Measuring Eddy Heat and Constituent Fluxes with High-Resolution Na and Fe Doppler Lidars

Chester S Gardner, University of Illinois at Urbana-Champaign
Alan Z Liu, Embry-Riddle Aeronautical University - Daytona Beach
Measuring eddy heat, constituent, and momentum fluxes with high-resolution Na and Fe Doppler lidars

Chester S. Gardner1 and Alan Z. Liu2
1Department of Electrical and Computer Engineering, University of Illinois at Urbana-Champaign, Urbana, Illinois, USA,
2Department of Physical Sciences, Embry-Riddle Aeronautical University, Daytona Beach, Florida, USA

Abstract Vertical transport by turbulent mixing plays a fundamental role in establishing the thermal and constituent structure of the upper mesosphere and lower thermosphere (MLT). Because of observational challenges, eddy heat, constituent, and momentum fluxes, and the associated coefficients for thermal ($k_{th}$), constituent ($k_{cz}$), and momentum ($k_{M}$), diffusion have not been well characterized in the MLT. We show that properly configured Na and Fe Doppler lidars, with sufficient resolution to observe the turbulence-induced wind, temperature, and density fluctuations, can make direct measurements of eddy fluxes throughout the mesopause region. When the horizontal ($\Delta \theta / \theta_z$ with $\theta = \text{altitude}$ and $\theta_z = \text{full width at half maximum laser divergence}$), vertical ($\Delta z$), and temporal ($\Delta t$) resolutions of the lidar satisfy the condition $L_c \approx \sqrt{(3\Delta \theta / \theta_z)^2 + (0.6\Delta \theta \Delta t)^2 + (1.5\Delta z)^2} \lesssim 270 \text{ m (125 m)}$, where $\theta$ is the mean horizontal wind velocity, the observations include more than 80% (90%) of the energy in the turbulence fluctuations, and the observed fluxes and derived diffusivities will be highly representative of the actual values. For existing Na and Fe Doppler lidars, which have modest power-aperture products of about 1 W m$^2$, long averaging times (5–20 h) are required to obtain statistically significant estimates of the eddy fluxes, $k_{cz}$, $k_{M}$, and $k_{M}$, profiles, and the turbulent Prandtl number ($Pr = k_M/k_{th}$) between about 85 and 100 km. These systems are capable of measuring the weekly or monthly mean flux and diffusivity profiles. Systems with power-aperture products of 5–100 W m$^2$ or larger could be used to study the eddy fluxes generated by the dissipation and breaking of individual gravity waves to altitudes as high as the turbopause (∼110 km).

1. Introduction

Gravity waves, tides, and planetary waves play major roles in establishing the thermal structure and general circulation of the upper mesosphere and lower thermosphere (MLT). Less well known and understood are the equally important roles that waves play in the vertical transport of atmospheric constituents, a fundamental process that has profound effects on the chemistry and composition of the atmosphere below the turbopause [e.g., Walterscheid, 1981; Walterscheid et al., 1987; Xu et al., 2003; Gardner and Liu, 2010; Zhu et al., 2010]. Gravity waves, tides, and planetary waves contribute to vertical transport by inducing advection, turbulent mixing, dynamical transport, and chemical transport. Although the physics of these four processes are fundamentally different, each mechanism gives rise to effective transport velocities as high as several cm/s in the mesopause region. Consequently, all four mechanisms can produce substantial vertical fluxes of atmospheric constituents, which directly affect the chemistry and structure of the MLT. Because wave-induced transport applies to every constituent of the atmosphere, knowledge of the magnitude, geographic distribution, and seasonal variability of all four mechanisms is important to a wide range of research problems, including general circulation modeling, atmospheric chemistry modeling, thermal balance calculations, and the study of the mesospheric metal and airglow layers.

Because constituent transport is difficult to measure, eddy diffusion parameterization schemes are commonly used to account for vertical transport in atmospheric chemistry models. Recent work has shown these schemes to be inadequate for modeling some MLT species, in large part, because the eddy diffusivity profiles and their seasonal and geographic variations are poorly known [e.g., Chabrillat et al., 2002; Gardner et al., 2010]. Fortunately, Doppler metal lidar techniques, coupled with large-aperture telescopes, now have the capability to directly measure the key transport processes including eddy heat and constituent fluxes, dynamical constituent fluxes, and, in special circumstances, the chemical fluxes of certain species. In addition, lidar, radar, and satellite measurements of horizontal winds and their latitudinal dependence can be used to infer the vertical advective transport associated with wave forcing of the global meridional circulation system.
[e.g., Fauliot et al., 1997]. While the direct measurement of dynamical and chemical transport with Doppler lidars has been achieved (for Na, Fe, and temperature) [e.g., Gardner and Liu, 2010], the measurement of eddy and advective transport continues to be an important but challenging goal for experimentalists.

In this paper, we address the observational requirements and resulting accuracies of Na and Fe Doppler lidar measurements of eddy heat, constituent, and momentum fluxes and the related coefficients for thermal \( (k_d) \), constituent \( (k_{d2}) \), and momentum \( (k_M) \) diffusion. In particular, we derive expressions for the vertical, horizontal, and temporal resolutions required to capture the majority of the energy in the turbulence-induced fluctuations. We also derive expressions for eddy flux and diffusivity errors caused by statistical and photon noise and evaluate the expected performance capabilities of modern Na and Fe lidar systems as a function of their power-aperture products. The results show that scientifically useful measurements of the eddy fluxes and diffusivity profiles can be obtained with existing Doppler lidar systems.

2. Wave-Induced Vertical Transport Processes

Vertical transport of heat and atmospheric constituents (i.e., gaseous species) is characterized by their vertical fluxes, which are defined as the expected values of the product of the instantaneous temperature \( (T) \) or constituent density \( (\rho) \) and the vertical wind \( (w) \). If we ignore, for the moment, the fluctuations associated with large-scale planetary waves and tides, the instantaneous vertical wind, temperature, and constituent density may be written as follows:

\[
\begin{align*}
    w &= \overline{w} + \omega_{GW} + \omega_{Turb} \\
    T &= \overline{T} + \overline{T_{GW}} + \overline{T_{Turb}} \\
    \rho &= \overline{\rho} + \rho_{GW} + \rho_{Turb}
\end{align*}
\]  

(1)

where the overbars denote the mean values and the primed quantities represent the gravity wave (GW) and turbulence-induced fluctuations. Planetary waves and tides can contribute to transport through wave–wave interactions and other mechanisms. Because it is not possible to distinguish the sources of the turbulence fluctuations observed by the lidar, the turbulence contributions by tides and planetary waves are included in (1) even though, for notational convenience, we have not explicitly accounted for these large-scale waves (i.e., for \( \omega_{\text{tide}} + \omega_{\text{PW}} \)). Because the wave and turbulence fluctuations are mutually independent (and uncorrelated) zero mean processes, the heat and constituent fluxes are given by

\[
\begin{align*}
    \overline{wT} &= (\overline{w} + \omega_{GW} + \omega_{Turb}) \overline{(T + T_{GW} + T_{Turb})} = \overline{wT} + \omega_{GW} \overline{T_{GW}} + \omega_{Turb} \overline{T_{Turb}} \\
    \overline{w\rho} &= (\overline{w} + \omega_{GW} + \omega_{Turb}) (\overline{\rho} + \rho_{GW} + \rho_{Turb}) = \overline{w\rho} + \omega_{GW} \rho_{GW} + \omega_{Turb} \rho_{Turb}
\end{align*}
\]  

(2)

Four mechanisms, based upon different physical processes and operating at different spatial scales, contribute to wave-induced heat and constituent transport, viz., advection, dynamical transport, chemical transport, and turbulent mixing.

**Advevtive transport** arises when the long-term average of the vertical winds is different from zero. Wave forcing of the meridional circulation system is a global-scale phenomenon that induces a summertime upwelling and wintertime downwelling in the mesopause region that is strongest at the poles and negligible at the equator [e.g., Garcia and Solomon, 1994]. This vertical motion exerts a significant influence on constituents and temperature at middle and high latitudes because the associated advection contributes to the vertical fluxes of all mesospheric species, while the concomitant adiabatic heating and cooling strongly influences the mesopause temperatures. The first term on the right-hand side of (2), \( \overline{w\rho} \), is the advective constituent flux associated with transport by the mean vertical wind. All constituents experience the same advective transport velocity.

The second term on the right-hand side of (2), \( \overline{w\rho_{GW}} \), is the constituent flux associated with local effects caused by nonbreaking waves propagating through the region, which perturb both the vertical winds and constituent densities. Two independent mechanisms contribute to this term, which we call the gravity wave dynamical and chemical fluxes.

\[
\overline{w_{GW}\rho_{GW}} = \text{dynamical flux} + \text{chemical flux}.
\]  

(3)

**Dynamical transport** is a mesoscale phenomenon that arises when the organized motions associated with dissipating but nonbreaking waves impart a net vertical displacement in each constituent when they
propagate through a region [e.g., Walterscheid et al., 1987; Liu and Gardner, 2004]. The dynamical constituent flux is proportional to the dynamical heat flux [Gardner and Liu, 2010]

\[
\left. w_{GW} R_{GW} \right|_{\text{dynamical}} = \frac{\left( g / R - \Gamma_{ad} \right)}{\left( \Gamma_{ad} + c T \partial T / \partial z \right)} \frac{w_{GW} T_{GW}}{T} \bar{p},
\]

where \( \Gamma_{ad} \approx 9.5 \) K/km is the adiabatic lapse rate of dry air, \( R = 287 \) m\(^2\)/K/s\(^2\) is the gas constant of the dry atmosphere, and \( g = 9.5 \) m/s\(^2\) is the acceleration of gravity. In the mesopause region dynamical transport of constituents is about 3 times faster than heat transport. Like advective transport, all constituents experience the same dynamical transport velocity.

Chemical transport of reactive species like Na, Fe, and atomic O is also a mesoscale phenomenon that occurs when the vertical advection and temperature perturbations caused by the wave-induced winds alter the chemical production and loss of the species. Both dissipating and nondissipating waves contribute to chemically induced transport whenever there is a nonzero correlation between the species fluctuations and the fluctuations in its chemical sources and sinks [Walterscheid and Schubert, 1989; Gardner and Liu, 2010]. If the wave-induced perturbations of the sources and sinks result in a net increase (decrease) of the constituent, the chemical flux is positive (negative) or upward (downward). In certain situations the chemical flux of a constituent can exceed the fluxes associated with all the other mechanisms. The effective chemical transport velocity is different for each constituent.

