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Abstract

We extend the result of Uzawa (1962) to settings which accommodate non-constant returns to scale. Therefore, the use of a cost function to estimate Allen/Uzawa elasticities of substitution is legitimate without assuming linear homogeneity in the production function.

1 Introduction

The Allen elasticity of substitution σ is a feature of the production technology that can be used to gauge how easily inputs can be substituted in production. It measures the responsiveness of relative input demand to changes in relative factor prices. If σ is high, then factor inputs can be easily substituted. In this case, a rapid rise in the price of one input will result in a large decrease in demand for that input in favour of other inputs. It should also lead to a small rise in overall cost, *ceteris paribus*.

The elasticity of substitution has been studied for 75 years. The elasticity between capital and labour is a key parameter in macroeconomics. It determines the level or growth rate of per-capita income, the speed of convergence and the relative importance of productive factors and technical efficiency in explaining cross-country differences in output per worker. In the short term, it has an effect on the monetary-policy transmission mechanism and the potential relationship between interest rates and employment (Chirinko, 2008). A recent volume of the Journal of Macroeconomics was dedicated to the study of this parameter (Klump & Papageorgiou, 2008).

It is not surprising that the oil price shocks in the 1970s spurred many studies on the elasticity of substitution between energy and other inputs in production.[1] Recent swings in energy and other commodity prices make this parameter no less relevant today. Furthermore, thinking of energy as an output, investigations of the elasticity of substitution in power generation are used to evaluate the effectiveness of policies to reduce pollution through substitution away from high-pollution inputs to low-pollution inputs.[2]

The elasticity of substitution between capital, skilled labour and unskilled labour informs the debate about causes of documented rising wage inequality. For example, estimates suggest that capital and unskilled labour are more substitutable than capital and less skilled labour. By this capital-skill-complementarity argument, due to Griliches (1969), increased availability of capital has decreased relative demand for unskilled labour. Through Marshall's Rules, together with information on a factor's share, σ can be used to estimate the elasticity of factor demand, which for example can inform the impact of a rise in minimum wages (Hamermesh, 1993). It can also be used to gauge the potential impact of education on wage inequality (Bowles, 1969).

Many studies compute σ by estimating cost functions rather than production functions. One reason is that the elasticities can be non-linear or inverse functions of the production parameters estimated, which can greatly magnify the standard errors (Hamermesh, 1993). When estimating flexible technologies, for example the translog, estimation of the cost function is far more tractable. Furthermore, standard profit-maximizing conditions can be used to improve the efficiency of the estimates (Binswanger, 1974). In the case where firm-level data is used, it is arguably more appropriate to estimate a cost function because the factor prices are exogenous and the inputs used are endogenous.

Estimation of the elasticity via a cost function makes use of a result due to Uzawa (1962). He exploits the duality between production function $q = f(x_1, x_2, ..., x_n)$ and cost function $C = C(w_1, w_2, ..., w_n, q)$ to express the elasticity of substitution as

$$\sigma_{ij} = \frac{CC_{ij}}{C_i C_j}, \tag{1}$$

where C_i, C_j are first derivatives with respect to the prices of factors x_i, xj and C_{ij} is the cross partial derivative. It is for this reason that σ is often referred to as the Allen/Uzawa elasticity.

The survey on labor demand by Hamermesh (1986) states equation (1) only holds if the production technology is linearly homogeneous (pg 433).

[1] A pioneering study is by Christensen & Greene (1976).

[2] For a review, see Soderholm (1998). Recent work on this includes Tuthull (2007).

Indeed, Uzawa's proof uses a unit cost function, which only uniquely represents the underlying production function under linear homogeneity (Varian, 1992) and his result thus appears strictly applicable to constant returns to scale only.

However, while it is common to impose constant returns in estimation, many studies use this result in more general settings. For example, of the twelve listed in the review by Chung (1994), only five have linearly homogeneous production technologies imposed. In a seminal paper on United States electric power generation, Christensen & Greene (1976) present a translog cost function that allows for varying returns to scale before invoking Uzawa's result to describe how to calculate the elasticity of substitution. There is evidence of non-constant returns to scale in electric power generation (Hisnanick & Kymn, 1999) and more generally in production (Basu & Fernald, 1997).

In fact, Binswanger (1974) does not rely on constant returns when presenting the cost fuction elasticity. However, the result is somewhat hidden in the paper. This may be why, to our knowledge, there is no explicit reference to a result of this type by individual papers or survey articles. Therefore it is instructive to confirm and document this in a note tailored for the purpose.

2 Generalizing Uzawa

For n factors, the elasticity of substitution between factors x_i and x_j is (Allen, 1938):

$$\sigma_{ij} = \frac{\mathbf{f}_{ij} \sum_k^n f_k x_k}{|\mathbf{f}| x_i x_j} \tag{2}$$

$|\mathbf{f}|$ is the determinant of the bordered Hessian of equilibrium conditions for a firm's cost-minimizing factor demands, holding output constant. $\mathbf{f}_{ij}$ is the cofactor of f_{ij} in $\mathbf{f}$. By Euler's theorem, the summation term equals q under constant returns to scale. Output and the prices of other factor prices are held constant.

Theorem 1 *For any production function $q = f(x_1, x_2, ..., x_n)$, we can use cost function $C = C(w_1, w_2, ..., w_n, q)$ to express the elasticity of substitution as:*

$$\sigma_{ij} = \frac{CC_{ij}}{C_i C_j} \tag{3}$$

where C_i, C_j are first derivatives with respect to the costs of factors i, j - w_i, w_j- and C_{ij} is the cross partial derivative.

