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Exact controllability of a Rayleigh beam with a single boundary control

Ahmet Ozkan Ozer, *Western Kentucky University*
Scott W Hansen, *Iowa State University*

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ORIGINAL ARTICLE

Further stabilization and exact observability results for voltage-actuated piezoelectric beams with magnetic effects

Ahmet Özkan Özer

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Abstract It is well known that magnetic energy of the piezoelectric beam is relatively small, and it does not change the overall dynamics. Therefore, the models, relying on electrostatic or quasi-static approaches, completely ignore the magnetic energy stored/produced in the beam. A single piezoelectric beam model without the magnetic effects is known to be exactly observable and exponentially stabilizable in the energy space. However, the model with the magnetic effects is proved to be not exactly observable/exponentially stabilizable in the energy space for almost all choices of material parameters. Moreover, even strong stability is not achievable for many values of the material parameters. In this paper, it is shown that the uncontrolled system is exactly observable in a space larger than the energy space. Then, by using a B^*-type feedback controller, explicit polynomial decay estimates are obtained for more regular initial data. Unlike the classical counterparts, this choice of feedback corresponds to the current flowing through the electrodes, and it matches better with the physics of the model. The results obtained in this manuscript have direct implications on the controllability/stabilizability of smart structures such as elastic beams/plates with piezoelectric patches and the active constrained layer (ACL) damped beams/plates.

Keywords Voltage-actuated piezoelectric beam · Current feedback · Strongly coupled wave system · Exact observability · Polynomial stabilization · Diophantine's approximation

A. Ö. Özer (✉)
Department of Mathematics and Statistics, University of Nevada-Reno,
Reno, NV 89557, USA
e-mail: aozer@unr.edu

1 Introduction

Piezoelectric material is an elastic beam/plate covered by electrodes at its top and bottom surfaces, insulated at the edges (to prevent fringing effects), and connected to an external electric circuit to create an electric field between the top and the bottom electrodes (See Fig. 1). It has a unique characteristic of converting mechanical energy to electrical and *magnetic energy*, and vice versa. Therefore these materials could be used as both actuators or sensors. Moreover, since they are generally scalable, smaller, less expensive and more efficient than traditional actuators, they have been employed in civil, industrial, automotive, aeronautic, and space structures.

In classical mechanics, it is very well known that equations of motion can be formulated either through a set of differential equations, or through a variational principle, so-called Hamilton's principle. In applying the Hamilton's principle, the functional is specified over a fixed time interval, and the admissible variations of the generalized coordinates (independent variables) are taken to be zero. The set of field equations for the piezoelectric beams/plates have been well established through the coupling of beam/plate equations and Maxwell's equations. There are many different mathematical models proposed in the literature depending on the type of actuation; voltage, charge or current.

The linear models of piezoelectric beams incorporate three major effects and their interrelations: mechanical, electrical, and magnetic effects. Mechanical effects are mostly modeled through Kirchhoff, Euler–Bernoulli, or Mindlin–Timoshenko small displacement assumptions. To include electrical and magnetic effects, there are mainly three approaches (due to Maxwell's equations): *electrostatic, quasi-static,* and *fully dynamic* [37]. Electrostatic approach is the most widely used among the others. It completely excludes magnetic effects and their couplings with electrical and mechanical effects ([10,17,33,36,37,43,45] and references therein). In this approach, even though the mechanical equations are dynamic, electric field is not dynamically coupled. In other words, the electrical effects are assumed to be stationary. In the case of quasi-static approach [22,45], magnetic effects are not completely ignored and electric charges have time dependence. The electromechanical coupling is still not dynamic though.

Due to the small displacement assumptions, the stretching and bending motions of a single piezoelectric beam are completely decoupled. The bending equation

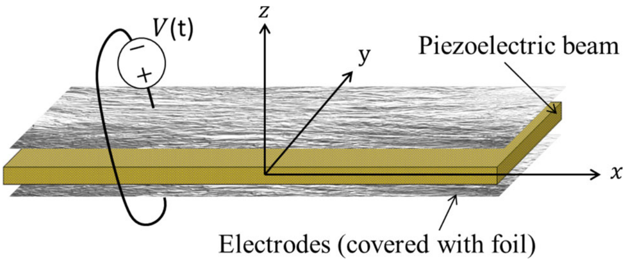

Fig. 1 For a voltage-actuated beam/plate, when voltage $V(t)$ is supplied to the electrodes, an electric field is created between the electrodes, and therefore the beam/plate either shrinks or extends

without the electrical effects corresponds to the fourth order Euler–Bernoulli or Rayleigh/Kirchhoff beam equations; see, i.e., [29]. Since the voltage control does not affect the bending equations at all, we only consider the stretching equations in this paper. For a beam of length L and thickness h, the beam model (no damping) derived by Euler–Bernoulli small displacement assumptions, and electrostatic/quasi-static assumptions describe the stretching motion as

$$\begin{cases} \rho\ddot{v} - \alpha_1 v_{xx} = 0, & (x,t) \in (0,L) \times \mathbb{R}^+ \quad \text{(1a)} \\ v(0,t) = 0, \ \alpha_1 v_x(L,t) = -\dfrac{\gamma V(t)}{h}, & t \in \mathbb{R}^+ \quad \text{(1b)} \\ (v,\dot{v})(x,0) = (v^0, v^1), & x \in [0,L] \quad \text{(1c)} \end{cases}$$

where ρ, α_1, γ denote mass density, elastic stiffness, and piezoelectric coefficients of the beam, respectively, $V(t)$ denotes the voltage applied at the electrodes, and v denotes the longitudinal displacement of centerline of the beam. Throughout this paper, we use dots to denote differentiation with respect to time.

