2007

A Note on the Maximal Gurov-Reshetnyak Condition

A. A. Korenovskyy, *Odessa National University*
A. K. Lerner, *Bar-Ilan University, Israel*
Alexander M. Stokolos, *Georgia Southern University*

Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 32, 2007, 461–470

A NOTE ON THE MAXIMAL GUROV–RESHETNYAK CONDITION

A. A. Korenovskyy, A. K. Lerner and A. M. Stokolos

Odessa National University, IMEM, Department of Mathematical Analysis
Dvoryanskaya, 2, 65026 Odessa, Ukraine; anakor@paco.net

Bar-Ilan University, Department of Mathematics
52900 Ramat Gan, Israel; aklerner@netvision.net.il

DePaul University, Department of Mathematical Sciences
Chicago, IL, 60614, U.S.A.; astokolo@depaul.edu

Dedicated to Professor Bogdan Bojarski on the occasion of his 75th birthday.

Abstract. In a recent paper [17] we established an equivalence between the Gurov–Reshetnyak and A_∞ conditions for arbitrary absolutely continuous measures. In the present paper we study a weaker condition called the *maximal* Gurov–Reshetnyak condition. Although this condition is not equivalent to A_∞ even for Lebesgue measure, we show that for a large class of measures satisfying Busemann–Feller type condition it will be self-improving as is the usual Gurov–Reshetnyak condition. This answers a question raised independently by Iwaniec and Kolyada.

1. Introduction

Throughout the paper, Q_0 will be a bounded cube from $\mathbf{R}^n$, and μ will be a non-negative Borel measure on Q_0 absolutely continuous with respect to Lebesgue measure. For $f \in L_\mu(Q_0)$ and for any subcube $Q \subset Q_0$ set

$$f_{Q,\mu} = \frac{1}{\mu(Q)} \int_Q f(x)\,d\mu, \quad \Omega_\mu(f;Q) = \frac{1}{\mu(Q)} \int_Q |f(x) - f_{Q,\mu}|\,d\mu.$$

A function $f \in L_\mu(Q_0)$ is said to belong $BMO(\mu)$ if

$$\sup_{Q\subset Q_0} \Omega_\mu(f;Q) < \infty.$$

Also we recall that the classes $A_p(\mu)$ and $RH_r(\mu), 1 < p, r < \infty$, consist of all non-negative $f \in L_\mu(Q_0)$ for which there exists $c > 0$ such that for all $Q \subset Q_0$,

$$\big(f_{Q,\mu}\big)\big((f^{-1/(p-1)})_{Q,\mu}\big)^{p-1} \le c \quad \text{and} \quad (f^r)_{Q,\mu} \le c(f_{Q,\mu})^r,$$

respectively. In the unweighted case (i.e., in the case when μ is Lebesgue measure) these objects were first considered in the classical works by John and Nirenberg [15],

2000 Mathematics Subject Classification: Primary 42B25.
Key words: Maximal Gurov–Reshetnyak condition, self-improving properties, non-doubling measures.

Muckenhoupt [20], Gehring [9], and Coifman and Fefferman [5]. It has been quickly realized that the theory developed in the unweighted case remains true for doubling measures (i.e., for those measures μ for which there exists a constant $c > 0$ such that $\mu(2Q) \leq c\mu(Q)$ for all cubes Q). Only recently, it was shown in the papers by Mateu *et al.* [18] and by Orobitg and Pérez [21] that most of important results concerning $BMO(\mu), A_p(\mu)$ and $RH_r(\mu)$ still hold for any absolutely continuous measure μ (and even for a wider class of measures). First of all, we mention that all these objects are closely related. Namely (see [21]),

$$A_\infty(\mu) \equiv \bigcup_{1<p<\infty} A_p(\mu) = \bigcup_{1<r<\infty} RH_r(\mu), \tag{1.1}$$

and $\log f \in BMO(\mu)$ for any $f \in A_\infty(\mu)$, and, conversely, given an $f \in BMO(\mu)$, there is $\lambda > 0$ such that $e^{\lambda f} \in A_\infty(\mu)$. Also, conditions expressed in the above definitions represent a kind of so-called self-improving properties. Indeed, if $f \in L_\mu(Q_0)$ belongs to $BMO(\mu)$, then f belongs to $L^p_\mu(Q_0)$ for any $1 < p < \infty$ (see [18]). Moreover, $A_p(\mu) \Rightarrow A_{p-\delta}(\mu)$ and $RH_r(\mu) \Rightarrow RH_{r+\delta}(\mu)$ for some small $\delta > 0$ which may differ in each implication (see [21]).