In this paper, we focus on turbulent mixing which is represented by the last terms on the right-hand sides of (2), \( w_{Turb} \nabla_{\text{Turb}} \) and \( w_{Turb} \nabla_{\text{Turb}} \). For notational simplicity we drop the subscripts so that in the subsequent text, prime quantities refer exclusively to turbulence-induced fluctuations. Eddy transport is the flux associated with the small-scale chaotic motions caused by breaking gravity waves, tides, and planetary waves [e.g., Hodges, 1969; Weinstock, 1978]. Turbulent mixing leads to a net vertical transport of constituents from regions of higher concentrations to those of lower concentrations. It has been studied extensively by treating the chemical transport whenever there is a nonzero correlation between the species fluctuations and the fluctuations in its chemical sources and sinks [Gardner and Liu, 2010]. Constituent transport is typically modeled as an eddy diffusion process, but there are large uncertainties in the assumed diffusion coefficients and their vertical profiles. Modelers employ sophisticated gravity wave parameterization schemes to predict \( k_{zz} \) and \( k_{T} \) and then make adjustments in the \( k_{zz} \) and \( k_{T} \) profiles so that the models generate realistic constituent and temperature profiles [e.g., Garcia and Solomon, 1994; Chabrillat et al., 2002; Plane, 2004; Marsh et al., 2007; Gardner et al., 2010]. Recent work has shown these schemes to be inadequate for modeling mesospheric CO\(_2\) and the...
meteoric metal layers, because the eddy diffusion parameterization is a crude proxy for the important
dynamical and chemical transport processes that are nondiffusive in nature and because the eddy diffusivity
profiles and their seasonal and geographic variations are poorly characterized [e.g., Chabrillat et al., 2002;
Gardner et al., 2010]. Fortunately, it is now possible to gain considerable insight about wave-induced
transport in the mesopause region using measurements collected with high power-aperture product Na and
Fe Doppler lidars.

Until now, measurements of eddy diffusivity profiles have been limited to observations of the spectral
broadening of backscattered radar pulses caused by turbulence-induced velocity fluctuations within the
radar scattering volume [e.g., Hocking, 1996; Latteck et al., 2005] to in situ observations of neutral density
fluctuations and their spectra made by rocket-borne ionization gauges [e.g., Lübken, 1997; Das and Sinha,
2010] and to observations of the expansion of chemiluminescent trails released in the upper atmosphere by
rockets. However, measurement of eddy heat and constituent flux profiles can also be made using ultrahigh-
resolution Na and Fe Doppler lidars that are sensitive to the small spatial and fast temporal scales of the
turbulence fluctuations (see Pformer and Hickson [2010] for examples of turbulence observed in Na
density profiles).

Recently, we reported direct measurements of the dynamical and chemical fluxes of atomic Na derived from
lidar observations of vertical winds, temperatures, and Na densities obtained at the Starfire Optical Range,
NM [Gardner and Liu, 2010]. We also presented indirect measurements of the vertical advective fluxes that
were derived from the continuity equation and the measured meridional winds. In this paper, we extend
those previous results by showing how to directly measure the eddy heat flux, the Na and Fe eddy fluxes, and
the associated diffusivity profiles \( k_\alpha \), and \( k_\beta \) using ultrahigh-resolution Doppler lidar data. We also show how to
measure the eddy momentum flux, the turbulent viscosity or equivalently, the coefficient of eddy
momentum diffusion \( (k_\alpha) \), and the turbulent Prandtl number \( (Pr = k_\alpha/k_\beta) \).

3. Spectra of Gravity Wave and Turbulence Fluctuations

Perturbations of atmospheric parameters arise from organized motions induced by propagating waves
and from chaotic, disordered motions caused by turbulence. Both motions contribute to vertical
classical transport. The smallest wave scales are associated with gravity waves. In the mesopause
region, the vertical wavelengths of gravity waves vary from several tens of kilometers to about 1 km. The
largest wavelengths are related to the vertical extent of the gravity wave sources in the lower atmosphere,
while diffusive and radiative damping eliminates all waves with vertical wavelengths shorter than about
1 km. Furthermore, gravity wave periods cannot be smaller than the buoyancy period, which is about
5 min. Turbulent eddies range in size from a few kilometers to a few tens of meters. The largest eddies are
generated by breaking gravity waves, which is the major source of turbulence in the mesopause region.
For scales smaller than a few tens of meters, the turbulent energy is dissipated as heat by viscous effects.
While there is no physical limit on turbulence time scales, the most energetic turbulence fluctuations occur
at times scales less than a few minutes. Note that we have ignored tides and planetary waves because their
spatial and temporal scales are much larger than the turbulence scales. However, the turbulence generated by
these large-scale disturbances affects the strength of the turbulence fluctuations and is included in our analysis.

Most theoretical and observational studies of gravity wave spectra have concentrated on explaining the
apparent universal power law form of the vertical wave number spectrum (see Gardner [1996] for an
overview of the leading theories). The canonical spectrum of wave-induced atmospheric density fluctuations
\( F_\nu (m) \), the power spectrum of the density fluctuations, is characterized by a vertical wave number \( m \) that
partitions the spectrum into a low wave number regime, which is dominated by the gravity wave source
characteristics. In this region the spectrum is usually assumed to be proportional to \( m^2 \) where \( s \sim 1 \). Because
\( m^2 = N/u^2 \), where \( N \) is the buoyancy frequency and \( u^2 \) is the root-mean-square (RMS) horizontal wind
perturbation, \( m^2 \) decreases with increasing altitude as \( u^2 \) increases. At mesopause altitudes where the RMS
horizontal wind fluctuations average about 25 m/s [Gardner and Liu, 2007], the vertical scale corresponding to
\( m = 2\pi/L_b^2 \) averages about 14 km [Senft and Gardner, 1991]. At large wave numbers greater than \( m_n \), but less
than the buoyancy wave number \( m_p = 2\pi/L_b \sim 2\pi/1 \) km, saturation and dissipation processes are believed to
control the spectrum. In this region the spectrum is proportional to \( N^2/l^2 \). At wave numbers larger than \( m_n \),

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diffusive and radiative damping eliminates the small vertical-scale waves so that spectrum rapidly falls to zero. This hypothetical gravity wave spectrum is plotted in Figure 1a for parameter values that are typical of the mesopause region. Numerous observations by atmospheric radars, lidars, and balloon sondes at many different sites have confirmed these general features of the gravity wave vertical wave number spectrum.

Shorter-scale fluctuations associated with vertical wave numbers greater than the buoyancy wave number arise primarily from turbulence. Turbulence is characterized by the formation of eddies of many different scale lengths. Most of the energy is contained in the large-scale eddies that are generated by breaking gravity waves, which then cascades to smaller and smaller eddies. Eventually, the eddies become so small that molecular diffusion becomes important, and the energy is dissipated by viscous effects. By using dimensional analysis, Kolmogorov [1941] was the first to provide a plausible explanation for the spectral shape of turbulence fluctuations, the famous $-5/3$ power law. Today, the turbulence spectral model proposed by Heisenberg [1948] is used widely to interpret observational data.

The 1-D spatial spectrum of turbulence fluctuations is characterized by a low wave number source region ($m \leq m_o=2\pi/l_o$), where it is assumed that the spectrum is proportional to $m^r$ ($0 < r$). In this region the spectrum is controlled by the source characteristics, and here the turbulent fluctuations are generally nonhomogeneous and highly anisotropic. At the smallest scales corresponding to large wave numbers in the inertial subrange ($m_o \leq m \leq m_i$) and the viscous dissipation range ($m_i=2\pi/l_i \leq m$), Kolmogorov and others have shown that the spectrum is proportional to $m^{-5/3}$ and $m^{-7}$, respectively. Here the fluctuations are statistically homogeneous and isotropic. The scale lengths corresponding to the breakpoints in the spectrum are called the outer ($l_o$) and inner ($l_i$) scales of turbulence. These scale lengths are variable as they depend on the kinematic viscosity $\nu$, the turbulent energy dissipation rate $\varepsilon$, and the buoyancy frequency $N$ [Weinstock, 1978, 1981; Lübken, 1997],

$$l_o = 9.97(\nu/N^3)^{1/2},$$
$$l_i = 9.90(\varepsilon/N)^{1/4}.\tag{7}$$

At mesopause heights, rocket-borne ionization gauge measurements of neutral density fluctuations have shown that the outer scale varies from a few hundred meters to almost 2 km, depending on the turbulence
strength, while the inner scale typically varies between about 10 and 30 m [e.g., Das and Sinha, 2010; Lübken et al., 1993; Lübken, 1997]. The turbulence $m$ spectrum is plotted in Figure 1a for parameter values that are typical of the mesopause region.

The gravity wave temporal spectrum extends from the inertial frequency ($f$) to the buoyancy frequency ($N$). Within this region observations have shown that the spectrum is approximately proportional to $\omega^{-p}$ where $p$ is typically between 5/3 and 2. The hypothetical $\omega$ spectrum for gravity waves is plotted in Figure 1b. The temporal spectrum of turbulence is usually modeled by invoking the well-known Taylor’s frozen turbulence hypothesis [Taylor, 1938], namely, the turbulent vortices are advected horizontally by the winds to generate the temporal fluctuations. This hypothesis is valid whenever the motions of the turbulent vortices are slow compared to the mean wind velocity. This condition is satisfied in the MLT where the turbulent wind fluctuations are typically a few m/s or less while the horizontal wind averages several tens of m/s. Consequently, the temporal spectrum is derived from the 1-D spatial spectrum by replacing the horizontal wave number by $\omega/|\bar{u}|$ and scaling the result by $1/|\bar{u}|$, where $\bar{u}$ is the mean horizontal wind velocity. This yields the familiar $\omega^{-5/3}$ spectral shape in the region $2\pi|\bar{u}|/l_0 \leq \omega \leq 2\pi|\bar{u}|/l$. The hypothetical $\omega$ spectrum for turbulence is plotted in Figure 1b for $|\bar{u}| = 10$ m/s, 20 m/s, and 50 m/s. Because gravity waves are the major source of turbulence in the MLT, the magnitudes of the gravity wave and turbulence spectra are related so that in reality both the total $m$ spectrum and total $\omega$ spectrum vary smoothly and continuously as the fluctuations transition from the larger scales dominated by gravity waves to the smaller scales dominated by turbulence.

Notice that wave and turbulence fluctuations occupy substantially different regions of the $m$ and $\omega$ spectra, with most of the gravity wave energy at scales larger than $l_g$ ($m < m_0$) and $T_g$ ($\omega < N$) and most of the turbulent energy at scales smaller than $l_g$ ($m_0 < m$) and $l_0/|\bar{u}|$ ($|\bar{u}|/m_0 < \omega$). Consequently, the measured data can be processed to isolate each component. For example, the Na, wind, and temperature observations measured at the Starfire Optical Range, NM [Gardner and Liu, 2007], were obtained with fundamental resolutions of 24 m and 90 s. In Gardner and Liu [2010], the data were low-pass filtered spatially to 500 m vertical resolution and low-pass filtered temporally to 2.5 min temporal resolution. Thus, the computed Na and heat fluxes only included perturbations in the spectral region $m < 2\pi/1$ km and $\omega < 2\pi/5$ min which are dominated by gravity waves, so the observations only included the gravity wave dynamical and chemical fluxes.