From the control theory point of view, it is well known that wave equation (1) can be exactly controlled in the natural energy space (therefore the uncontrolled problem is exactly observable if the observability time is large enough). If we have the choice of a feedback in the form of a boundary damping $V(t) = -k\,\dot{v}(L,T)$ with $k > 0$, the solution of the closed-loop system is exponentially stable in the energy space (see, for instance [24]).

In the fully dynamic approach, magnetic effects are included, and therefore the wave behavior of the electromagnetic fields are accounted for, i.e., see [27]. These effects are experimentally observed to be minor on the overall dynamics for polarized ceramics (see the review article [46]). For a beam of length L and thickness h, the Euler–Bernoulli model with magnetic effects is derived in [29] as

$$\begin{cases} \rho\ddot{v} - \alpha v_{xx} + \gamma\beta p_{xx} = 0, & \quad \text{(2a)} \\ \mu\ddot{p} - \beta p_{xx} + \gamma\beta v_{xx} = 0, & (x,t) \in (0,L) \times \mathbb{R}^+, \quad \text{(2b)} \end{cases}$$

$$\begin{cases} v(0) = p(0) = \alpha v_x(L) - \gamma\beta p_x(L) = 0, & \quad \text{(2c)} \\ \beta p_x(L) - \gamma\beta v_x(L) = -\dfrac{V(t)}{h}, & t \in \mathbb{R}^+ \quad \text{(2d)} \\ (v,p,\dot{v},\dot{p})(x,0) = (v^0,p^0,v^1,p^1), & x \in [0,L] \quad \text{(2e)} \end{cases}$$

where $\rho, \alpha, \gamma, \mu, \beta$, and V denote mass density per unit volume, elastic stiffness, piezoelectric coefficient, magnetic permeability, impermittivity coefficient of the beam, and voltage prescribed at the electrodes of the beam, respectively, and

$$\alpha = \alpha_1 + \gamma^2\beta. \tag{3}$$

Moreover, $p = \int_0^x D_3(x,t)\,\mathrm{d}t$ is the total charge at the point x with $D_3(x,t)$ being the electric displacement along the transverse direction. Observe that the term $\mu\ddot{p}$ in (2) is due to the dynamic approach. If this term is ignored, an elliptic-type differential equation is obtained, and once this equation is solved and back substituted to the mechanical equations, the system (2) boils down to the system (1) obtained in electrostatic and quasi-static approaches.

By using (3), the boundary conditions (2d) can be simplified as the following

$$v_x(L) = -\frac{\gamma V(t)}{\alpha_1 h}, \quad p_x(L) = -\frac{\alpha V(t)}{\beta\alpha_1 h}. \tag{4}$$

The system (2) with the simplified boundary conditions (4) is a simultaneous controllability problem with the control $V(t)$. Simultaneous controllability problems were first introduced by [24,34]. Controllability and stabilizability of the beam/plate with a control applied to *a point/a curve* in the beam/plate cases were investigated by a number of researchers including [7–9,16,21,41], and references therein. By using a generalization of Ingham's inequality (with a weakened gap condition) (i.e., see [23]) and Diophantine's approximations [14], exact controllability (observability) in finite time, and stabilizability are obtained depending on the Diophantine approximation properties of the joints in the beam case, and how strategic the controlled curve is in the plate case. Simultaneous controllability for general networks and trees is considered in [16]. The controllability of two interconnected beams (including the rotational inertia) by a point mass is considered in [15]. In this problem the weakened gap condition is a necessity. Notice that the system (2) is a strongly coupled wave system, whereas in [1–3] various other weakly coupled systems are considered. The methodology used in these papers is slightly different from ours. There is also research done in proving the controllability of various coupled parabolic systems, i.e., see [4,5,25]. The use of number theoretical results is unavoidable in [25].

In this paper, we consider a coupled wave system (2) where the coupling terms are at the order of the principal terms. The eigenvalues of the uncontrolled system ($V(t) \equiv 0$), are all on the imaginary axis, and for almost all choices of parameters, they get arbitrarily close to each other (see Theorem 4). In other words, eigenvalues do not have a uniform gap. Our first goal is to obtain the observability inequality for the uncontrolled system in a less regular space. Next, we choose a B^*-type feedback, i.e., $V(t) = \frac{1}{2h}\dot{p}(L)$ in (2), to obtain the closed-loop system

$$\begin{cases} \rho\ddot{v} - \alpha v_{xx} + \gamma\beta p_{xx} = 0 & (5a)\\ \mu\ddot{p} - \beta p_{xx} + \gamma\beta v_{xx} = 0, \qquad (x,t) \in (0,L)\times\mathbb{R}^+, & (5b)\end{cases}$$

$$\begin{cases} v(0) = p(0) = \alpha v_x(L) - \gamma\beta p_x(L) = 0, & (5c)\\ \beta p_x(L) - \gamma\beta v_x(L) = -\dfrac{\dot{p}(L)}{2h^2}, \qquad t\in\mathbb{R}^+ & (5d)\\ (v,p,\dot{v},\dot{p})(x,0) = (v^0,p^0,v^1,p^1), \qquad x\in[0,L]. & (5e)\end{cases}$$

In fact, the system (5) is shown to be strongly stable [28], but not exponentially stable in the energy space for almost all choices of parameters [29]. Based on the observability inequality, we use the methods in [6,12] to obtain decay estimates for the solutions of the closed loop system (5). Notice that this type of feedback is very practical since it corresponds to the current flowing through the electrodes.