In the mid 70's, Gurov and Reshetnyak [10, 11] introduced in analogy with the definition of $BMO(\mu)$ (in the unweighted case) the class $GR_\varepsilon(\mu), 0 < \varepsilon < 2$, which consists of all non-negative $f \in L_\mu(Q_0)$ such that for any $Q \subset Q_0$,

$$\Omega_\mu(f; Q) \leq \varepsilon f_{Q,\mu}. \tag{1.2}$$

This class has found interesting applications in quasi-conformal mappings and PDE's (see, e.g., [3, 14]). Observe that (1.2) trivially holds for $\varepsilon = 2$, and therefore only the case $0 < \varepsilon < 2$ is of interest. It turned out that (1.2) also represents a kind of self-improving property. It was established in [3, 10, 11, 14, 19, 26] for Lebesgue measure and in [7, 8] for doubling measures that if ε is small enough, namely $0 < \varepsilon < c2^{-n}$, then the $GR_\mu(\varepsilon)$ implies $f \in L^p_\mu(Q_0)$ for some $p > 1$. In [16], it was shown that in the case of $n = 1$ and Lebesgue measure this self-improvement holds for the whole range $0 < \varepsilon < 2$. In a recent paper [17], the authors have established a rather surprising analogue of (1.1), namely for any absolutely continuous μ,

$$A_\infty(\mu) = \bigcup_{0<\varepsilon<2} GR_\varepsilon(\mu). \tag{1.3}$$

First, this result shows a close relation between the classes $GR_\varepsilon(\mu)$ and $A_p(\mu)$. Second, it follows immediately that for any absolutely continuous μ and $n \geq 1$ and for all $0 < \varepsilon < 2$ the $GR_\varepsilon(\mu)$ condition implies higher integrability properties of f.

Iwaniec and Kolyada independently asked the authors whether a weaker variant of (1.2):

$$f^{\#}_{\mu,Q_0}(x) \leq \varepsilon M_{\mu,Q_0} f(x) \quad \mu\text{-a.e. in } Q_0 \tag{1.4}$$

has an analogous self-improving property for all $\varepsilon < 2$. Here, as usual,

$$f^{\#}_{\mu,Q_0}(x) = \sup_{Q \ni x, Q \subset Q_0} \Omega_\mu(f; Q) \quad \text{and} \quad M_{\mu,Q_0} f(x) = \sup_{Q \ni x, Q \subset Q_0} |f|_{Q,\mu}.$$

The property expressed in (1.4) we call the *maximal* Gurov–Reshetnyak condition, and denote it by $MGR_\varepsilon(\mu)$.

Observe that the passing to maximal operators is a quite natural and well-known approach to many questions mentioned above. For example, Gehring's approach to the reverse Hölder inequality [9] as well as Bojarski's proof of the Gurov–Reshetnyak Lemma for small ε [3] were based on maximal function estimates. Actually, many papers establishing the self-improving property of $GR_\varepsilon(\mu)$ for small ε contain implicitly the same for $MGR_\varepsilon(\mu)$ (see, e.g., [3, 7, 19]). On the other hand, author's proof of (1.3) cannot be directly generalized to the class $MGR_\varepsilon(\mu)$. Therefore, the question of Iwaniec and Kolyada is of interest for $c < \varepsilon < 2$.

In this paper we show that for a large class of measures, including any doubling measures in $\mathbf{R}^n$ and any absolutely continuous measures in $\mathbf{R}^1$, the maximal Gurov–Reshetnyak condition $MGR_\varepsilon(\mu)$ is self-improving for any $0 < \varepsilon < 2$. The relevant class of measures will be given in the following definition.

Definition 1.1. We say that a measure μ satisfies the Busemann-Feller type condition (BF-condition) if

$$\varphi_\mu(\lambda) \equiv \sup_E \frac{\mu\{x : M_{\mu,Q_0}\chi_E(x) > \lambda\}}{\mu(E)} < \infty \quad (0 < \lambda < 1),$$

where the supremum is taken over all measurable sets $E \subset Q_0$ of positive μ-measure.