In theory, the turbulence contributions can be isolated by high-pass filtering the data temporally and perhaps, also spatially, to retain only those fluctuations with periods shorter than $2\pi/N = T_g \approx 5$ min and vertical wavelengths shorter than $l_g \sim 1$ km. In practice, this higher-resolution data will be contaminated by larger photon noise, which can impair the accuracy of the eddy flux measurements. However, if the lidar has a sufficiently large power-aperture product and the data are averaged long enough, accurate, scientifically useful measurements of the eddy fluxes and eddy diffusivity profiles ($k_{zz}$ and $k_{nn}$) can be obtained. The feasibility of this approach is analyzed in detail in the next section.

### 4. Measuring Eddy Constituent and Heat Fluxes

The eddy constituent and heat fluxes are defined, respectively, as the expected value of the product of the vertical wind and constituent density fluctuations caused by turbulence $\overline{\omega w \rho'}$ and the expected value of the product of the vertical wind and temperature fluctuations $\overline{\omega w T'}$. $w$, $\rho'$, and $T'$ are jointly Gaussian-distributed, zero mean random processes [Gardner and Yang, 1998]. Gaussian probability distributions are completely determined by the means, variances, and cross-correlation coefficients of the random variables. Measuring the eddy fluxes is equivalent to measuring the cross correlations between $w'$ and $\rho'$ and between $w'$ and $T'$. In practice, the wind, temperature, and density fluctuations are measured by the lidar over the height range of the Na and Fe layers and over long time periods. The fluxes are estimated by first computing the products of the measured vertical wind fluctuations and the temperature and Na/Fe density fluctuations. These instantaneous, point flux values are random variables whose means are the desired fluxes. The data processing goal is to obtain accurate estimates of the mean values of these products, as well as estimates of their uncertainties (i.e., variances of the means). The means are estimated by averaging the instantaneous, point flux values over altitude and time. The accuracies of the computed mean values depend on the variances of the individual flux estimates, which in turn depend on the variances of the measured wind, temperature, and densities, the height range and time period over which the instantaneous fluxes are
averaged, and the vertical correlation length and correlation time of the instantaneous fluxes. The variance of the means is simply equal to the variance of the instantaneous, point fluxes divided by the number of statistically independent measurements that were averaged. Of course, the measurements are contaminated by photon noise, which contributes to the flux variances and exhibits different correlation properties than the fluxes. In this section we derive expressions for the measured flux variances, including the effects of photon noise, and for the flux correlation length and correlation time, which depend on the turbulence spectrum.

The point estimate of the eddy fluxes (EF), their means, and variances are

$$\text{EF}_p = (w' + \Delta w)(\rho' + \Delta \rho),$$
$$\text{EF}_T = (w' + \Delta w)(T' + \Delta T),$$

$$\text{EF}_p = (w' + \Delta w)(\rho' + \Delta \rho) = \overline{w'\rho'} + \Delta w\Delta \rho,$$
$$\text{EF}_T = (w' + \Delta w)(T' + \Delta T) = \overline{w'T'} + \Delta w\Delta T$$

and

$$\text{Var}(\text{EF}_p) = \left( \overline{w'\rho'} + \Delta w\Delta \rho \right)^2 + \overline{\text{Var}(w')} + \overline{\text{Var}(\Delta w)} + \overline{\text{Var}(\rho')} + \overline{\text{Var}(\Delta \rho)}$$
$$\approx \overline{\text{Var}(w')} + \overline{\text{Var}(\Delta w)} + \overline{\text{Var}(\rho')} + \overline{\text{Var}(\Delta \rho)},$$
$$\text{Var}(\text{EF}_T) = \left( \overline{w'T'} + \Delta w\Delta T \right)^2 + \overline{\text{Var}(w')} + \overline{\text{Var}(\Delta w)} + \overline{\text{Var}(\rho')} + \overline{\text{Var}(\Delta \rho)}$$

where $$\Delta w$$, $$\Delta T$$, and $$\Delta \rho$$ are the zero mean measurement errors caused by photon noise in the lidar measurements and $$\overline{\text{Var}()}$$ denotes the variance of the parameter. The right-hand sides of (10) follow from the fact that the fluxes and their photon noise biases are very small compared to the variances of the turbulence fluctuations. Equations (9) and (10) are derived from (8) by noting that the measurement errors and the wind, temperature, and density fluctuations are mutually independent, zero mean, Gaussian-distributed random processes. To derive (10) we used the following identity for Gaussian processes.

$$\overline{(w'\rho')^2} = \overline{w'\rho'}^2 = 2\overline{(w')^2} + 2\overline{(\rho')^2} + \overline{\text{Var}(w')}\overline{\text{Var}(\rho')}$$
$$w' = w' + \Delta w$$
$$\rho' = \rho' + \Delta \rho.$$

Because the wind, temperature, and density are derived from the same three lidar photon count profiles, measured at three different frequencies within the Na or Fe fluorescence spectrum [e.g., Gardner, 2004], the photon noise errors for these parameters are slightly correlated. Hence, the estimated fluxes include the small biases $$\Delta w\Delta \rho$$ and $$\Delta w\Delta T$$. These biases can be calculated using the photon count data and subtracted from the estimated fluxes to provide a bias-free measurement.

To reduce the variance of the flux estimates, the instantaneous fluxes are averaged over time or altitude or both

$$\text{EF}_p = \frac{1}{L} \int_{z-L/2}^{z+L/2} \overline{dz} \int_{t-L/2}^{t+L/2} \overline{dt} (w' + \Delta w)(\rho' + \Delta \rho),$$
$$\text{EF}_T = \frac{1}{L} \int_{z-L/2}^{z+L/2} \overline{dz} \int_{t-L/2}^{t+L/2} \overline{dt} (w' + \Delta w)(T' + \Delta T),$$

where $$t$$ and $$L$$ are, respectively, the averaging period and height range and $$t$$ and $$z$$ are, respectively, the time and altitude of the flux estimates. To proceed we follow the approach of Gardner and Yang [1998] and express the variances of the spatially and temporally averaged fluxes in terms of the two-sided power spectra of the wind, temperature, and density fluctuations caused by turbulence and photon noise.

$$\overline{\text{Var}(\text{EF}_p)} \approx \frac{1}{(2\pi)^2 L} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \overline{dm} \overline{F_w(\omega, m) + F_{\Delta w}(\omega, m)} \overline{F_{\rho'}(\omega, m) + F_{\Delta \rho}(\omega, m)}$$
$$\overline{\text{Var}(\text{EF}_T)} \approx \frac{1}{(2\pi)^2 L} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \overline{dm} \overline{F_w(\omega, m) + F_{\Delta w}(\omega, m)} \overline{F_T(\omega, m) + F_{\Delta T}(\omega, m)}.$$
The spectra of the measurement errors introduced by photon noise are white and therefore separable.

\[
F_{w}(\omega, m) = \Delta t \Delta \omega \text{Var}(\Delta w), \quad 0 \leq |\omega| \leq \pi/\Delta t \text{ and } 0 \leq |m| \leq \pi/\Delta z,
\]

\[
F_{\rho}(\omega, m) = \Delta t \Delta \omega \text{Var}(\Delta \rho), \quad 0 \leq |\omega| \leq \pi/\Delta t \text{ and } 0 \leq |m| \leq \pi/\Delta z, \tag{14}
\]

\[
F_{\Delta T}(\omega, m) = \Delta t \Delta \omega \text{Var}(\Delta T), \quad 0 \leq |\omega| \leq \pi/\Delta t \text{ and } 0 \leq |m| \leq \pi/\Delta z.
\]

Because the turbulence fluctuations are assumed to be homogeneous and isotropic, the vertical and temporal resolutions of the fundamental wind, temperature, and density spectra all have the same shape. But, unlike the photon noise, the joint turbulence spectra are not separable. Furthermore, because of Taylor's frozen turbulence hypothesis, the joint \((\omega, m)\) spectra are derived from the joint \((k, m)\) spectra, where \(k\) is the zonal wave number, by setting \(k = \omega/|m|\) and scaling the result by \(1/|\mathcal{N}|\). To simplify the integrals in (13) we approximate the two-sided turbulence \((\omega, m)\) spectrum by the following piecewise continuous model (see Appendix A).

\[
F_{w}(\omega, m) = \frac{F_{w}(|k| = \omega/|\mathcal{N}|, m)}{|\mathcal{N}|} = \frac{F_{w}(k = \sqrt{(\omega/|\mathcal{N}|)^{2} + m^{2}})}{|\mathcal{N}|} H(k/\kappa_{0})
\]

\[
F_{w}(\omega, m) = \frac{3 \text{Var}(w')^{2} / (\pi |\mathcal{N}|)}{(9r + 15)/(r + 1) - 8(l_{0}/l_{0})^{2/3}} \left(\frac{\kappa_{0}}{\kappa}\right)^{r-1} H(k/\kappa_{0})
\]

\[
\left(\frac{\kappa_{0}}{\kappa}\right) H(k/\kappa_{0}) = \begin{cases} \left(\frac{\kappa_{0}}{\kappa}\right) \left(\frac{(\omega/|\mathcal{N}|)^{2} + m^{2}}{\kappa_{0}}\right)^{r-1} & 0 \leq (\omega/|\mathcal{N}|)^{2} + m^{2} \leq \kappa_{0} \\
\left(\frac{\kappa_{0}}{\kappa}\right) \left(\frac{(\omega/|\mathcal{N}|)^{2} + m^{2}}{\kappa_{0}}\right)^{8/3} & \kappa_{0} \leq (\omega/|\mathcal{N}|)^{2} + m^{2} \leq \kappa_{1} \\
\left(\frac{\kappa_{1}/\kappa_{0}}{\kappa}\right)^{16/3} \left(\frac{\kappa_{0}}{\kappa}\right) \left(\frac{(\omega/|\mathcal{N}|)^{2} + m^{2}}{\kappa_{0}}\right)^{8} & \kappa_{1} \leq (\omega/|\mathcal{N}|)^{2} + m^{2}
\end{cases} \tag{15}
\]

The temperature and density spectra are obtained from (15) by replacing the vertical wind variance with the variance of the temperature and density, respectively.