This paper is organized as follows: In Sect. 2, we first prove that the uncontrolled system is well-posed in the interpolation spaces. In Sect. 3, we prove the exact observability results. In Sect. 4, we give explicit decay rates for the solutions of the closed-loop system with the current feedback at the electrodes. Finally, in the Appendix, we briefly mention known results from number theory which are needed to prove our observability inequalities.

2 Well-posedness

The energy associated with (2) is given by

$$\mathrm{E}(t)=\frac{1}{2}\int_0^L\left\{\rho|\dot{v}|^2+\mu|\dot{p}|^2+\alpha_1|v_x|^2+\beta\,|\gamma v_x-p_x|^2\right\}\,\mathrm{d}x,\quad t\in\mathbb{R}. \tag{6}$$

We define the Hilbert space

$$H^1_L(0,L)=\{v\in H^1(0,L):v(0)=0\},\qquad \mathbb{X}=(\mathbb{L}^2(0,L))^2 \tag{7}$$

and the complex linear space

$$\mathrm{H}=\left(H^1_L(0,L)\right)^2\times\mathbb{X} \tag{8}$$

equipped with the energy inner product

$$\begin{aligned}
&\left\langle\begin{pmatrix}u_1\\u_2\\u_3\\u_4\end{pmatrix},\begin{pmatrix}v_1\\v_2\\v_3\\v_4\end{pmatrix}\right\rangle_{\mathrm{H}}=\left\langle\begin{pmatrix}u_3\\u_4\end{pmatrix},\begin{pmatrix}v_3\\v_4\end{pmatrix}\right\rangle_{(\mathbb{L}^2(0,L))^2}+\left\langle\begin{pmatrix}u_1\\u_2\end{pmatrix},\begin{pmatrix}v_1\\v_2\end{pmatrix}\right\rangle_{\left(H^1_L(0,L)\right)^2}\\
&:=\int_0^L\{\rho u_3\bar{v}_3+\mu u_4\bar{v}_4\}\,\mathrm{d}x+\int_0^L\{\alpha_1(u_1)_x(\bar{v}_1)_x+\beta\,(\gamma(u_1)_x-(u_2)_x)\,(\gamma(\bar{v}_1)_x-(\bar{v}_2)_x)\}\,\mathrm{d}x\\
&=\int_0^L\left\{\rho u_3\bar{v}_3+\mu u_4\bar{v}_4+\left\langle\begin{pmatrix}\alpha_1+\gamma^2\beta & -\gamma\beta\\-\gamma\beta & \beta\end{pmatrix}\begin{pmatrix}u_{1x}\\u_{2x}\end{pmatrix},\begin{pmatrix}v_{1x}\\v_{2x}\end{pmatrix}\right\rangle_{\mathbb{C}^2}\right\}\,\mathrm{d}x
\end{aligned} \tag{9}$$

where $\langle\cdot,\cdot\rangle_{\mathbb{C}^2}$ is the inner product on $\mathbb{C}^2$. Indeed, (9) is an inner product since the matrix $\begin{pmatrix}\alpha_1+\gamma^2\beta & -\gamma\beta\\-\gamma\beta & \beta\end{pmatrix}$ is positive definite.

Interpolation spaces

Define the operator

$$A : \mathrm{Dom}(A) \subset \mathbb{X} \to \mathbb{X}, \quad A = \begin{pmatrix} -\frac{\alpha}{\rho} D_x^2 & \frac{\gamma\beta}{\rho} D_x^2 \\ \frac{\gamma\beta}{\mu} D_x^2 & -\frac{\beta}{\mu} D_x^2 \end{pmatrix}$$

where

$$\begin{aligned} \mathrm{Dom}(A) = (H^2(0, L))^2 \bigcap \{(w_1, w_2)^{\mathrm{T}} \in (H_L^1(0, L))^2 \\ : \ w_{1x}(L) = w_{2x}(L) = 0\}. \end{aligned} \tag{10}$$

The operator A can be easily shown to be a positive and self-adjoint operator, and since the Dom(A) is compactly embedded in $\mathbb{X}$, the operator A^{-1} is compact, and therefore A^{-1} has only countable many positive eigenvalues in its point spectrum, and all of its eigenvalues converge to zero. Therefore, the operator A has only countable many positive eigenvalues $\{\lambda_j\}_{j\in\mathbb{N}}$ in its point spectrum, and $|\lambda_j| \to \infty$ as $j \to \infty$.