In the case of Lebesgue measure and the maximal operator associated with the homothety-invariant differential basis, this condition coincides with the well-known Busemann–Feller density condition (see, e.g., [4] or [12, p. 122]).

Our main result is the following.

Theorem 1.2. *Let μ satisfy the BF-condition, and let $0 < \varepsilon < 2$. Assume that a non-negative $f \in L_\mu(Q_0)$ satisfies the maximal Gurov–Reshetnyak condition $MGR_\varepsilon(\mu)$. Then there is $p_0 > 1$ depending on μ, ε and n such that for all $1 \le p < p_0$ one has*

$$(f^p)_{Q_0,\mu} \le c(f_{Q_0,\mu})^p, \tag{1.5}$$

where c depends on μ, ε, p and n.

Some comments about this result are in order. By (1.1) and (1.3), the usual Gurov–Reshetnyak condition $GR_\varepsilon(\mu)$ implies (1.5) with any subcube $Q \subset Q_0$ instead of Q_0. Theorem 1.2 shows that although we cannot obtain from the $MGR_\varepsilon(\mu)$ condition such a nice conclusion, we still have a higher integrability result. In fact, $MGR_\varepsilon(\mu)$ is really much weaker than $GR_\varepsilon(\mu)$. Indeed, let $n = 1, Q_0 = (0, 1)$, μ is Lebesgue measure, and f, for example, is the characteristic function of the interval $(0, 7/8)$. Then it is easy to see that $f^{\#}_{Q_0,\mu}(x) \le 1/2$, while $M_{Q_0,\mu}f(x) > 7/8$ for all $x \in (0, 1)$. Thus, (1.4) holds for this function with $\varepsilon = 4/7$. However, $f \notin A_\infty(\mu)$, since it is zero on a set of positive measure. Hence, in view of (1.3), (1.2) cannot hold for this f with any $\varepsilon < 2$. This example shows also that the class $GR_\varepsilon(\mu)$ in (1.3) cannot be replaced by $MGR_\varepsilon(\mu)$.

We make several remarks about the BF-condition. It is easy to see that this condition holds provided M_μ has a weak type (p,p) property with respect to μ for some $p>0$. A standard argument shows that if μ is doubling in $\mathbf{R}^n$, then M_μ is of weak type $(1,1)$. It is well-known that in the case $n=1$ the doubling condition can be completely removed; namely in this case, M_μ is of weak type $(1,1)$ for arbitrary Borel measure μ (see [22]). It has been recently shown in [23] that in the case $n\ge 2$ for a large class of radial (and non-doubling, in general) measures, including, for example, a Gaussian measure, M_μ will be of strong type (p,p) for any $p>1$. On the other hand, it was mentioned in [25] that there exists μ for which M_μ will not be of strong type (p,p) for any $p>1$. In Section 3 below we give an example of μ (in the case $n=2$) for which the BF-condition does not hold.

We would like to emphasize that we still do not know whether the BF-condition in Theorem 1.2 is really necessary. In other words we do not know whether there exist an absolutely continuous measure μ on a cube $Q_0\subset\mathbf{R}^n, n\ge 2$, and a function $f\in L_\mu(Q_0)$ such that μ does not satisfy the BF-condition, f satisfies (1.4) for some $\varepsilon<2$ and $f\notin L^p_\mu(Q_0)$ for any $p>1$.

Let us mention also that Theorem 1.2 gives yet another proof of the Gurov–Reshetnyak Lemma for the whole range of ε if μ is a BF-measure.

The paper is organized as follows. In the next section we prove our main result. Section 3 contains a detailed analysis of the BF-condition.