By substituting (14) and (15) into (13) and carrying out the integrations, we obtain for the eddy flux variances

\[
\text{Var}(\mathcal{N}F_{T}) = \text{Var}(w' T) = \frac{\Delta z_{\mathcal{N}} \Delta t_{\mathcal{N}}}{L_{t}} \text{Var}(w' \cdot \text{Var}(T')
\]

\[
+ \frac{\Delta z_{\Delta t}}{L_{t}} \left[ \text{Var}(\Delta w) \cdot \text{Var}(\Delta \rho) + \text{Var}(w') \cdot \text{Var}(\Delta \rho) + \text{Var}(\Delta \omega) \cdot \text{Var}(\rho') \right]
\]

\[
\text{Var}(\mathcal{N}F_{T}) = \frac{\Delta z_{\mathcal{N}} \Delta t_{\mathcal{N}}}{L_{t}} \text{Var}(w' \cdot \text{Var}(T')
\]

\[
+ \frac{\Delta z_{\Delta t}}{L_{t}} \left[ \text{Var}(\Delta w) \cdot \text{Var}(\Delta T) + \text{Var}(w') \cdot \text{Var}(\Delta T) + \text{Var}(\Delta \omega) \cdot \text{Var}(T') \right]. \tag{16}
\]

where \(\Delta z\) and \(\Delta t\) are, respectively, the vertical and temporal resolutions of the fundamental wind, temperature, and density measurements. \(\Delta \mathcal{N} \Delta t_{\mathcal{N}}\) is the product of the effective flux correlation length and correlation time. This parameter, which is the same for the constituent and heat fluxes, is given by

\[
\Delta \mathcal{N} \Delta t_{\mathcal{N}} = \frac{1}{(2\pi)^{2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \text{d}m \text{d}m' \text{Var}(w') \text{Var}(\rho')
\]

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \text{d}m \text{d}m' \text{Var}(w') \text{Var}(T')
\]

\[
= \frac{1}{(2\pi)^{2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \text{d}m \text{d}m' \text{Var}(w') \text{Var}(T')
\]

\[
= \frac{1}{(2\pi)^{2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \text{d}m \text{d}m' \text{Var}(w') \text{Var}(T')
\]

\[
= \frac{1}{(2\pi)^{2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \text{d}m \text{d}m' \text{Var}(w') \text{Var}(T')
\]

\[
= \frac{1}{(2\pi)^{2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \text{d}m \text{d}m' \text{Var}(w') \text{Var}(T')
\]

\[
= \frac{1}{(2\pi)^{2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \text{d}m \text{d}m' \text{Var}(w') \text{Var}(T')
\]

\[
= \frac{1}{(2\pi)^{2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \text{d}m \text{d}m' \text{Var}(w') \text{Var}(T')
\]

\[
= \frac{1}{(2\pi)^{2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \text{d}m \text{d}m' \text{Var}(w') \text{Var}(T')
\]

\[
= \frac{1}{(2\pi)^{2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \text{d}m \text{d}m' \text{Var}(w') \text{Var}(T')
\]

The first terms on the right-hand sides of (16) are the statistical errors associated with estimating the mean values of the instantaneous fluxes. These statistical flux errors arise even if the wind, temperature, and density are measured perfectly, with no error. Statistical error can be reduced by averaging many statistically independent samples of the instantaneous fluxes. The number of independent flux samples is \(L_{t}/(\Delta \mathcal{N} \Delta t_{\mathcal{N}})\) = \(30 L_{t}/|\mathcal{N}|/l_{0}^{2}\). The second sets of terms on the right-hand sides of (16) are the additional flux errors associated
with the errors in the wind, temperature, and density measurements caused by photon noise. The photon noise errors can also be reduced by averaging.

5. Impact of Lidar Resolution

To measure the eddy heat and constituent fluxes, the lidar must be capable of observing the temporal and spatial scales of the fluctuations that make the largest contribution to turbulent mixing. In other words, the lidar must observe a significant fraction of the energy in the turbulence spectrum. While modern Doppler metal lidars can easily achieve vertical resolutions comparable to the inner scale of turbulence (10–20 m), the effects of the finite integration time of the measurements and the horizontal resolution, which is related to the laser divergence angle (θL), beam shape of the laser and field of view of the receiving telescope (θFOV), also need to be addressed. Horizontal averaging of the turbulence fluctuations within the beam, temporal averaging of the laser pulses, and range binning of the photon counts will reduce the lidar sensitivity to turbulence as the shorter spatial scales and faster temporal scales are eliminated from the measurements. Fortunately, the vertical wave number (m), horizontal wave number (k = √(k^2 + p^2)), and temporal frequency (ω) spectra of the turbulence fluctuations are red and proportional to m−5/3, k−8/3 = (k^2 + p^2)^−8/3, and ω−5/3, respectively, so that most of the turbulent energy is contained in the larger slower scales which are most responsible for the turbulent mixing. By properly designing the experiment, that is by choosing θL, Δz, and Δt to ensure adequate sensitivity to the turbulence fluctuations, while employing sufficiently large averaging intervals (L) and periods (τ) to minimize the error, it is possible to obtain accurate estimates of the eddy fluxes and eddy diffusivities for both heat and constituents.

The fundamental vertical wind, temperature, and species density measurements are made by integrating the backscattered photon counts over a time period Δt, altitude range Δz, and over the beam width of the laser Δx = Δy = zθL and field of view of the receiving telescope θFOV. We assume that the laser operates in the TEM00 mode (Gaussian beam shape) and that the full width at half maximum (FWHM) beam width zθL is related to the RMS beam width σL as follows:

\[ z\theta_L = 2\sqrt{2}\ln 2\sigma_L. \]  

(18)

We also assume that the telescope field of view is large compared to σL. In this case the observed density variance can be expressed in terms of the density spectrum and the filter functions associated with averaging the observed density fluctuations over time, altitude, and the laser beam width,

\[ \text{Var}(\rho_{\text{obs}}) = \frac{1}{2\pi} \int \int F_{\rho}(k, l, m) e^{-i(kx + ly)} \frac{\sin^2(k|\theta|\Delta t/2) \sin^2(m\Delta z/2)}{(k|\theta|\Delta t/2)^2 (m\Delta z/2)^2} \text{d}m\text{d}k, \]

\[ = \frac{1}{2\pi} \int \kappa^2 F_{\rho}(\kappa) G(\kappa, \sigma_L, \Delta z, |\theta|\Delta t) \text{d}\kappa, \]  

(19)

where \( F_{\rho}(\kappa) \) is the 3-D turbulence spectrum given by (A3). Equation (19) was derived by expressing the density fluctuations in terms of their Fourier transform, integrating over the spatial and temporal regions probed by the lidar beam, squaring the result, and then computing its expectation in terms of the power spectrum of the density fluctuations. The composite filter function, \( G(\kappa) \), is a positive function of \( \kappa^2 \) that generally decreases with increasing \( \kappa \) and so acts as a low-pass filter, eliminating the contributions of small-scale (i.e., large \( \kappa \)) turbulent vortices. Because we are mainly concerned with its behavior in the region \( \kappa \leq \kappa_C \), where \( \kappa_C \) is the effective cutoff wave number, we approximate \( G(\kappa) \) as follows.

\[ G(\kappa, \sigma_L, \Delta z, |\theta|\Delta t) \approx \begin{cases} 4\pi \left[ 1 - (\kappa/\sqrt{2}\kappa_C)^2 \right] & 0 \leq \kappa \leq \sqrt{2}\kappa_C \\ 0 & \text{otherwise} \end{cases} \]  

(20)
The cutoff wave number is computed by expanding the formula for \( G(\kappa) \) given by (19) in a power series in \( \kappa^2 \) and retaining only the first two terms.

\[
G(\kappa, \sigma_\ell, \Delta z, |\Delta|t) \approx \sum_{0}^{\infty} \sin \theta \left( 1 - \kappa^2 \left[ \frac{2^2 \sin^2 \theta}{12} + \frac{(|\Delta|t)^2}{\sin^2 \theta \cos^2 \phi + \frac{1}{12} \cos^2 \theta} \right] \right) \text{d}\phi \text{d}\theta
\]

\[
= 4\pi \left( 1 - \kappa^2 \left( \frac{2 \Delta^2}{3 \sigma^2_\ell} + \frac{(|\Delta|t)^2}{72\pi} \right) \right) = 4\pi \left( 1 - \left( \kappa/\sqrt{2\kappa_C} \right)^2 \right)
\]

\[\kappa_C = \left[ \frac{4 \sigma^2_\ell + \frac{(|\Delta|t)^2}{27\pi}}{36\pi} \right]^{1/2} = \frac{2\pi}{L_C}
\]

\[L_C = \sqrt{\frac{\Delta^2 \sin^2 \theta_0}{6} + \frac{\pi}{9} (|\Delta|t)^2 + \frac{2\pi^2}{9} \sigma^2_\ell} \approx \sqrt{\frac{\Delta^2 \sin^2 \theta_0}{6} + (0.6|\Delta|t)^2 + (1.5\Delta z)^2}
\]

\(L_C\) is the effective cutoff wavelength. The cutoff wave number \(\kappa_C\) is approximately equal to the half-maximum spatial bandwidth of the low-pass filter \(G(\kappa)\).

The fraction of the total density variance captured by the lidar is

\[
\text{Var}
\left( \frac{\rho_{\text{obs}}}{\rho} \right) \approx \left( \frac{1}{2\pi^3} \right)^2 \int_0^{\infty} \kappa^2 F_\rho(\kappa) G(\kappa, \sigma_\ell, \Delta z, |\Delta|t) \text{d}\kappa \left/ \frac{\text{Var}(\rho)}{\text{Var}(\rho)} \right.
\]

\[\approx \left\{ \begin{array}{ll}
\frac{6/(r + 1)}{(9r + 15)/(r + 1) - 8(l_0/l_0)^{2/3}} \left( \frac{l_0}{L_C} \right)^{r + 1} & l_0 \leq L_C \\
\frac{6/(r + 1)}{(9r + 15)/(r + 1) - 9(L_C/l_0)^{2/3}} & l_C \leq l_0
\end{array} \right.
\]

\[\approx \left[ 1 - \frac{(l_0/l_0)^{2/3}}{(9r + 15)/(r + 1) - 8(l_0/l_0)^{2/3}} \left( \frac{L_C}{l_0} \right)^6 \right] \leq l_C \leq l_0
\]

Because the wind and temperature spectra have the same shape as the density spectra, (22) also represents the fraction of the total wind or temperature variances captured by the lidar measurements. To illustrate lets consider typical Na and Fe lidar parameters of \( \Delta z = 50 \text{ m} \), \( \Delta t = 5 \text{ s} \), and \( \theta_0 = 0.5 \text{ mrad FWHM} \). In this case \( L_C = 180 \text{ m} \) for \( z = 90 \text{ km} \) and \( |\Delta|t = 30 \text{ m/s} \). For typical values of the turbulence parameters, \( r = 1 \), \( l_0 = 1.5 \text{ km} \), and \( l_1 = 30 \text{ m} \), (22) predicts that approximately 86% of the turbulent energy would be reflected in the wind, density, and temperature observations. This value increases to almost 90% if the laser divergence is reduced to 0.25 mrad FWHM (\( L_C = 135 \text{ m} \)). In both cases, the measured heat and constituent fluxes should be highly representative of the actual fluxes. The value of \( L_C \) is tabulated in Table 1 versus the percentage of the turbulent energy captured by the lidar observations. The observations will include more than 80% of the turbulent energy when \( L_C \) is less than 270 m. The percentage rises to more than 90% when \( L_C \) is less than 125 m. While the mean horizontal wind velocity in the mesopause region is typically on the order of 30 m/s or less, large-scale planetary waves and tides can generate winds approaching 100 m/s or more. In these circumstances, \( L_C \) will increase and the sensitivity to turbulence fluctuations will decrease unless \( \Delta t \) is reduced to compensate for the higher wind velocity.