Now we find the eigenvalues of A. Consider the eigenvalue problem

$$A \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \lambda \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}. \tag{11}$$

Solving (11) is equivalent to solving

$$\begin{cases} \alpha z_{1xx} - \gamma\beta z_{2xx} = -\rho\lambda z_1 & \text{(12a)} \\ \beta z_{2xx} - \gamma\beta z_{1xx} = -\mu\lambda z_2, & \text{(12b)} \\ z_1(0) = z_2(0) = z_{1x}(L) = z_{2x}(L) = 0. & \text{(12c)} \end{cases}$$

Define

$$\zeta_1 = \frac{1}{\sqrt{2}}\sqrt{\frac{\gamma^2\mu}{\alpha_1} + \frac{\mu}{\beta} + \frac{\rho}{\alpha_1} + \sqrt{\left(\frac{\gamma^2\mu}{\alpha_1} + \frac{\mu}{\beta} + \frac{\rho}{\alpha_1}\right)^2 - \frac{4\rho\mu}{\beta\alpha_1}}} \tag{13}$$

$$\zeta_2 = \frac{1}{\sqrt{2}}\sqrt{\frac{\gamma^2\mu}{\alpha_1} + \frac{\mu}{\beta} + \frac{\rho}{\alpha_1} - \sqrt{\left(\frac{\gamma^2\mu}{\alpha_1} + \frac{\mu}{\beta} + \frac{\rho}{\alpha_1}\right)^2 - \frac{4\rho\mu}{\beta\alpha_1}}} \tag{14}$$

$$b_1 = \frac{1}{\gamma\mu}(\alpha_1\zeta_1^2 - \rho) = \frac{1}{2}\left(\gamma + \frac{\alpha_1}{\gamma\beta} - \frac{\rho}{\gamma\mu} + \sqrt{\left(\gamma + \frac{\alpha_1}{\gamma\beta} - \frac{\rho}{\gamma\mu}\right)^2 + \frac{4\rho}{\mu}}\right) \tag{15}$$

$$b_2 = \frac{1}{\gamma\mu}(\alpha_1\zeta_2^2 - \rho) = \frac{1}{2}\left(\gamma + \frac{\alpha_1}{\gamma\beta} - \frac{\rho}{\gamma\mu} - \sqrt{\left(\gamma + \frac{\alpha_1}{\gamma\beta} - \frac{\rho}{\gamma\mu}\right)^2 + \frac{4\rho}{\mu}}\right). \tag{16}$$

Obviously, $\zeta_1, \zeta_2 > 0$ since

$$\left(\frac{\gamma^2\mu}{\alpha_1} + \frac{\mu}{\beta} + \frac{\rho}{\alpha_1}\right)^2 - \frac{4\rho\mu}{\beta\alpha_1} = \left(\frac{\gamma^2\mu}{\alpha_1} + \frac{\mu}{\beta} - \frac{\rho}{\alpha_1}\right)^2 + \frac{4\rho\mu\gamma^2}{\alpha_1^2} > 0,$$

and

$$b_1, b_2 \neq 0, \quad b_1 \neq b_2, \quad b_1 b_2 = -\frac{\rho}{\mu}.$$

Theorem 1 *Let* $\sigma_j = \frac{(2j-1)\pi}{2L}, \quad j \in \mathbb{N}$. *The eigenvalue problem* (11) *has distinct eigenvalues*

$$\lambda_{1j} = \frac{\sigma_j^2}{\zeta_1^2}, \quad \lambda_{2j} = \frac{\sigma_j^2}{\zeta_2^2}, \quad j \in \mathbb{N} \tag{17}$$

with the corresponding eigenfunctions

$$y_{1j} = \begin{pmatrix} 1 \\ b_1 \end{pmatrix} \sin\sigma_j x, \qquad y_{2j} = \begin{pmatrix} 1 \\ b_2 \end{pmatrix} \sin\sigma_j x, \quad j \in \mathbb{N}. \tag{18}$$

Proof Using $\alpha = \alpha_1 + \gamma^2\beta$ reduces (12a) and (12b) to

$$\begin{cases} z_{1xx} = \dfrac{-\lambda}{\alpha_1}(\rho z_1 + \gamma\mu z_2) & \text{(19a)} \\ z_{2xx} = -\lambda\left(\dfrac{\gamma\rho}{\alpha_1} z_1 + \left(\dfrac{\gamma^2\mu}{\alpha_1} + \dfrac{\mu}{\beta}\right) z_2\right). & \text{(19b)} \end{cases}$$

First, we find the eigenvalues of (17). It is obvious that $\lambda = 0$ is not an eigenvalue since the solution of (19) with (12c) is $z_1 = z_2 = 0$.

We look for solutions of the form

$$z_{1j} = f_j \sin\sigma_j x, \quad z_{2j} = g_j \sin\sigma_j x. \tag{20}$$

Solutions of this form satisfy all the homogeneous boundary conditions (12c). We seek f_j, g_j and λ_j so that the system (19) is satisfied. Substituting (20) into (19) we obtain

$$\begin{cases} \sigma_j^2 f_j = \dfrac{\lambda}{\alpha_1}\left(\rho f_j + \gamma\mu g_j\right) \\ \sigma_j^2 g_j = \lambda\left(\dfrac{\gamma\rho}{\alpha_1} f_j + \left(\dfrac{\gamma^2\mu}{\alpha_1} + \dfrac{\mu}{\beta}\right) g_j\right). \end{cases}$$