2. Proof of main result

2.1. Some auxiliary propositions. We first recall that the non-increasing rearrangement of a measurable function f on Q_0 with respect to μ is defined by

$$f^*_\mu(t)=\inf\{\alpha>0:\mu\{x\in Q_0:|f(x)|>\alpha\}\le t\}\quad (0<t<\mu(Q_0)).$$

Set also $f^{**}_\mu(t)=t^{-1}\int_0^t f^*_\mu(\tau)d\tau$. Note that [2, pp. 43, 53]

$$\int_0^t f^*_\mu(\tau)\,d\tau=\sup_{E\subset Q_0:\mu(E)=t}\int_E|f|\,d\mu \tag{2.1}$$

and

$$\int_{Q_0}|f|^p\,d\mu=\int_0^{\mu(Q_0)}f^*_\mu(\tau)^p\,d\tau\quad (p>0). \tag{2.2}$$

We will need a local variant of the well-known Herz-type estimate

$$f^{**}_\mu(t)\le c(M_\mu f)^*_\mu(t).$$

In the case of $Q_0=\mathbf{R}^n$ and Lebesgue measure this result can be found in [2, p. 122]. It was extended to arbitrary absolutely continuous measures μ in [1]. The case of the bounded cube Q_0 requires a slightly modified argument based on the following covering lemma from [18].

Lemma 2.1. *Let E be a subset of Q_0 with $\mu(E)\le\rho\mu(Q_0)$, $0<\rho<1$. Then there exists a sequence $\{Q_i\}$ of cubes contained in Q_0 such that*

(i) $\mu(Q_i \cap E) = \rho\mu(Q_i)$;

(ii) $\bigcup_i Q_i = \bigcup_{k=1}^{B_n} \bigcup_{i\in F_k} Q_i$, *where each of the family* $\{Q_i\}_{i\in F_k}$ *is formed by pairwise disjoint cubes and a constant* B_n *depends only on* n;

(iii) $E' \subset \cup_i Q_i$, *where* E' *is the set of* μ*-density points of* E.

Proposition 2.2. *For any* $f \in L_\mu(Q_0)$ *we have*

$$f^{**}_\mu(t) \le c_n (M_{\mu,Q_0} f)^*_\mu(t) \quad (0 < t < \mu(Q_0)), \tag{2.3}$$

where c_n *depends only on* n.

Proof. Let $\Omega = \{x \in Q_0 : M_{\mu,Q_0} f(x) > (M_{\mu,Q_0} f)^*_\mu(t)\}$. Then for some δ_0 and for any $\delta < \delta_0$ we have $\mu(\Omega) \le t \le (1-\delta)\mu(Q_0)$. Fix such a δ and apply Lemma 2.1 to the set Ω and number $\rho = 1 - \delta$. We get a sequence $\{Q_i\}$ satisfying properties (i)–(iii) of the lemma. It follows easily from (i) that $\mu(Q_i \cap \Omega^c) > 0$, and hence, $|f|_{Q_i,\mu} \le (M_{\mu,Q_0} f)^*_\mu(t)$. From this and from properties (i)–(iii) we obtain

$$\int_\Omega |f|\, d\mu \le \int_{\cup_i Q_i} |f|\, d\mu \le \sum_{k=1}^{B_n} \sum_{i\in F_k} \mu(Q_i)|f|_{Q_i,\mu} \le \frac{B_n}{1-\delta} t (M_{\mu,Q_0} f)^*_\mu(t).$$

Letting $\delta \to 0$ yields $\int_\Omega |f|\, d\mu \le B_n t (M_{\mu,Q_0} f)^*_\mu(t)$. Therefore, for any measurable set $E \subset Q_0$ with $\mu(E) = t$,

$$\begin{aligned}\int_E |f|\, d\mu \le \int_{E\setminus\Omega} |f|\, d\mu + \int_\Omega |f|\, d\mu &\le (\mu(E\setminus\Omega) + B_n t)(M_{\mu,Q_0} f)^*_\mu(t) \\ &\le (B_n + 1) t (M_{\mu,Q_0} f)^*_\mu(t).\end{aligned}$$

Taking the supremum over all $E \subset Q_0$ with $\mu(E) = t$ and using (2.1) completes the proof. □

Given a measurable f, define the local maximal function $m_{\lambda,\mu} f$ (cf. [24]) by

$$m_{\lambda,\mu} f(x) = \sup_{Q\ni x, Q\subset Q_0} \big(f\chi_Q\big)^*_\mu\big(\lambda\mu(Q)\big) \quad (0 < \lambda < 1).$$

Proposition 2.3. *Suppose that* μ *satisfies the BF-condition. Then for any measurable* f,

$$(m_{\lambda,\mu} f)^*_\mu(t) \le f^*_\mu\big(t/\varphi_\mu(\lambda)\big) \quad (0 < t < \mu(Q_0)). \tag{2.4}$$