### Table 1. Cutoff Wavelength \( L_C \) Versus Percentage of Observed Turbulent Energy \( r = 1 \) and \( l_0/l_0 = 1/50 \)

| Var(\( \rho_{\text{obs}} \)) | \( L_C \) | \( L_0 = \sqrt{(3z\theta_0)^2 + (0.6|\Delta|t)^2 + (1.5\Delta z)^2} \) at \( l_0 = 1.5 \text{ km} \) |
|--------------------------|--------|----------------------------------|
| 95%                      | 0.0464 | 69 m                             |
| 90%                      | 0.0843 | 126 m                            |
| 85%                      | 0.1293 | 194 m                            |
| 80%                      | 0.1802 | 270 m                            |
| 75%                      | 0.2365 | 355 m                            |
| 70%                      | 0.2977 | 447 m                            |
| 65%                      | 0.3634 | 545 m                            |
| 60%                      | 0.4333 | 650 m                            |
6. Eddy Flux Measurement Accuracy

The magnitude of the eddy heat flux at mesopause heights varies from about 0.2 to 2 K m/s. The magnitudes of the Na and Fe eddy fluxes vary from about 1000 to 10,000/cm²/s. Thus, to obtain scientifically useful measurements, the RMS flux errors should be at least several times smaller than these values. At mesopause heights, the variances of the turbulence-induced fluctuations are [Lübken et al., 1993; Lübken, 1997]

\[
\begin{align*}
\text{Var}(w') &= \text{Var}(u') \approx 0.1 - 4 \text{ m}^2/\text{s}^2 \\
\text{Var}(T') &= 0.04 - 4 \text{ K}^2 \\
\text{Var}(\rho_{\text{Atmos}}')/\langle \rho_{\text{Atmos}} \rangle^2 &= 0.01 - 1\%^2
\end{align*}
\]

Note that while the total atmospheric density variance induced by turbulence ranges from 0.01 to 1%², the relative constituent variances can be up to 10–15 times larger if the mean constituent profiles exhibit large density gradients such as exist on the top and bottom sides of the Na and Fe layers. If we use the worst-case variance values in (16) to estimate the eddy flux variances, then we need to average about 400 or more statistically independent flux samples to reduce the statistical errors to sufficiently small values. This requires \( L_T = 10^6 \) ms, depending on the values of the turbulence outer scale and the mean horizontal wind velocity, which could be accomplished by averaging the flux measurements for about 10 min and 1.5 km in altitude. Thus, when the lidar signal levels are so large that photon noise is negligible and statistical noise dominates the observations, the resolution of the measured eddy fluxes (\( l \) and \( r \)) is sufficient to study the turbulent mixing caused by the dissipation and breaking of even the smallest-scale individual gravity waves.

To assess the flux errors caused by photon noise, we need to consider how the data are processed to derive the wind, temperature, and density fluctuations. Measurements are made with Na and Fe Doppler lidars by sequentially tuning the laser to three different frequencies, \( f_0 - f_\delta f_\delta \) and \( f_0 + f_\delta \) and recording the backscattered photon counts. \( f_0 \) is nominally near the peak of the fluorescence spectrum and \( f_\delta \) is the frequency shift, which depends upon the lidar and is chosen to minimize some combination of the measurement errors [Gardner, 2004]. Numerous authors have analyzed the performance characteristics of three-frequency Doppler Na and Fe lidars [e.g., Bills et al., 1991; She et al., 1992; She and Yu, 1994; Papen et al., 1995; Gardner, 2004; Chu and Papen, 2005; Su et al., 2008; Gardner and Vargas, 2014]. To derive accurate estimates of winds, temperatures, and densities, the data processing algorithms must account for constituent isotopes, multiple hyperfine lines, the laser spectral shape, and, in certain situations, saturation of the fluorescence line [e.g., Welsh and Gardner, 1989; von der Gathen, 1991]. In practice, numerical physical models are used to map the observed photon counts into the correct temperature, wind, and density values. This approach works well as long as the signal levels are large and the photon noise is small. However, to measure the eddy fluxes, data must be acquired at very high temporal and vertical resolutions. Typically, the photon counts in each altitude-time bin will be small, even near the peaks of the Na and Fe layers. Because of nonlinear distortion caused by the much stronger photon noise, the usual data processing approaches cannot be easily adapted to derive \( w' \), \( T' \), and \( \rho' \).

Since the turbulence perturbations in the photon count data are small compared to the gravity wave perturbations and the mean values of the photon counts, it is much more robust to derive \( w' \), \( T' \), and \( \rho' \) directly from the concomitant photon count perturbations by linearizing the system equations. The first step in the process is to low-pass filter each of the three photon count profiles, temporally, and perhaps, also vertically, to retain only the mean values plus the gravity wave fluctuations. This can be accomplished by using a suitable smoothing function, such as a two-dimensional Hamming window. For example, applying a 2-D Hamming window with a temporal width of 2.5 min FWHM and vertical width of 500 m FWHM to each of the photon count profiles would eliminate most of the turbulence fluctuations but retain the mean plus all the gravity wave fluctuations with periods greater than 5 min and vertical wavelengths greater than 1 km. Therefore, the difference between the original high-resolution profile and the smoothed low-resolution profile includes only turbulence fluctuations and photon noise. If we denote the smoothed photon count profile acquired when the laser was tuned to the frequency \( f \) as \( \bar{N}(f) \), then the relative turbulence-induced fluctuations are given by

\[
\frac{N(f) - \bar{N}(f)}{\bar{N}(f)} = \frac{N(f) + \Delta N(f)}{\bar{N}(f)} \approx \frac{1}{\bar{N}(f)} \frac{\partial \bar{N}(f)}{\partial w} w' + \frac{1}{\bar{N}(f)} \frac{\partial \bar{N}(f)}{\partial T} T' + \frac{1}{\bar{N}(f)} \frac{\partial \bar{N}(f)}{\partial \rho} \rho' + \frac{\Delta N(f)}{\bar{N}(f)},
\]

(24)
where $\Delta N(f)$ is the photon noise. In deriving (24) we assume that the small turbulence fluctuations result in small linear perturbations of the photon counts. The sensitivity parameters, viz., the derivatives of the smoothed photon counts in (24), are computed directly from the theoretical models for the Na and Fe fluorescence spectra. These derivatives, which depend upon the wind, temperature, and density, are evaluated by employing the wind, temperature, and density values derived from the smoothed photon count profiles using the conventional data processing approach. These values include the means plus gravity wave fluctuations, viz., $\mathbf{\bar{w}} + w_{GW}', \mathbf{\bar{T}} + T_{GW}'$, and $\mathbf{\bar{\rho}} + \rho_{GW}'$.

The measured turbulence perturbations at the three frequencies can be written in matrix form as a system of three linear equations,

$$
\begin{bmatrix}
\frac{N'}{N} \\
\frac{N'_0}{N_0} \\
\frac{N'_+}{N_+}
\end{bmatrix} =
\begin{bmatrix}
\frac{\partial N}{\partial w} & \frac{\partial N}{\partial T} & \frac{\partial N}{\partial \rho} \\
\frac{\partial N_0}{\partial w} & \frac{\partial N_0}{\partial T} & \frac{\partial N_0}{\partial \rho} \\
\frac{\partial N_+}{\partial w} & \frac{\partial N_+}{\partial T} & \frac{\partial N_+}{\partial \rho}
\end{bmatrix}
\begin{bmatrix}
w' \\
T' \\
\rho'
\end{bmatrix} +
\begin{bmatrix}
\frac{\Delta N'}{N} \\
\frac{\Delta N'_0}{N_0} \\
\frac{\Delta N'_+}{N_+}
\end{bmatrix}
$$

(25)

where the subscripts denote the frequency of the laser. The solutions for the turbulence fluctuations and the photon noise errors are computed by multiplying both sides of (25) by the inverse of the sensitivity matrix.

$$
\begin{bmatrix}
w' \\
T' \\
\rho'
\end{bmatrix} =
\begin{bmatrix}
\frac{\partial N}{\partial w} & \frac{\partial N}{\partial T} & \frac{\partial N}{\partial \rho} \\
\frac{\partial N_0}{\partial w} & \frac{\partial N_0}{\partial T} & \frac{\partial N_0}{\partial \rho} \\
\frac{\partial N_+}{\partial w} & \frac{\partial N_+}{\partial T} & \frac{\partial N_+}{\partial \rho}
\end{bmatrix}^{-1}
\begin{bmatrix}
\frac{N'}{N} \\
\frac{N'_0}{N_0} \\
\frac{N'_+}{N_+}
\end{bmatrix}
$$

$$
\begin{bmatrix}
\Delta w \\
\Delta T \\
\Delta \rho
\end{bmatrix} =
\begin{bmatrix}
\frac{\partial N}{\partial w} & \frac{\partial N}{\partial T} & \frac{\partial N}{\partial \rho} \\
\frac{\partial N_0}{\partial w} & \frac{\partial N_0}{\partial T} & \frac{\partial N_0}{\partial \rho} \\
\frac{\partial N_+}{\partial w} & \frac{\partial N_+}{\partial T} & \frac{\partial N_+}{\partial \rho}
\end{bmatrix}^{-1}
\begin{bmatrix}
\frac{\Delta N'}{N} \\
\frac{\Delta N'_0}{N_0} \\
\frac{\Delta N'_+}{N_+}
\end{bmatrix}
$$

(26)

We consider two representative Doppler lidar systems, the Fe lidar developed by Chu and Huang [2010] at the University of Colorado and the Na lidar systems developed by the Colorado State University and University of Illinois groups [e.g., She and Yu, 1994; Gardner and Yang, 1998]. The key system parameters for these lidars and

| Table 2. Photon Noise Errors for Typical Three-Frequency Na/Fe Doppler Lidars Nighttime Operation\textsuperscript{a} |
|---|---|---|
| Lidar | Na | Fe |
| Wavelength $\Delta \lambda$ | 589 nm | 372 nm |
| Frequency shift $f_0$ | 630 MHz | 742 MHz |
| RMS laser line width $\sigma_{Laser}$ | 60 MHz | $< 20$ MHz |
| Mean temperature $T$ | 185 K | 185 K |
| Mean vertical velocity $\mathbf{\bar{w}}$ | 0 m/s | 0 m/s |
| RMS wind error $\Delta w_{RMS}$ | 406 m/s | 263 m/s |
| RMS temperature error $\Delta T_{RMS}$ | 463 K | 458 K |
| RMS density error $\Delta \rho_{RMS}$ | 1.31 $\rho_0$ | 1.39 $\rho_0$ |
| Heat flux bias $\Delta w_{\Delta T}$ | $-3.455$ K/m/s | $-12.930$ K/m/s |
| Constituent flux bias $\Delta w_{\Delta \rho}$ | $2.724 \rho_0 \text{cm/s}$ | $-966 \rho_0 \text{cm/s}$ |

\textsuperscript{a}Dwell times at each frequency are identical.
their measurement accuracies are summarized in Table 2 for nighttime observations. We define the signal-to-noise ratio (SNR) as the square of the mean signal count, when the laser is tuned to the peak of the species absorption spectrum for the whole integration period $\Delta t$, divided by the total variance of the detected photon count including background noise. For nighttime observations, when the background noise count ($N_B$) is negligible, the SNR is approximately equal to the mean signal count.

$$\text{SNR} = \frac{\mathbb{E}[f_0(\Delta z, \Delta t)]^2}{\text{Var}[N(f_0, \Delta z, \Delta t)] + \text{Var}[N_B]} \approx \mathbb{E}[f_0(\Delta z, \Delta t)]$$  \hspace{1cm} (27)

Notice that for this definition the SNR is independent of the laser frequency shift and of the dwell times for each of the three frequencies. In deriving the RMS measurement errors and the covariances between the wind, temperature, and density errors, we assume that the laser is tuned to each frequency for 1/3 of the time. In practice, the dwell times and the frequency shift may be adjusted to minimize the errors [Gardner, 2004]. Since the heat flux is the most difficult atmospheric quantity to measure, these parameters should be chosen to minimize the product of the vertical wind and temperature errors. The optimum values depend on temperature and are different for day and night operation. Since a detailed discussion of the optimization problem is beyond the scope of this paper, we restrict our analysis to the lidar parameters tabulated in Table 2.