The system above has nontrivial solutions if the following characteristic equation is satisfied

$$y_j^2 - \left(\frac{\gamma^2\mu}{\alpha_1} + \frac{\mu}{\beta} + \frac{\rho}{\alpha_1}\right) y_j + \frac{\rho\mu}{\beta\alpha_1} = 0.$$

where $y_j = \frac{\sigma_j^2}{\lambda}$. Since $\left(\frac{\gamma^2\mu}{\alpha_1} + \frac{\mu}{\beta} + \frac{\rho}{\alpha_1}\right)^2 - \frac{4\rho\mu}{\beta\alpha_1} = \left(\frac{\gamma^2\mu}{\alpha_1} + \frac{\mu}{\beta} - \frac{\rho}{\alpha_1}\right)^2 + \frac{4\rho\gamma^2\mu}{\alpha_1^2} > 0$, a simple calculation shows that we have solutions $y_{j1} = \zeta_1^2, \; y_{j2} = \zeta_2^2$ where $\zeta_1, \zeta_2 \in \mathbb{R}$ are defined by (13) and (14), respectively. Therefore $\lambda_{1j} = \frac{\sigma_j^2}{\zeta_1^2}, \quad \lambda_{2j} = \frac{\sigma_j^2}{\zeta_2^2}, \quad j \in \mathbb{N}$, and (17) follows.

Now we find the eigenvectors (18). Let $\lambda = \lambda_{1j}$. Choosing $f_j = 1$ yields $g_j = b_1$. The first eigenvector y_{1j} follows from the solution $z_{1j} = \sin\sigma_j(x)$ and $z_{2j} = b_1 \sin\sigma_j(x)$. Similarly, let $\lambda = \lambda_{2j}$. Choosing $g_j = 1$ yields $f_j = 1/b_2$. Hence the second eigenvector y_{2j} follows from the solution $z_{1j} = \frac{1}{b_2}\sin\sigma_j(x)$ and $z_{2j} = \sin\sigma_j(x)$. □

Obviously, the eigenvectors (18) of A are mutually orthogonal in $(H_L^1(0,L))^2$ by using the inner product defined by (9). Therefore, they form a Riesz basis in $(H_L^1(0,L))^2$. Now we introduce the space $\mathbb{X}_\theta = \mathrm{Dom}(A^\theta)$ for all $\theta \geq 0$ with the norm $\|\cdot\|_\theta = \|A^\theta \cdot\|_{\mathbb{X}}$. For example, using the definition of inner product $\langle\cdot,\cdot\rangle_{(H_L^1(0,L))^2}$ in (9) yields

$$\langle z_1, z_2\rangle_{\mathbb{X}_{1/2}} = \left\langle A^{1/2} z_1, A^{1/2} z_2\right\rangle_{\mathbb{X}} = \langle A z_1, z_2\rangle_{\mathbb{X}} = \langle z_1, z_2\rangle_{(H_L^1(0,L))^2}\,.$$

The space $\mathbb{X}_{-\theta}$ is defined to be the dual of $\mathbb{X}_\theta$ pivoted with respect to $\mathbb{X}$. For example, the inner product on $\mathbb{X}_{-1/2}$ is defined by

$$\langle z_1, z_2\rangle_{\mathbb{X}_{-1/2}} := \left\langle A^{-1/2} z_1, A^{-1/2} z_2\right\rangle_{\mathbb{X}} = \left\langle A^{-1} z_1, z_2\right\rangle_{\mathbb{X}}.$$

Defining $(H_L^1(0,L))^*$ to be the dual space of $H_L^1(0,L)$ pivoted with respect to $\mathbb{L}^2(0,L)$, we have

$$\mathbb{X}_0 = \mathbb{X}, \quad \mathbb{X}_{1/2} = (H_L^1(0,L))^2, \quad \mathbb{X}_{-1/2} = ((H_L^1(0,L))^*)^2 \tag{22}$$

Moreover, $\mathbb{X}_1 = \mathrm{Dom}(A)$ by the definition above. Note that the operator $A : \mathbb{X}_\theta \to \mathbb{X}_{\theta-1}$ can be boundedly extended or restricted for each $\theta \in \mathbb{R}$.

In fact, since the eigenvectors (18) are mutually orthogonal in $\mathbb{X}_\theta$ for all $\theta \in \mathbb{R}$, every $U \in \mathbb{X}_\theta$ has a unique expansion $U = \sum_{k=1,2}\sum_{j\in\mathbb{N}} c_{kj} y_{kj}$ where c_{1j}, c_{2j} are complex numbers. Define the operator A^θ for all $\theta \in \mathbb{R}$ by

$$A^\theta U := \sum_{k=1,2}\sum_{j\in\mathbb{N}} c_{kj}\lambda_{kj}^\theta y_{kj}.$$

Then

$$\|U\|^2_{\mathbb{X}_{\theta/2}} = \langle A^\theta U, U\rangle_{\mathbb{X}} = \sum_{k=1,2}\sum_{j\in\mathbb{N}} \lambda_{kj}^\theta |c_{kj}|^2 \|y_{kj}\|^2_{\mathbb{X}}. \tag{23}$$

Similarly,

$$\|U\|^2_{\mathbb{X}_{-\theta/2}} = \langle A^{-\theta} U, U\rangle_{\mathbb{X}} = \sum_{k=1,2}\sum_{j\in\mathbb{N}} \lambda_{kj}^{-\theta} |c_{kj}^2| \|y_{kj}\|^2_{\mathbb{X}}.$$