Proof. It follows from the definitions that

$$\{x \in Q_0 : m_{\lambda,\mu} f(x) > \alpha\} = \{x \in Q_0 : M_{\mu,Q_0} \chi_{\{|f|>\alpha\}}(x) > \lambda\}.$$

Therefore,

$$\mu\{x \in Q_0 : m_{\lambda,\mu} f(x) > \alpha\} \le \varphi_\mu(\lambda) \mu\{x \in Q_0 : |f(x)| > \alpha\},$$

which is equivalent to (2.4). □

Lemma 2.4. ([20, Lemma 4]) *Let h be a non-negative and non-increasing function on the interval $[0,a]$, and assume that*

$$\frac{1}{s}\int_0^s h(\tau)\,d\tau \le Dh(s) \quad (0<s<a/r)$$

for some $D,r>1$. Then if $1\le p<D/(D-1)$,

$$\int_0^a h^p(\tau)\,d\tau \le c\Big(\int_0^a h(\tau)\,d\tau\Big)^p,$$

where c depends on r,p and D.

Actually, this lemma was proved in [20] with $r=20$ but exactly the same argument works for any $r>1$.

2.2. Proof of Theorem 1.2. Choose some constants $\alpha,\lambda\in(0,1)$ such that $(1-\alpha)\lambda>\varepsilon/2$. Take an arbitrary $\delta\in(0,1-\lambda)$. Given a cube $Q\subset Q_0$, set $E_Q=\{x\in Q: f(x)>f_{Q,\mu}\}$, and let

$$\mathscr{Q}=\{Q\subset Q_0:\mu(E_Q)\ge\alpha\mu(Q)\}.$$

Observe that if $Q\subset\mathscr{Q}$, then $f_{Q,\mu}\le(f\chi_Q)^*_\mu(\alpha\mu(Q))$. Therefore,

$$M_{\mu,Q_0}f(x)\le\max\Big(m_{\alpha,\mu}f(x),\sup_{Q\ni x,Q\not\subset\mathscr{Q}}f_{Q,\mu}\Big). \tag{2.5}$$

Assume that $Q\not\subset\mathscr{Q}$. Set $E_Q^c=Q\setminus E_Q$,

$$A_1(Q)=\big\{x\in E_Q^c: f_{Q,\mu}-f(x)>\big((f_{Q,\mu}-f)\chi_{E_Q^c}\big)^*_\mu\big(\lambda\mu(E_Q^c)\big)\big\}$$

and

$$A_2(Q)=\big\{x\in E_Q^c: f(x)>\big(f\chi_{E_Q^c}\big)^*_\mu\big((1-\lambda-\delta)\mu(E_Q^c)\big)\big\}.$$

Then $\mu\big(A_1(Q)\cup A_2(Q)\big)\le(1-\delta)\mu(E_Q^c)$ and $\mu(E_Q^c)\ge(1-\alpha)\mu(Q)$. Therefore we obtain

$$\begin{aligned}
f_{Q,\mu}&\le\inf_{y\in E_Q^c\setminus\{A_1(Q)\cup A_2(Q)\}}\big((f_{Q,\mu}-f(y))+f(y)\big)\\
&\le\big((f_{Q,\mu}-f)\chi_{E_Q^c}\big)^*_\mu\big(\lambda\mu(E_Q^c)\big)+\big(f\chi_{E_Q^c}\big)^*_\mu\big((1-\lambda-\delta)\mu(E_Q^c)\big)\\
&\le\frac{\mu(Q)}{\mu(E_Q^c)}\frac{1}{2\lambda}\frac{2}{\mu(Q)}\int_{E_Q^c}(f_{Q,\mu}-f)\,d\mu+\big(f\chi_{E_Q^c}\big)^*_\mu\big((1-\lambda-\delta)\mu(E_Q^c)\big)\\
&\le\frac{1}{2\lambda(1-\alpha)}\Omega_\mu(f;Q)+(f\chi_Q)^*_\mu((1-\lambda-\delta)(1-\alpha)\mu(Q)).
\end{aligned}$$