The expected signal count can be estimated using the lidar equation [e.g., Gardner, 2004]. For zenith observations, the nighttime SNRs for modern Na and Fe lidars are given by

$$\text{SNR}_{\text{Na}} \approx \mathbb{E}[N_{\text{Na}}] = 2 \times 10^{-3} \left(\frac{100 \text{ km}}{z}\right)^2 p_{\text{Laser}}A_{\text{Tele}}P_{\text{Na}}(z)\Delta z\Delta t$$

$$\text{SNR}_{\text{Fe}} \approx \mathbb{E}[N_{\text{Fe}}] = 10^{-4} \left(\frac{100 \text{ km}}{z}\right)^2 p_{\text{Laser}}A_{\text{Tele}}P_{\text{Fe}}(z)\Delta z\Delta t$$  \hspace{1cm} (28)

$z =$ altitude (km)

$p_{\text{Laser}} =$ laser power (W)

$A_{\text{Tele}} =$ telescope aperture area (m$^2$)

$P_{\text{Na}}(z) =$ Na density (cm$^{-3}$)

$P_{\text{Fe}}(z) =$ Fe density (cm$^{-3}$)

$\Delta z =$ vertical resolution (m)

$\Delta t =$ temporal resolution (s)

Note that when the stated units are used in these equations, the SNRs are dimensionless. These equations are valid whenever the product of the two-way atmospheric transmittance and the combined optical efficiencies of the laser and telescope are about 6%, a value that should be achievable with careful design and alignment. The SNRs are tabulated in Table 3 versus altitude along with the nominal values for the Na and Fe densities. The University of Illinois and Utah State University Na Doppler lidars employ lasers with an average power of 1.5 W and telescopes with diameters varying between 0.7 and 1 m. The nominal power-aperture products of these systems are between 0.6 and 1.2 W m$^2$. The University of Colorado Fe Doppler lidar includes a laser with an average power of more than 2 W and 0.8 m diameter telescope for a power-aperture product of about 1 W m$^2$. Thus, to estimate the flux errors, we assume that both the Na and Fe lidars have power-aperture products of 1 W m$^2$. 

**Table 3.** Nominal Na and Fe Lidar SNRs for $\Delta z =$ 50 m and $\Delta t =$ 9 s

<table>
<thead>
<tr>
<th>$z$ (km)</th>
<th>$\rho_{\text{Na}}$ (cm$^{-3}$)</th>
<th>$\text{SNR}_{\text{Na}}$</th>
<th>$\rho_{\text{Fe}}$ (cm$^{-3}$)</th>
<th>$\text{SNR}_{\text{Fe}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>110</td>
<td>30</td>
<td>$22 \cdot \frac{p_{\text{Laser}}A_{\text{Tele}}}{\rho_{\text{Na}}(z)}$</td>
<td>145</td>
<td>$5.4 \cdot \frac{p_{\text{Laser}}A_{\text{Tele}}}{\rho_{\text{Fe}}(z)}$</td>
</tr>
<tr>
<td>105</td>
<td>85</td>
<td>$69 \cdot \frac{p_{\text{Laser}}A_{\text{Tele}}}{\rho_{\text{Na}}(z)}$</td>
<td>300</td>
<td>$12 \cdot \frac{p_{\text{Laser}}A_{\text{Tele}}}{\rho_{\text{Fe}}(z)}$</td>
</tr>
<tr>
<td>100</td>
<td>550</td>
<td>$495 \cdot \frac{p_{\text{Laser}}A_{\text{Tele}}}{\rho_{\text{Fe}}(z)}$</td>
<td>800</td>
<td>$36 \cdot \frac{p_{\text{Laser}}A_{\text{Tele}}}{\rho_{\text{Fe}}(z)}$</td>
</tr>
<tr>
<td>95</td>
<td>2500</td>
<td>$2490 \cdot \frac{p_{\text{Laser}}A_{\text{Tele}}}{\rho_{\text{Fe}}(z)}$</td>
<td>3000</td>
<td>$150 \cdot \frac{p_{\text{Laser}}A_{\text{Tele}}}{\rho_{\text{Fe}}(z)}$</td>
</tr>
<tr>
<td>90</td>
<td>3300</td>
<td>$3670 \cdot \frac{p_{\text{Laser}}A_{\text{Tele}}}{\rho_{\text{Fe}}(z)}$</td>
<td>8250</td>
<td>$458 \cdot \frac{p_{\text{Laser}}A_{\text{Tele}}}{\rho_{\text{Fe}}(z)}$</td>
</tr>
<tr>
<td>85</td>
<td>1300</td>
<td>$1620 \cdot \frac{p_{\text{Laser}}A_{\text{Tele}}}{\rho_{\text{Fe}}(z)}$</td>
<td>6750</td>
<td>$420 \cdot \frac{p_{\text{Laser}}A_{\text{Tele}}}{\rho_{\text{Fe}}(z)}$</td>
</tr>
<tr>
<td>80</td>
<td>150</td>
<td>$211 \cdot \frac{p_{\text{Laser}}A_{\text{Tele}}}{\rho_{\text{Fe}}(z)}$</td>
<td>1500</td>
<td>$105 \cdot \frac{p_{\text{Laser}}A_{\text{Tele}}}{\rho_{\text{Fe}}(z)}$</td>
</tr>
</tbody>
</table>
The estimated flux errors and biases are tabulated versus altitude in Tables 4 and 5. The errors were computed by assuming that the instantaneous flux estimates were averaged over $L = 2.5$ km in altitude and $\tau = 5$ h in time for Na and $\tau = 20$ h in time for Fe. We employ a longer integration period for the Fe lidar measurements because the Fe signal level (SNR) is 5–10 times smaller than the Na signal. Because of the relatively small power-aperture product for both lidars, photon noise dominates the flux errors, while the photon noise bias can be comparable to the actual flux values near the peak of the Na layer. Even so, scientifically useful eddy flux estimates can be derived from the observations since the biases are known and can be computed from the measured photon counts and subtracted from the observed fluxes.

In the mesopause region, the heat and constituent transport velocities typically vary between 1 and 5 cm/s [Gardner and Yang, 1998; Gardner and Liu, 2007, 2010]. To obtain scientifically useful measurements, the biases should be 5–10 times smaller, but this requires a high SNR, which might be difficult to achieve when measuring eddy fluxes. If the biases are not too large, they can be calculated from the measured values of SNR using the formulas listed in Table 2 and then subtracted from the derived fluxes. Alternatively, the lidar data can be collected at half the required resolution (i.e., $\Delta t/2$ or $\Delta z/2$) to yield two interleaved sets of wind, temperature, and density measurements. The wind data from one set can be combined with the temperature and density data from the other set to compute the heat and constituent fluxes. In this case the flux biases would be zero because the wind errors from one set of data would be statistically independent of the temperature and density errors from the other set. This approach yields two bias-free, statistically independent estimates of the heat and constituent fluxes that exhibit twice the RMS error of a single measurement because dividing the data into two sets reduces the effective SNR by $1/2$. However, the two independent flux measurements could be averaged so that the resulting heat and constituent fluxes would exhibit RMS errors that are $\sqrt{2}$ larger than the values given in Tables 4 and 5, but their photon noise biases would be zero.

One can adjust the resolution and the smoothing intervals to improve the observations. For example, if the vertical and temporal resolutions are both doubled ($\Delta z = 100$ m and $\Delta t = 18$ s), the biases listed in Tables 4 and 5 would be reduced by a factor of 4 and the RMS errors would be reduced by a factor of 2. In this case the fundamental wind, temperature, and density measurements would include about 74% of the turbulent energy ($L_c = 378$ m) rather than about 83% for the 50 m and 9 s resolution data ($L_c = 224$ m; see (18) and (19).

### Table 4. Eddy Flux Biases and RMS Errors for a 1 W m$^2$ Na Doppler Lidar $L = 2.5$ km, $\tau = 5$ h, $\Delta z = 50$ m, $\Delta t = 9$ s, $l_0 = 1.5$ km, $l_1 = 30$ m, and $|\mathbf{u}| = 30$ m/s

<table>
<thead>
<tr>
<th>z(km)</th>
<th>$\Delta \omega_{\nu_{Na}}$</th>
<th>Std($\omega_{\nu_{Na}}$)</th>
<th>$\Delta \omega T$</th>
<th>Std($\omega T$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>110</td>
<td>3710/cm$^2$/s</td>
<td>230/cm$^2$/s</td>
<td>-160 Km/s</td>
<td>27 Km/s</td>
</tr>
<tr>
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<td>3350/cm$^2$/s</td>
<td>210/cm$^2$/s</td>
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<td>8.6 Km/s</td>
</tr>
<tr>
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<td>3020/cm$^2$/s</td>
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<td>1.2 Km/s</td>
</tr>
<tr>
<td>95</td>
<td>2730/cm$^2$/s</td>
<td>170/cm$^2$/s</td>
<td>-1.4 Km/s</td>
<td>0.24 Km/s</td>
</tr>
<tr>
<td>90</td>
<td>2450/cm$^2$/s</td>
<td>155/cm$^2$/s</td>
<td>-0.94 Km/s</td>
<td>0.16 Km/s</td>
</tr>
<tr>
<td>85</td>
<td>2180/cm$^2$/s</td>
<td>135/cm$^2$/s</td>
<td>-2.1 Km/s</td>
<td>0.37 Km/s</td>
</tr>
<tr>
<td>80</td>
<td>1930/cm$^2$/s</td>
<td>120/cm$^2$/s</td>
<td>-16 Km/s</td>
<td>2.8 Km/s</td>
</tr>
</tbody>
</table>

### Table 5. Eddy Flux Biases and RMS Errors for a 1 W m$^2$ Fe Doppler Lidar $L = 2.5$ km, $\tau = 20$ h, $\Delta z = 50$ m, $\Delta t = 9$ s, $l_0 = 1.5$ km, $l_1 = 30$ m, and $|\mathbf{u}| = 30$ m/s

