) \to \text{Ext}^{i}_{\mathcal{S}}(C, L) \to \cdots\) is an exact sequence and \(\text{Ext}^{i}_{\mathcal{E}}(-, E) = 0\) for all \(i \geq 1\) it follows that \(\text{Ext}^{i}_{\mathcal{E}, \mathcal{S}}(M, E) \cong \text{Ext}^{i}_{\mathcal{S}}(M, E)\), for all \(i \geq 1\). So

\[
\text{Ext}^{i}_{\mathcal{E}, \mathcal{S}}(M, E) \cong \text{Ext}^{i}_{\mathcal{S}}(M, E) \tag{9}
\]

for all \(i \geq 1\), for any \(E \in \mathcal{E}\).

By (7), (8), (9) we have \(\text{Ext}^{-1}_{\mathcal{E}, \mathcal{S}}(M, N) \cong \text{Ext}^{-1}_{\mathcal{E}, \mathcal{S}}(M, N)\) for any \(_R M, _R N\).

By Proposition 1 the \(\text{Hom}(\mathcal{E}, -)\) exact sequence \(0 \to K \to C \to M \to 0\) \((K \in \mathcal{L}, C \in \mathcal{E})\) gives us an exact sequence \(0 \to \text{Ext}^{-1}_{\mathcal{E}, \mathcal{S}}(M, L) \to \text{Ext}^{-1}_{\mathcal{E}, \mathcal{S}}(C, L) \to \text{Ext}^{-1}_{\mathcal{E}, \mathcal{S}}(K, L) \to \text{Ext}^{-2}_{\mathcal{E}, \mathcal{S}}(M, L) \to \text{Ext}^{-2}_{\mathcal{E}, \mathcal{S}}(C, L) \to \cdots\) for any \(_R L\), in particular for any \(L \in \mathcal{L}\). Since \(\text{Ext}^{-1}_{\mathcal{E}, \mathcal{S}}(T, L) = 0\) for any \(_R T\) and \(\text{Ext}^{-i}_{\mathcal{S}}(C, L) = 0\) for all \(i \geq 1\) it follows that \(\text{Ext}^{-1}_{\mathcal{S}}(M, L) = 0\) for any left \(R\)-module \(M\), for any \(L \in \mathcal{L}\).

By Proposition 2 the \(\text{Hom}(\mathcal{E}, -)\) exact sequence \(0 \to N \to E \to L \to 0\) \((E \in \mathcal{E}, L \in \mathcal{L})\) gives us an exact sequence \(0 \to \text{Ext}^{-1}_{\mathcal{E}, \mathcal{S}}(M, N) \to \text{Ext}^{-1}_{\mathcal{E}, \mathcal{S}}(M, E) \to \cdots\)


\(\hat{\text{Ext}}^1_{e,\mathcal{P}}(M, L) \to \hat{\text{Ext}}^2_{e,\mathcal{P}}(M, N) \to \hat{\text{Ext}}^2_{e,\mathcal{P}}(M, E) \to \hat{\text{Ext}}^2_{e,\mathcal{P}}(M, L) \to \cdots\), for each \(R M\).

Since \(L \in \mathcal{L}\) we have \(\hat{\text{Ext}}^1_{e,\mathcal{P}}(M, L) = \hat{\text{Ext}}^1_{e,\mathcal{P}}(M, L) = 0\) and therefore

\[\hat{\text{Ext}}^2_{e,\mathcal{P}}(M, N) \simeq \hat{\text{Ext}}^2_{e,\mathcal{P}}(M, E) \simeq \text{Ext}^2_R(M, E)\] (10)

A dual argument shows that

\[\hat{\text{Ext}}^2_{e,\mathcal{P}}(M, N) \simeq \hat{\text{Ext}}^2_{e,\mathcal{P}}(M, E) \simeq \text{Ext}^2_R(M, E)\] (11)

By (10) and (11) we have \(\hat{\text{Ext}}^2_{e,\mathcal{P}}(M, N) \simeq \text{Ext}^2_R(M, E) \simeq \hat{\text{Ext}}^2_{e,\mathcal{P}}(M, N)\). Similarly \(\hat{\text{Ext}}^n_{e,\mathcal{P}}(M, N) \simeq \hat{\text{Ext}}^n_{e,\mathcal{P}}(M, N)\) for all \(n \geq 1\).

\[\Box\]

**Example 1.** Let \(R\) be a Gorenstein ring. Let \(\mathcal{C} = \text{Gor proj}\) and let \(\mathcal{L}\) be the class of modules of finite projective dimension. By Enochs and Jenda (2000, Remark 11.5.10), \((\text{Gor proj}, \mathcal{L})\) is a cotorsion theory over \(R\) and has enough injectives and projectives. By Enochs and Jenda (2000, Proposition 11.5.9) \((\text{Gor proj}, \mathcal{L})\) is a hereditary cotorsion theory. So \((\text{Gor proj}, \mathcal{L})\) is a complete hereditary cotorsion theory. Also \((\mathcal{L}, \text{Gor inj})\) is a complete hereditary cotorsion theory in this case. By Theorem 1 we have \(\hat{\text{Ext}}^i_{\text{Gor proj}, \text{Proj}}(M, N) \simeq \hat{\text{Ext}}^i_{\text{Gor inj}, \text{Inj}}(M, N)\) for all \(i \geq 1\) for any \(RM, RN\).

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