This along with the maximal Gurov–Reshetnyak condition (1.4) yields

$$\sup_{Q\ni x,Q\not\subset\mathscr{Q}}f_{Q,\mu}\le\frac{\varepsilon}{2\lambda(1-\alpha)}M_{\mu,Q_0}f(x)+m_{\gamma,\mu}f(x),$$

where $\gamma=(1-\lambda-\delta)(1-\alpha)$. From this and from (2.5) we obtain

$$M_{\mu,Q_0}f(x)\le cm_{\alpha',\mu}f(x),$$

where $c = \frac{2\lambda(1-\alpha)}{2\lambda(1-\alpha)-\varepsilon}$ and $\alpha' = \min(\alpha, \gamma)$. Taking the rearrangements of both parts and using Propositions 2.2 and 2.3, we have

$$f_\mu^{**}(t) \leq c f_\mu^*(t/\varphi_\mu(\alpha')) \quad (0 < t < \mu(Q_0)).$$

This implies easily

$$f_\mu^{**}(t) \leq c\varphi_\mu(\alpha') f_\mu^*(t) \quad (0 < t < \mu(Q_0)/\varphi_\mu(\alpha')),$$

which along with (2.2) and Lemma 2.4 completes the proof. □

3. On the BF-condition

First of all, we observe that we do not know whether for some absolutely continuous μ the function φ_μ can take both finite and infinite values. The following proposition represents only a partial answer to this question.

Proposition 3.1. *There exists a constant $\gamma < 1$ depending only on n such that if $\varphi_\mu(\gamma) < \infty$, then $\varphi_\mu(\lambda) < \infty$ for all $0 < \lambda < 1$.*

Proof. Given a set $E \subset Q_0$ and $0 < \lambda < 1$, let $E_\lambda = \{x \in Q_0 : M_{\mu,Q_0}\chi_E(x) > \lambda\}$. By Proposition 2.2, for any $E \subset Q \subset Q_0$,

$$\min(1, \mu(E)/t) \leq c_n (M_{\mu,Q}\chi_E)_\mu^*(t)$$

or, equivalently,

$$\frac{\mu(E)}{c_n\lambda} \leq \mu\{x \in Q : M_{\mu,Q}\chi_E(x) > \lambda\} \leq \mu(Q \cap E_\lambda).$$

Hence,

$$M_{\mu,Q_0}\chi_E(x) \leq c_n\lambda M_{\mu,Q_0}\chi_{E_\lambda}(x) \quad (x \in Q_0),$$

which yields

$$\mu\{x \in Q_0 : M_{\mu,Q_0}\chi_E > c_n\xi\lambda\} \leq \mu\{x \in Q_0 : M_{\mu,Q_0}\chi_{E_\lambda} > \xi\}. \tag{3.1}$$

Therefore,

$$\varphi_\mu(c_n\lambda\xi) \leq \varphi_\mu(\lambda)\varphi_\mu(\xi) \quad (\lambda, \xi \in (0,1) : \lambda\xi < 1/c_n).$$

This clearly implies the desired result if $0 < \lambda < 1/c_n$. The case $1/c_n < \lambda < 1$ follows from the monotonicity of $\varphi_\mu(\lambda)$. □

Remark 3.2. The last proposition means that Theorem 1.2 still holds if one relaxes the BF-condition to

$$\mu\{x \in Q_0 : M_{\mu,Q_0}\chi_E > \gamma\} \leq c\mu(E) \quad \forall\, E \subset Q_0$$

with some $0 < \gamma < 1$. Note that a similar condition with $\gamma = 1/2$ for the directional maximal operator appeared in [6] (see also [12, p. 372]), where it was called a Tauberian condition.

Remark 3.3. Inequality (3.1) is a full analogue of the Lemma from [13], which was proved there in a different context.

We give now an example of the absolutely continuous measure μ on the unit cube $(0,1)^2$ which does not satisfy the BF-condition.