<table>
<thead>
<tr>
<th>z(km)</th>
<th>$\Delta \omega_{\nu_{Fe}}$</th>
<th>Std($\omega_{\nu_{Fe}}$)</th>
<th>$\Delta \omega T$</th>
<th>Std($\omega T$)</th>
</tr>
</thead>
<tbody>
<tr>
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<td>-180/cm$^2$/s</td>
<td>1500/cm$^2$/s</td>
<td>-2400 Km/s</td>
<td>33 Km/s</td>
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<td>1400/cm$^2$/s</td>
<td>-1100 Km/s</td>
<td>14 Km/s</td>
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<tr>
<td>100</td>
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<td>-360 Km/s</td>
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<tr>
<td>95</td>
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<td>-28 Km/s</td>
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<tr>
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<tr>
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<td>-9.2/cm$^2$/s</td>
<td>800/cm$^2$/s</td>
<td>-125 Km/s</td>
<td>1.7 Km/s</td>
</tr>
</tbody>
</table>
and Table 1). Further reductions in the RMS flux errors could be achieved by simply increasing \( L \) and \( r \). When the flux errors are dominated by photon noise, from (16) and (28), we find that
\[
\text{Std}(\mathbf{w}^T) = \text{Var}(\mathbf{w}) \propto \frac{1}{\sqrt{Lr \cdot \text{SNR}}} \propto \frac{1}{\sqrt{Lr} \left( P_{\text{Laser}} A_{\text{Telescope}} / C \right)^2}
\]
(29)

\[ C = \text{Na or Fe} \]

Increasing the power-aperture product of the lidars, by utilizing larger telescopes and more powerful lasers, can also improve the measurements by either reducing the error or reducing the required averaging intervals \( L \) and \( r \), thereby improving the spatial and temporal resolution of the derived flux profiles. For example, if the power-aperture product of the Fe lidar is increased from 1 to 2 W m\(^2\), the averaging interval can be reduced from 20 to 5 h while still achieving the flux errors listed in Table 5 at the vertical resolution \( L = 2.5 \text{ km} \). A larger power-aperture product can also extend the observations to regions where the Na and Fe densities are much smaller than their values near the layer peaks. However, to push the observations above 100 km altitude to the turbopause (\( \sim 110 \text{ km} \)), where the Na and Fe densities are at most a few percent of their values at the layer peaks, would require lidars with power-aperture products of 100 W m\(^2\) or larger.

From (16) it is easy to show that photon noise makes a negligible contribution to the heat flux errors whenever all the following conditions are satisfied.

\[
\text{Var}(\Delta w) \ll \frac{\Delta z \Delta T}{\Delta z \Delta T} \text{Var}(w) = \frac{\rho_0}{30 (L)} \frac{\Delta z \Delta T}{\Delta z \Delta T} \text{Var}(T) \\
\text{Var}(\Delta T) \ll \frac{\Delta z \Delta T}{\Delta z \Delta T} \text{Var}(T) = \frac{\rho_0}{30 (L)} \frac{\Delta z \Delta T}{\Delta z \Delta T} \text{Var}(w) \cdot \text{Var}(T) \\
\text{Var}(\Delta w) \cdot \text{Var}(\Delta T) \ll \frac{\Delta z \Delta T}{\Delta z \Delta T} \text{Var}(w) \cdot \text{Var}(T) = \frac{\rho_0}{30 (L)} \frac{\Delta z \Delta T}{\Delta z \Delta T} \text{Var}(w) \cdot \text{Var}(T)
\]
(30)

This requires a SNR (computed with \( \Delta z = 50 \text{ m} \) and \( \Delta T = 9 \text{ s} \)) of about \( 10^5 \) or greater (see (23)). From Table 3 we find that this condition is satisfied at 90 km altitude when the power-aperture product is about 30 W m\(^2\) for the Na lidar and 200 W m\(^2\) for the Fe lidar. Since these values are much larger than the \( \sim 1 \text{ W m}^2 \) value for existing systems, photon noise will be the dominant error source for the flux measurements made with these more modest lidars.

The eddy diffusivity \((k_{zz})\) is related to the Na and Fe eddy fluxes (see (5))

\[
k_{zz} = \frac{-w \rho_{\text{Na}}}{\rho_{\text{Na}}} = \frac{-w \rho_{\text{Fe}}}{\rho_{\text{Fe}}} \\
\text{Var}(\Delta k_{zz}) \approx \frac{\text{Var} \left( w \rho_{\text{Na}} / \rho_{\text{Na}} \right)}{\rho_{\text{Na}}} \\
\text{Var}(\Delta k_{zz}) \approx \frac{\text{Var} \left( w \rho_{\text{Fe}} / \rho_{\text{Fe}} \right)}{\rho_{\text{Fe}}}
\]
(31)

while the eddy coefficient for thermal diffusion \((k_{th})\) is related to the eddy heat flux (see (6))

\[
k_{th} = \frac{-w T}{(\Gamma_{sd} + \partial T / \partial z)} \\
\text{Var}(\Delta k_{th}) = \frac{\text{Var} \left( w T \right)}{(\Gamma_{sd} + \partial T / \partial z)}
\]
(32)
Both parameters provide key insights into the turbulent mixing and heating of the atmosphere. The ratio $k_{\text{eff}}/k_M$ is equal to one for passive tracers in an adiabatic motion and could be larger for chemically active species [Strobel, 1989]. Given the accuracies of the flux measurements tabulated in Tables 3 and 4, Na and Fe lidars should be capable of providing useful measurements of $k_{\text{eff}}$ and $k_M$ at least throughout the mesopause region where the Na and Fe densities are large.

7. Measuring the Eddy Momentum Fluxes

Another parameter of interest is the turbulent viscosity $k_M$, also known as the coefficient of momentum diffusion. $k_M$ is related to the divergence of the turbulent momentum flux [H.-L. Liu, 2000; A. Z. Liu, 2009]

$$k_M = -\frac{\mathbf{W} \cdot \partial \mathbf{u}}{\partial z},$$

$$\text{Var}(\Delta k_M) = \frac{\text{Var}(\mathbf{W} \cdot \mathbf{u})}{c^2},$$

where $u$ refers to the wind in the dominant direction of mean momentum flux and the overbar represents mean background value. The turbulent momentum flux profile can be measured using the dual coplanar beam technique that was pioneered by Vincent and Reid [1983] for measuring the gravity wave momentum fluxes with radars. If the lidar employs two sets of dual coplanar beams, one set pointing $\theta^\circ$ off zenith due east and west, and the other pointing due north and south, then the variance of the zonal momentum flux is given by

$$\text{Var}(\mathbf{W} \cdot \mathbf{u}) = \frac{\Delta z \Delta t}{L_t} \left[ \frac{\text{Var}(w) \text{Var}(u)}{2} + \frac{\text{Var}(w) \tan^2 \theta}{4} + \frac{\text{Var}(u) \tan^2 \theta}{4} \right] + \frac{\Delta z \Delta t}{L_t} \left[ \frac{\text{Var}(w)}{2 \sin^2 \theta} + \frac{\text{Var}(u)}{2 \cos^2 \theta} \text{Var}(\Delta V_{\text{EW}}) + \frac{\text{Var}^2(\Delta V_{\text{EW}})}{4 \sin^2 \theta \cos^2 \theta} \right],$$

where $\text{Var}(\Delta V_{\text{EW}})$ is the variance of the photon noise error for the radial wind measurement for the east and west beams. The first bracketed term on the right-hand side of (34) is from Thorsen et al. [2000] for the special case where the line-of-sight winds on the two coplanar beams are uncorrelated. It is the statistical error associated with estimating the variance of the turbulence-induced radial wind fluctuations on the two beams. The second term is the additional error that arises from the photon noise associated with the lidar measurements of the radial winds. For large SNRs, the photon noise component is small and the variance is dominated by statistical fluctuations represented by the first term. The momentum flux variance is minimum for the zenith angle satisfying

$$\theta_{\text{min}} = \tan^{-1} \left[ \frac{\text{Var}(w)}{\text{Var}(u)} \right] = 45^\circ$$

and is given by

$$\text{Var}(\mathbf{W} \cdot \mathbf{u}) |_{\theta_{\text{min}}} = \frac{\Delta z \Delta t}{L_t} \text{Var}(w) \text{Var}(u) = \frac{L_t}{30 \sigma_t} \text{Var}^2(w).$$

The variance formula for the meridional momentum flux is similar. Because the turbulence is largely isotropic (Note that it is the subtle departures from isotropy that result in a net momentum flux), the variances of the vertical and horizontal velocity fluctuations are approximately equal so that the optimum zenith angle, which minimizes the eddy momentum flux variance, is 45°.

When photon noise dominates, the momentum flux variance is given by

$$\text{Var}(\mathbf{W} \cdot \mathbf{u}) \approx \frac{\Delta z \Delta t}{L_t} \frac{\text{Var}^2(\Delta V_{\text{EW}})}{4 \sin^2 \theta \cos^2 \theta}, \text{Var}(\Delta V_{\text{EW}}) \approx \frac{\Delta z \Delta t}{L_t} \frac{\text{Var}^2(\Delta w)}{4 \sin^2 \theta \cos^4 \theta}$$

$$\text{Var}(\Delta V_{\text{EW}}) \approx \frac{r^2 \Delta z}{2^2 \Delta t} \text{Var}(\Delta w) \approx \frac{\text{Var}(\Delta w)}{\cos \theta}.$$
where $r$ is the range to altitude $z$ and $\Delta r$ is the range resolution. The zenith angle that minimizes the variance is

$$\theta_{\text{min}} = \tan^{-1} \left( \frac{1}{\sqrt{2}} \right) \approx 35.26^\circ$$

and the minimum variance is given by

$$\text{Var}(\omega \nu)_{\theta = \theta_{\text{min}}} \approx \frac{27}{16} \frac{\Delta z}{L_T} \text{Var}^2(\Delta \omega) = 1.6875 \frac{\Delta z}{L_T} \text{Var}^2(\Delta \omega).$$ (39)

The variances of the eddy heat, constituent, and momentum fluxes are summarized in Table 6 for the limiting cases when statistical noise dominates (high SNR) and when photon noise dominates (low SNR). Although the forms for the flux variances are similar, measuring the momentum flux is more challenging than measuring the heat and constituent fluxes because the signal levels are weaker at the large zenith angles and because four coplanar lidar beams are required. However, the measurement is feasible provided the lidar is powerful enough to achieve the required signal levels or the averaging period is long enough to reduce the variance to acceptable values. A factor of 4–5 increase, in either the power-aperture product or averaging period, is required to measure eddy momentum fluxes compared to the values required for the heat and constituent flux observations.

The turbulent Prandtl number can also be estimated by calculating the ratio $Pr = k_{aw}/k_{aw}$. If $Pr$ is small ($<1$), then the net effect of turbulence on the thermal budget is cooling associated with downward heat transport. Conversely, if $Pr$ is large ($>1$), the net effect is heating [Strobel et al., 1985; H.-L. Liu, 2000; A. Z. Liu, 2009]. Since little is known about $k_{aw}, k_{aw}, k_{aw}$ and $Pr$ in the middle and upper atmosphere, even long-term average measurements of these quantities would be enormously helpful in gauging the validity of parameterization schemes currently employed in global circulation models of the Earth's atmosphere.