Semigroup formulation

Let $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^{\mathrm{T}} = (v, p, \dot{v}, \dot{p})^{\mathrm{T}}$. Then the system (2) with the output $y(t) = \frac{1}{h}\dot{p}(L,t)$ can be put into the following state-space formulation

$$\begin{cases} \dot{\psi} = \mathcal{A}\psi + BV(t) = \begin{pmatrix} 0 & I_{2\times 2} \\ -A & 0 \end{pmatrix}\psi + \begin{pmatrix} 0_{2\times 2} \\ B_0 \end{pmatrix} V(t), & \text{(24a)} \\ \psi(x,0) = \psi^0 & \text{(24b)} \\ y(t) = -B^*\psi = (0_{2\times 2} \quad B_0^*)\ \psi & \text{(24c)} \end{cases}$$

where

$$\begin{aligned} &B_0 \in \mathcal{L}(\mathbb{C}, \mathbb{X}_{-1/2}), \text{ with } B_0 V(t) = \begin{pmatrix} 0 \\ -\frac{1}{h}\delta(x-L) \end{pmatrix} V(t), \\ &B_0^* \in \mathcal{L}(\mathbb{X}_{1/2}, \mathbb{C}), \text{ with } B^*\psi = (0_{2\times 2} \quad B_0^*)^{\mathrm{T}}\psi = -\frac{1}{h}\psi_4(L), \end{aligned} \tag{25}$$

By the notation above we write $\mathrm{H} = \mathbb{X}_{1/2} \times \mathbb{X}$. The operator $\mathcal{A} : \mathrm{Dom}(\mathcal{A}) \subset \mathrm{H} \to \mathrm{H}$ with the choice of the domain

$$\mathrm{Dom}(\mathcal{A}) = \mathbb{X}_1 \times \mathbb{X}_{1/2} \tag{26}$$

$$= (H^2(0,L))^2 \times (H_L^1(0,L))^2 \bigcap \{\psi \in \mathrm{H} : \ \psi_{1x}(L) = \psi_{2x}(L) = 0\} \tag{27}$$

is densely defined in H.

Lemma 1 [29] *The infinitesimal generator $\mathcal{A}$ satisfies $\mathcal{A}^* = -\mathcal{A}$ on* H, *and $\mathcal{A}$ and $\mathcal{A}^*$ are unitary, i.e.,*

$$\operatorname{Re}\langle \mathcal{A}\psi, \psi\rangle_{\mathrm{H}} = \operatorname{Re}\left\langle \mathcal{A}^*\psi, \psi\right\rangle_{\mathrm{H}} = 0. \tag{28}$$

Also, $\mathcal{A}$ has a compact resolvent.

Consider the uncontrolled system

$$\begin{cases} \dot{\varphi} = \mathcal{A}\varphi, & \text{(29a)} \\ \varphi(x,0) = \varphi^0 & \text{(29b)} \\ y(t) = -B^*\varphi. & \text{(29c)} \end{cases}$$

Definition 1 The operator $B \in \mathcal{L}(\mathbb{C}, \mathrm{H}_{-1})$ is an admissible control operator for $\{\mathrm{e}^{\mathcal{A}t}\}_{t\geq 0}$ if there exists a positive constant $c(T)$ such that for all $u \in H^1(0,T)$,

$$\left\| \int_0^T \mathrm{e}^{\mathcal{A}(T-t)} Bu(t)\mathrm{d}t \right\|_{\mathrm{H}} \leq c(T)\|u\|_{\mathbb{L}^2(0,T)}.$$

Definition 2 The operator $B^* \in \mathcal{L}(\mathrm{Dom}(\mathcal{A}), \mathbb{C})$ is an admissible observation operator for $\{\mathrm{e}^{\mathcal{A}^*t}\}_{t\geq 0}$ if there exists a positive constant $c(T)$ such that for all $\varphi^0 \in \mathrm{Dom}(\mathcal{A})$

$$\int_0^T \|B^*\mathrm{e}^{\mathcal{A}^*t}\varphi^0\|^2 \,\mathrm{d}t \leq c(T)\|\varphi^0\|_{\mathrm{H}}^2.$$

The operator B^* is an admissible observation operator for $\{\mathrm{e}^{\mathcal{A}^*t}\}_{t\geq 0}$, if and only if B is an admissible control operator for $\{\mathrm{e}^{\mathcal{A}t}\}_{t\geq 0}$ [42, pg.127]).

It is proved in [29] that both B and B^* operators are admissible. Now the theorem on well-posedness of (24) is now immediate.

Theorem 2 [29] *Let $T > 0$, and $V(t) \in \mathbb{L}^2(0,T)$. For any $\psi^0 \in$ H, there exists positive constants $c_1(T)$, $c_2(T)$ and a unique solution to* (24) *with $\psi \in C([0,T];\mathrm{H})$, and*

$$\|\psi\|_{\mathrm{H}}^2 \leq c_1(T)\left\{\|\psi^0\|_{\mathrm{H}}^2 + \|V(t)\|_{\mathbb{L}^2(0,T)}^2\right\}, \tag{30}$$

$$\|y(t)\|_{\mathbb{L}^2(0,T)}^2 \,\mathrm{d}t \leq c_2(T)\left\{\|y(0)\|_{\mathrm{H}}^2 + \|V(t)\|_{\mathbb{L}^2(0,T)}^2\right\}. \tag{31}$$

We have the following theorem characterizing the eigenvalues and eigenfunctions of $\mathcal{A}$.