Let $\delta, L > 0$ and

$$d\mu = \big(\delta\chi_{(-L,0)^2}(x,y) + \chi_{(0,L)^2}(x,y)\big)dxdy.$$

Given a point $(x,y) \in (0,L)^2$, let Q be a minimal cube contained in $(-2L,2L)^2$ and containing (x,y) and the cube $(-L,0)^2$. Then $\mu(Q) = xy + \delta L^2$. Hence, setting $E = (-L,0)^2$, we get

$$M_{\mu,(-2L,2L)^2}\chi_E(x,y) \geq \frac{\mu(Q\cap E)}{\mu(Q)} = \frac{\delta L^2}{xy + \delta L^2}. \tag{3.2}$$

Denote $c_\lambda = \frac{1}{\lambda} - 1$. Assuming $\delta c_\lambda < 1$, we get, by (3.2),

$$\begin{aligned} \frac{\mu\{M_{\mu,(-2L,2L)^2}\chi_E(x,y) > \lambda\}}{\mu(E)} &\geq \frac{\mu\{(x,y)\in(0,L)^2 : xy < \delta L^2 c_\lambda\}}{\delta L^2} \\ &\geq c_\lambda \int_{\delta c_\lambda L}^{L} \frac{dx}{x} = c_\lambda \log\frac{1}{\delta c_\lambda}. \end{aligned} \tag{3.3}$$

Choose now a sequence of cubes $\{Q_i\}$ such that the cubes $2Q_i$ are pairwise disjoint and $\bigcup_{i=1}^\infty 2Q_i \subset (0,1)^2$. We divide each cube Q_i into four equal quadrants, and let Q_i' and Q_i'' be the first and the third quadrants respectively. Let $\{\delta_i\}$ be a sequence of positive numbers such that $\delta_i \to 0$ as $i\to\infty$. Set

$$d\mu = \sum_{i=1}^{\infty}\big(\delta_i\chi_{Q_i''}(x,y) + \chi_{Q_i'}(x,y)\big)dxdy.$$

Given a $\lambda\in(0,1)$, there is an N such that $\delta_i c_\lambda < 1$ for all $i \geq N$. Hence, by (3.3),

$$\frac{\mu\{M_{\mu,(0,1)^2}\chi_{Q_i''} > \lambda\}}{\mu(Q_i'')} \geq c_\lambda \log\frac{1}{c_\lambda\delta_i} \quad (i\geq N). \tag{3.4}$$

This shows that μ does not satisfy the BF-condition, since the right-hand side of (3.4) tends to ∞ as $i\to\infty$.

In conclusion we give one more proposition concerning the function φ_μ which is probably of some independent interest. We recall that the operator M_μ is said to be of restricted weak type (p,p) if there exists $c>0$ such that

$$\varphi_\mu(\lambda) \leq c\lambda^{-p} \quad (0<\lambda<1). \tag{3.5}$$

By the well-known interpolation theorem of Stein and Weiss [2, p. 233], (3.5) implies the strong type (q,q) of M_μ for $q>p$. The following proposition shows first that a slightly better estimate than (3.5) allows us to get the strong type (p,p), and, second, it yields a very simple proof of the Stein–Weiss theorem for M_μ.

Proposition 3.4. *If $\int_0^1 \varphi_\mu(\lambda)^{1/q}\,d\lambda < \infty$, then M_μ is bounded on L^q_μ and*

$$\|M_\mu f\|_{L^q_\mu} \leq \Big(\int_0^1 \varphi_\mu(\lambda)^{1/q}\,d\lambda\Big)\|f\|_{L^q_\mu} \quad (1\leq q<\infty).$$

Proof. By (2.2),

$$\frac{1}{\mu(Q)}\int_Q |f|\,d\mu = \int_0^1 (f\chi_Q)^*_\mu(\lambda\mu(Q))\,d\lambda,$$

and thus,

$$M_\mu f(x) \le \int_0^1 m_{\lambda,\mu} f(x)\,d\lambda.$$

Applying Minkowski's inequality along with Proposition 2.3 yields

$$\|M_\mu f\|_{L^q_\mu} \le \int_0^1 \|m_{\lambda,\mu} f\|_{L^q_\mu}\,d\lambda \le \Big(\int_0^1 \varphi_\mu(\lambda)^{1/q}\,d\lambda\Big)\|f\|_{L^q_\mu},$$

as required. □

References

[1] Asekritova, I. U., N. Ya. Krugljak, L. Maligranda, and L.-E. Persson: Distribution and rearrangement estimates of the maximal function and interpolation. - Studia Math. 124:2, 1997, 107–132.

[2] Bennett, C., and R. Sharpley: Interpolation of operators. - Academic Press, New York, 1988.