### 8. Conclusions

Rocket-borne ionization gauge measurements of the spectrum of neutral density fluctuations are a proven technique for observing turbulence in the upper atmosphere. This approach, which provides only a snapshot in time, relies on complex turbulence theory to relate features of the fitted spectra to key parameters like the eddy diffusion coefficient and energy dissipation rate [e.g., Weinstock, 1978, 1981; Lübken, 1997]. Radars can make turbulence measurements continuously. However, because of the large radar beam width and concomitant large sampling volume, the broadening of the temporal frequency spectrum of the backscattered radar pulses by turbulence-induced velocity fluctuations is susceptible to contamination by gravity waves and large anisotropic eddies which can dominate and distort the spectral width measurements. In addition, the derivation of turbulence parameters is sensitive to the assumed shape of the turbulence spectrum at scales larger than the outer scale (i.e., source region). Plane [2004] and Hocking [1996] suggest that current radar estimates of eddy diffusivities and energy dissipation rates may be 5–10 times too large because of wave contamination, turbulence anisotropies, and potentially questionable assumptions about the shape of the turbulence spectrum in the source region.

Doppler lidar measurement of atmospheric turbulence offers several important advantages compared to the radar and in situ rocket techniques. Like radar, lidar observations can be made continuously for long periods of time thereby enabling studies of the long-term morphology of turbulence variations. Furthermore, lidar winds, densities, and temperatures can be acquired and processed at the high vertical, horizontal, and temporal resolutions that capture the dominant turbulence scales that are responsible for eddy mixing, while
the contamination by gravity waves can be eliminated by spatially and temporally filtering the data before the eddy fluxes are computed. In addition, lidars can measure all three fluxes, not just the eddy constituent flux. The most important advantage of the lidar technique is that the actual constituent, heat, and momentum fluxes are measured directly so that no complex turbulence theory or postprocessing is required to interpret the observations. In addition, the eddy coefficients for thermal (k_u), constituent (k_o), and momentum (k_u) diffusion can provide insights to the dynamical and chemical processes associated with the measured constituents and turbulent heating/cooling of the background atmosphere.

Based upon the analysis above, it is clear that modern Na and Fe Doppler lidars, with modest power-aperture products of just 1 W m^2, should be able to measure the eddy fluxes and associated eddy diffusion profiles in 85–100 km altitude region where the Na and Fe densities are largest. Like the measurements of gravity wave fluxes [Gardner and Yang, 1998; Gardner and Liu, 2007], long-term averaging, on the order of 5–20 h, would be required to obtain statistically significant results. Momentum flux measurements will require longer averaging times (4–5 times longer) because the observations must be made in four different beam directions at large zenith angles where the signals are weaker. To insure that the measured fluctuations include a substantial fraction of the turbulent energy, the raw measurements should be collected at vertical and horizontal resolutions comparable to the inner scale (l_i) and a temporal resolution comparable to l_i/|\omega|. A vertical resolution of 50–100 m, a field of view of 0.25–0.5 mrad FWHM, and a temporal resolution of 5–10 s would enable the lidar to capture more than 80–90% of the energy in the turbulence spectrum and should produce highly representative heat and constituent flux measurements.

Unlike the measurement of gravity wave fluxes, which require precise zenith pointing of the laser beam [Gardner and Yang, 1998], the vertical pointing accuracy required for eddy flux measurements is only a degree or so. Gravity wave horizontal wind variances can be 100 times larger than the vertical wind variance, so precise zenith pointing is required to eliminate contamination of the measured radial winds by the strong horizontal velocities, which are highly correlated with the temperature and density fluctuations. Because turbulence in the important inertial subrange (2\pi/l_o < |m| < 2\pi/l_i and 2\pi|\omega|/l_o < |\omega| < 2\pi|\omega|/l_i) is isotropic, the turbulence-induced horizontal and vertical velocity fluctuations are comparable. The mesoscale (|m| < 2\pi/l_o and |\omega| < 2\pi/T_D) gravity wave fluctuations are removed from the data by high-pass filtering before the eddy fluxes are computed. Hence, the pointing requirement for the eddy flux measurements can be relaxed considerably. However, it would be quite valuable to measure both the gravity wave and turbulence fluxes simultaneously, and so we recommend that the lidar beam be pointed to zenith with an accuracy comparable to the mrad beam width so that reliable gravity wave heat and constituent fluxes can also be derived from the data.

Tables 4 and 5 show the predicted RMS errors for the eddy heat and constituent fluxes throughout the mesopause region for representative turbulence, Na and Fe lidar, and signal processing parameters. These results suggest that scientifically useful measurements of the eddy fluxes may be possible on a nightly basis between 85 and 100 km, depending on the resolution employed and the signal level of the lidar. Certainly, high-quality observations can be obtained on weekly and monthly schedules, periods during which it is highly likely that the required 5–20 h of observations can be acquired. It is important to note that our calculations were made by assuming that these lidars were operating close to their theoretical limits, which is achievable with careful system design and alignment under clear atmospheric conditions (see discussion immediately following (28)). If not, then the signal levels will be lower than the values used in our analysis so that longer averaging periods would then be required to compensate for the smaller SNR.

Of course, the resolution, accuracy, and altitude coverage of the measured eddy fluxes and derived eddy diffusivities can be improved considerably if the power-aperture products of the lidars are increased above the modest 1 W m^2 value employed in this study. Systems with power-aperture products of 5 W m^2 or larger could be used to study the eddy fluxes generated by the dissipation and breaking of individual gravity waves. To push the observations above 100 km altitude through the turbopause (\sim110 km), where the Na and Fe densities are only a few percent of the densities near the layer peaks, would require lidars with much higher power-aperture products (50 W m^2 or larger), especially if the goal is to study the turbulence generated in this region by the smallest-scale gravity waves, which have periods of a few minutes and vertical wavelengths of about 1 km. The large power-aperture product is required to compensate for the low Na and Fe densities and for the small values of the flux averaging period r and height range L, required to observe the effects of the smallest-scale waves.
Appendix A: Turbulence Spectra

We employ piecewise continuous models for the 1-D spatial spectra of turbulence that are based upon the theoretical models developed by Kolmogorov [1941] and others using dimensional analysis. We assume that the turbulence is homogeneous and isotropic so that the 1-D vertical frequency spectrum is related to the 1-D horizontal wave number spectrum. Under these conditions, the two-sided $m$ and $\omega$ spectral models for the turbulence-induced fluctuations of atmospheric density are given by

$$F_{\nu}(m) = \frac{3\text{Var}(\nu'|\mid \omega)}{(9r + 15)/(r + 1) - 8(l_i/l_o)^{2/3}} H(m/\kappa_o).$$

$$F_{\nu}(\omega) = \frac{1}{|\omega|} \int \left[ F_{\nu}(k = \omega/|\omega|) \right] \left( \frac{3\text{Var}(\nu'|\mid \omega)}{(9r + 15)/(r + 1) - 8(l_i/l_o)^{2/3}} H(\omega/|\omega|/\kappa_o) \right) \frac{d\omega}{|\omega|}.$$

$$H(\kappa/\kappa_o) = \begin{cases} (\kappa/\kappa_o)^{5/3} & \kappa_o \leq \kappa_i \leq \kappa \\ (\kappa_i/\kappa_o)^{-1} & 0 \leq \kappa \leq \kappa_o \\ (\kappa_i/\kappa_o)^{16/3}(\kappa_o/\kappa)^8 & \kappa_i \leq \kappa \end{cases} \quad (A1)$$

$$\text{Var}(\nu') = \frac{1}{2\pi} \int \int F_{\nu}(m) \, dm = \frac{1}{2\pi} \int \int F_{\nu}(\omega) \, d\omega.$$

$l_o$ is the outer scale of turbulence and $l_i$ is the inner scale. The wind and temperature spectra have the same shape as the density spectra.

Because the turbulence fluctuations are isotropic, the 2-D and 3-D spatial spectra are related to the 1-D spatial spectrum as follows.

$$F_{\nu}(k, m) = F_{\nu}(k = \sqrt{k^2 + m^2}) = \frac{3\text{Var}(\nu'|\mid \omega)}{(9r + 15)/(r + 1) - 8(l_i/l_o)^{2/3}} \left( \frac{\kappa_o}{k} \right) H(\kappa/\kappa_o).$$

$$F_{\nu}(\omega, m) = \frac{\int \left[ F_{\nu}(k = \omega/|\omega|) \right] \left( \frac{\int \left[ F_{\nu}(k = \sqrt{(\omega/|\omega|)^2 + m^2}) \right]}{|\omega|} \left( \frac{3\text{Var}(\nu'|\mid \omega)}{(9r + 15)/(r + 1) - 8(l_i/l_o)^{2/3}} \left( \frac{\kappa_o}{k} \right) H(\kappa/\kappa_o) \right) \frac{d\omega}{|\omega|} \right) \frac{dk}{|\omega|} \frac{dm}{|\omega|} = \frac{1}{2\pi} \int \int \int F_{\nu}(k, m) \, dk \, d\omega \, dm \quad (A2)$$

$$\text{Var}(\nu') = \frac{1}{(2\pi)^2} \int \int \int F_{\nu}(\omega, m) \, d\omega \, dm = \frac{1}{(2\pi)^2} \int \int \int \left[ \frac{3\text{Var}(\nu'|\mid \omega)}{(9r + 15)/(r + 1) - 8(l_i/l_o)^{2/3}} \left( \frac{\kappa_o}{k} \right) H(\kappa/\kappa_o) \right] \frac{d\omega}{|\omega|} \frac{dk}{|\omega|} \frac{dm}{|\omega|}.$$
\[
F_\nu (k, l, m) = F_\nu (\kappa) = \sqrt{k^2 + l^2 + m^2} = \frac{3\text{Var}(\nu/c)}{(2\pi)^2} \frac{\left(\kappa_o / \kappa\right)^2 H(\kappa/\kappa_o)}{(9r + 15)/(r + 1) - 8(l/\kappa_o)^2/3}
\]

\[
\text{Var}(\nu) = \frac{1}{(2\pi)^2} \int \int \int F_\nu (k, l, m) \, dm \, dk \, dk = \frac{2}{(2\pi)^2} \int_0^\infty k^2 F_\nu (\kappa) \, dk
\]

\[
\theta = \kappa \sin \theta \cos \phi
\]

\[
l = \kappa \sin \theta \sin \phi
\]

\[
m = \kappa \cos \theta
\]

\[
k^2 = k^2 + l^2 + m^2
\]

\[
dk/dm = k^2 \sin \phi \, d\kappa \, d\phi / d\theta
\]

Notice that in the spectral region \( \kappa_o \leq \kappa \leq \kappa_i \), the 1-D spatial spectra are proportional to \( \kappa^{-5/3} \) \( (\kappa = k, l, \text{or} m) \), the 2-D spectra are proportional to \( \kappa^{-8/3} \) \( (\kappa = (k^2 + l^2)^{1/2}, (k^2 + m^2)^{1/2}, \text{or} (l^2 + m^2)^{1/2}) \), and the 3-D spectrum is proportional to \( \kappa^{-11/3} \) \( (\kappa = (k^2 + l^2 + m^2)^{1/2}) \).

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References