Theorem 3 *Let* $\sigma_j = \frac{(2j-1)\pi}{2L}$, $j \in \mathbb{N}$. *The eigenvalue problem* $\mathcal{A}Y = \lambda Y$ *has distinct eigenvalues*

$$\tilde{\lambda}_{1j}^{\mp} = \mp i\sqrt{\lambda_{1j}} = \frac{\mp i\sigma_j}{\zeta_1}, \quad \tilde{\lambda}_{2j}^{\mp} = \mp i\sqrt{\lambda_{2j}} = \frac{\mp i\sigma_j}{\zeta_2}, \quad j \in \mathbb{N} \tag{32}$$

Since $\tilde{\lambda}_{1j}^{-} = -\tilde{\lambda}_{1j}^{+}$, $\tilde{\lambda}_{2j}^{-} = -\tilde{\lambda}_{2j}^{+}$, $j \in \mathbb{N}$, *the corresponding eigenfunctions are*

$$Y_{1j} = \begin{pmatrix} \frac{1}{\tilde{\lambda}_{1j}^{+}} \\ \frac{b_1}{\tilde{\lambda}_{1j}^{+}} \\ 1 \\ b_1 \end{pmatrix} \sin\sigma_j x, \quad Y_{-1j} = \begin{pmatrix} \frac{1}{\tilde{\lambda}_{1j}^{+}} \\ \frac{b_1}{\tilde{\lambda}_{1j}^{+}} \\ -1 \\ -b_1 \end{pmatrix} \sin\sigma_j x,$$

$$Y_{2j} = \begin{pmatrix} \frac{1}{\tilde{\lambda}_{2j}^{+}} \\ \frac{b_2}{\tilde{\lambda}_{2j}^{+}} \\ 1 \\ b_2 \end{pmatrix} \sin\sigma_j x, \quad Y_{-2j} = \begin{pmatrix} \frac{1}{\tilde{\lambda}_{2j}^{+}} \\ \frac{b_2}{\tilde{\lambda}_{2j}^{+}} \\ -1 \\ -b_2 \end{pmatrix} \sin\sigma_j x, \quad j \in \mathbb{N} \tag{33}$$

where ζ_1, ζ_2, b_1 *and* b_2 *are defined by* (13)–(16). *The function*

$$\varphi(x,t) = \sum_{j\in\mathbb{N}} \left[c_{1j} Y_{1j} e^{\tilde{\lambda}_{1j}^{+}} t + d_{1j} Y_{-1j} e^{-\tilde{\lambda}_{1j}^{+} t} + c_{2j} Y_{2j} e^{\tilde{\lambda}_{2j}^{+} t} + d_{2j} Y_{-2j} e^{-\tilde{\lambda}_{2j}^{+} t} \right] \tag{34}$$

solves (29) *for the initial data*

$$\begin{aligned} \varphi^0 &= \sum_{j\in\mathbb{N}} \left[c_{1j} Y_{1j} + d_{1j} Y_{-1j} + c_{2j} Y_{2j} + d_{2j} Y_{-2j} \right] \\ &= \sum_{j\in\mathbb{N}} \begin{pmatrix} \frac{1}{\tilde{\lambda}_{1j}^{+}}(c_{1j}+d_{1j}) + \frac{1}{\tilde{\lambda}_{2j}^{+}}(c_{2j}+d_{2j}) \\ \frac{b_1}{\tilde{\lambda}_{1j}^{+}}(c_{1j}+d_{1j}) + \frac{b_2}{\tilde{\lambda}_{2j}^{+}}(c_{2j}+d_{2j}) \\ (c_{1j}-d_{1j}) + (c_{2j}-d_{2j}) \\ b_1(c_{1j}-d_{1j}) + b_2(c_{2j}-d_{2j}) \end{pmatrix} \sin\sigma_j x \end{aligned} \tag{35}$$

where $\{c_{kj}, d_{kj}, \quad k = 1, 2, \quad j \in \mathbb{N}\}$ *are complex numbers such that*

$$\|\varphi^0\|_{\mathrm{H}}^2 \asymp \sum_{j\in\mathbb{N}} \left(|c_{1j}|^2 + |d_{1j}|^2 + |c_{2j}|^2 + |d_{2j}|^2 \right), \quad \text{i.e.,} \tag{36}$$

$$\tilde{C}_1 \, \|\varphi^0\|_{\mathrm{H}}^2 \leq \sum_{j\in\mathbb{N}} \left(|c_{1j}|^2 + |d_{1j}|^2 + |c_{2j}|^2 + |d_{2j}|^2 \right) \leq \tilde{C}_2 \, \|\varphi^0\|_{\mathrm{H}}^2 \tag{37}$$

with two positive constants $\tilde{C}_1, \tilde{C}_2$ *which are independent of the particular choice of* $\Psi^0 \in \mathrm{H}$.