[3] Bojarski, B.: Remarks on the stability of reverse Hölder inequalities and quasi-conformal mappings. - Ann. Acad. Sci. Fenn. Ser. A I Math. 10, 1985, 89–94.

[4] Busemann, H., and W. Feller: Zur Differentiation der Lebesgueschen Integrale. - Fund. Math. 22, 1934, 226–256.

[5] Coifman, R. R., and C. Fefferman: Weighted norm inequalities for maximal functions and singular integrals. - Studia Math. 15, 1974, 241–250.

[6] Cordoba, A., and R. Fefferman: On the equivalence between the boundedness of certain classes of maximal and multiplier operators in Fourier analysis. - Proc. Natl. Acad. Sci. USA 74, 1977, 423–425.

[7] Franciosi, M.: Weighted rearrangements and higher integrability results. - Studia Math. 92:2, 1989, 131–139.

[8] Franciosi, M.: On weak reverse integral inequalities for mean oscillations. - Proc. Amer. Math. Soc. 113:1, 1991, 105–112.

[9] Gehring, F. W.: The L^p-integrability of the partial derivatives of a quasiconformal mapping. - Acta Math. 130, 1973, 265–277.

[10] Gurov, L. G.: On the stability of Lorentz mappings. Estimates for derivatives. - Dokl. Akad. Nauk 220, 1975, 273–276 (Russian); Engl. transl.: Sov. Math. Dokl. 16, 1975, 56–59.

[11] Gurov, L. G., and Yu. G. Reshetnyak: A certain analogue of the concept of a function with bounded mean oscillation. - Sibirsk. Mat. Zh. 17:3, 1976, 540–546 (Russian).

[12] de Guzman, M.: Real variable methods in Fourier analysis. - North-Holland Math. Stud. 46, North-Holland Publishing Company, Amsterdam–New York–Oxford, 1981.

[13] Hunt, R., and D. S. Kurtz: The Hardy–Littlewood maximal function on $L(p,1)$. - Indiana Univ. Math. J. 32, 1983, 155–158.

[14] Iwaniec, T.: On L^p-integrability in PDEs and quasiregular mappings for large exponents. - Ann. Acad. Sci. Fenn. Ser. A I Math. 7:2, 1982, 301–322.

[15] John, F., and L. Nirenberg: On functions of bounded mean oscillation. - Comm. Pure Appl. Math. 14, 1961, 415–426.

[16] Korenovskii, A, A.: The connection between mean oscillations and exact exponents of summability of functions. - Mat. Sb. 181:12, 1990, 1721–1727 (Russian); Engl. transl.: Sb. Math. 71:2, 1992, 561–567.

[17] Korenovskyy, A. A., A. K. Lerner, and A. M. Stokolos: A note on the Gurov–Reshetnyak condition. - Math. Res. Lett. 9:5-6, 2002, 579–585.

[18] Mateu, J., P. Mattila, A. Nicolau, and J. Orobitg: BMO for nondoubling measures. - Duke Math. J. 102:3, 2000, 533–565.

[19] Milman, M.: A note on interpolation and higher integrability. - Ann. Acad. Sci. Fenn. Math. 23:1, 1998, 169–180.

[20] Muckenhoupt, B.: Weighted inequalities for the Hardy maximal function. - Trans. Amer. Math. Soc. 165, 1972, 207–227.

[21] Orobitg, J., and C. Pérez: A_p weights for nondoubling measures in $\mathbf{R}^n$ and applications. - Trans. Amer. Math. Soc. 354:5, 2002, 2013–2033.

[22] Sjögren, P.: A remark on the maximal function for measures in $\mathbf{R}^n$. - Amer. J. Math. 105, 1983, 1231–1233.

[23] Sjögren, P., and F. Soria: Sharp estimates for the non-centered maximal operator associated to Gaussian and other radial measures. - Adv. Math. 181:2, 2004, 251–275.

[24] Strömberg, J.-O.: Bounded mean oscillation with Orlicz norms and duality of Hardy spaces. - Indiana Univ. Math. J. 28, 1979, 511–544.

[25] Vargas, A.: On the maximal function for rotation invariant measures in $\mathbf{R}^n$. - Studia Math. 110:1, 1994, 9–17.

[26] Wik, I.: Note on a theorem by Reshetnyak–Gurov. - Studia Math. 86:3, 1987, 287–290.

Received 15 June 2006