Proof. The conditional factor demands are derived from the cost minimization problem:

$$\min \sum_i w_i x_i \text{ subject to } q = f(x_1, x_2, ..., x_n) \tag{4}$$

The firm's first order conditions are, where μ is the Lagrange Multiplier,

$$\begin{aligned} w_i &= \mu f_i (i = 1, ..., n), && (5) \\ q &= f(x_1, x_2, ..., x_n) && (6) \end{aligned}$$

and the cost function is

$$C(w_1, w_2, ..., w_n, q) = \sum_i w_i x_i(w_1, w_2, ..., w_n, q) \tag{7}$$

Following Allen (1938) but without assuming constant returns to scale, differentiate the first order conditions with respect to w_1 and divide each equation by μ:

$$\begin{array}{cccccc} 0 & +f_1 \frac{\partial x_1}{\partial w_1} & +f_2 \frac{\partial x_2}{\partial w_1} & +. & +f_n \frac{\partial x_n}{\partial w_1} & = 0 \\ \frac{1}{\mu} f_1 \frac{\partial \mu}{\partial w_1} & +f_{11} \frac{\partial x_1}{\partial w_1} & +f_{12} \frac{\partial x_1}{\partial w_1} & +. & +f_{1n} \frac{\partial x_1}{\partial w_1} & = \frac{1}{\mu} \\ \frac{1}{\mu} f_2 \frac{\partial \mu}{\partial w_1} & +f_{21} \frac{\partial x_1}{\partial w_1} & +f_{22} \frac{\partial x_1}{\partial w_1} & +. & +f_{2n} \frac{\partial x_1}{\partial w_1} & = 0 \\ . & . & . & . & . & . \\ \frac{1}{\mu} f_n \frac{\partial \mu}{\partial w_1} & +f_{n1} \frac{\partial x_1}{\partial w_1} & +f_{n2} \frac{\partial x_1}{\partial w_1} & +. & +f_{nn} \frac{\partial x_1}{\partial w_1} & = 0 \end{array} \tag{8}$$

By Cramer's Rule,

$$\frac{\partial x_2}{\partial w_1} = \begin{vmatrix} 0 & f_1 & 0 & . & f_n \\ q_1 & f_{11} & \frac{1}{\mu} & . & f_{1n} \\ q_2 & f_{12} & 0 & . & f_{2n} \\ . & . & . & . & . \\ q_n & f_{1n} & 0 & . & f_{nn} \end{vmatrix} \div \begin{vmatrix} 0 & f_1 & f_2 & . & f_n \\ f_1 & f_{11} & f_{12} & . & f_{1n} \\ . & . & . & . & . \\ . & . & . & . & . \\ f_n & f_{n1} & f_{n2} & . & f_{nn} \end{vmatrix}, \tag{9}$$

so

$$\frac{\partial x_2}{\partial w_1} = \frac{1}{\mu} \frac{\mathbf{f}_{12}}{|\mathbf{f}|} \tag{10}$$

As in (2), $|\mathbf{f}|$ is the determinant of the bordered Hessian of equilibrium conditions for a firm's cost-minimizing factor demands, holding output constant, and $\mathbf{f}_{12}$ is the cofactor of f_{12} in $\mathbf{f}$. By (2):

$$\sigma_{21} = \frac{\sum_n f_i x_i}{x_1 x_2} \mu \frac{\partial x_2}{\partial w_1} \tag{11}$$

By the first order conditions,

$$\mu \sum_{n} f_i x_i = C \tag{12}$$

and, by Shephard's Lemma,

$$x_j = C_j \tag{13}$$

such that $\frac{\partial x_2}{\partial w_1} = C_{21}$. Thus $\sigma_{21} = \frac{CC_{21}}{C_1 C_2}$. ■

We have generalized the Uzawa (1962) result, which allows us to use cost function parameters to find σ.

By Slutsky symmetry (and more directly Euler's Theorem), $\frac{\partial x_2}{\partial w_1} = \frac{\partial x_1}{\partial w_2}$ and $C_{21} = C_{12}$, so $\sigma_{21} = \sigma_{12.}$ Furthermore, using equations (11) - (13), $\sigma_{12} = \frac{C}{x_1 x_2}\frac{\partial x_1}{\partial w_2} = \frac{C}{w_2 x_2}\frac{\partial \log x_1}{\partial \log w_2}$. Defining $s_j = \frac{w_j x_j}{C}$ as the cost share of x_j and $\bar{\lambda}_{ij}$ as the constant output cross elasticity of factor demand, it follows that $\sigma_{12} = \frac{\bar{\lambda}_{12}}{s_2}$ or, for any pair i, j,

$$\bar{\lambda}_{ij} = s_j \sigma_{ij} \tag{14}$$

in accordance with Marshall's Rules. Some texts (eg Heathfield & Wibe, 1987:61) assert the relationship between $\bar{\lambda}$ and σ holds only under conditions of constant returns, but we have confirmed it holds for more general technologies.

3 Conclusion

The use of cost functions to estimate elasticities of substitution is widespread. Applications include the analysis of substitutability between labour types and, with renewed interest, between methods of generating power. The legitimate use of cost functions to estimate this technological parameter rests on a result due to Uzawa (1962), which assumes a linearly homogeneous technology. On the one hand, it has been asserted that the result only holds under constant returns to scale. On the other, empirical studies have applied Uzawa's result to more general technologies. We have documented that the elasticity of substitution can be expressed in terms of the cost function for more general technologies. It is thus legitimate to use cost functions to estimate this parameter even in industries where returns to scale may not be assumed constant.

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