Proof Let $W=(W_1,W_2)^{\mathrm{T}}$. Solving the eigenvalue problem $\mathcal{A}W=\tilde{\lambda}W$ is equivalent to solving $AW_1=-\tilde{\lambda}^2 W_1$ and $W_2=\tilde{\lambda}W_1$. Since $\{\lambda_{1j},\ \lambda_{2j},\ j\in\mathbb{N}\}$ defined by (17) are the eigenvalues of A, it follows that $\tilde{\lambda}_{1j}^{\mp}=\mp i\sqrt{\lambda_{1j}}$ and $\tilde{\lambda}_{2j}^{\mp}=\mp i\sqrt{\lambda_{2j}},\quad j\in\mathbb{N}$, and therefore (32) follows. (34) and (35) follow from (32), (33) and Theorem 1. For the proof of (37), see [29]. □

It is easy to show that the eigenfunctions $\{Y_{kj}, k=-2,-1,1,2, j\in\mathbb{N}\}$ are mutually orthogonal in H [with respect to the inner product (9)]. Therefore, they form a Riesz basis in H. This result also follows from the fact that we have a skew-symmetric operator $\mathcal{A}$ with a compact resolvent (see Lemma 1).

For $\theta\in\mathbb{R}$, we define the space

$$\mathcal{S}_\theta:=\left\{\sum_{k=1,2}\sum_{j\in\mathbb{N}}c_{kj}Y_{kj}+d_{kj}Y_{-kj}\ :\ \sum_{k=1,2}\sum_{j\in\mathbb{N}}|\tilde{\lambda}_{kj}|^{2\theta}\left(\left|c_{kj}\right|^2+\left|d_{kj}\right|^2\right)<\infty\right\} \tag{38}$$

by the completion of eigenvectors $\{Y_{kj},\quad k=-2,-1,1,2,\quad j\in\mathbb{N}\}$ with respect to the norm

$$\|U\|^2_{\mathcal{S}_\theta}=\sum_{k=1,2}\sum_{j\in\mathbb{N}}|\tilde{\lambda}_{kj}|^{2\theta}\left(\left|c_{kj}\right|^2+\left|d_{kj}\right|^2\right). \tag{39}$$

Remark 1 For the simplicity of the calculations in the next sections, we use the equivalent norm $\|U\|_{\mathcal{S}_\theta}=\left(\sum_{k=1,2}\sum_{j\in\mathbb{N}}|2j-1|^{2\theta}\left(\left|c_{kj}\right|^2+\left|d_{kj}\right|^2\right)\right)^{\frac{1}{2}}$. This follows from $\zeta_1,\zeta_2>0$.

Denote the space $\mathcal{S}_{-\theta}$ to be dual of $\mathcal{S}_\theta$ pivoted with respect to $\mathcal{S}_0:=\mathrm{H}=(H^1_L(0,L))^2\times(\mathbb{L}^2(0,L))^2$. By (23)

$$\begin{aligned}\mathcal{S}_1&=\mathbb{X}_1\times\mathbb{X}_{1/2}=\mathrm{Dom}(\mathcal{A})\\ \mathcal{S}_0&=\mathbb{X}_{1/2}\times\mathbb{X}=\mathrm{H}\\ \mathcal{S}_{-1}&=\mathbb{X}\times\mathbb{X}_{-1/2}\end{aligned}$$

Let $0<\varepsilon<\frac{1}{2}$. By (38), we can also define the interpolation spaces

$$\mathbb{X}_{1/2+\varepsilon/2}=[\mathbb{X}_1,\mathbb{X}_{1/2}]_{1-\varepsilon/2},\mathbb{X}_\varepsilon=[\mathbb{X}_{1/2},\mathbb{X}]_{1-\varepsilon/2}$$

so that

$$[\mathcal{S}_1,\mathcal{S}_0]_{1-\varepsilon}=\mathcal{S}_\varepsilon=\mathbb{X}_{1/2+\varepsilon/2}\times\mathbb{X}_{\varepsilon/2},$$

$$[\mathcal{S}_2,\mathcal{S}_1]_{1-\varepsilon}=\mathcal{S}_{1+\varepsilon}=\mathbb{X}_{1+\varepsilon/2}\times\mathbb{X}_{1/2+\varepsilon/2}$$

and their duals $\mathcal{S}_{-1-\varepsilon}$ and $\mathcal{S}_{-\varepsilon}$ pivoted with respect to $\mathcal{S}_0=\mathrm{H}$; see [38] for more information on interpolation spaces. We have the following dense compact embeddings

$$\mathcal{S}_{1+\varepsilon} \subset \mathcal{S}_1 \subset \mathcal{S}_\varepsilon \subset \mathcal{S}_0 \subset \mathcal{S}_{-\varepsilon} \subset \mathcal{S}_{-1} \subset \mathcal{S}_{-1-\varepsilon}.$$

With the notation above $\mathcal{S}_{-1-\varepsilon} = \mathbb{X}_{-\varepsilon/2} \times \mathbb{X}_{-1/2-\varepsilon/2}$.

Now we have the following result from [42]:

Since $A : \mathbb{X}_1 \to \mathbb{X}$ can be uniquely extended (or restricted) to $\tilde{\tilde{A}} : \mathbb{X}_\theta \to \mathbb{X}_{\theta-1}$ for any $\theta \in \mathbb{R}$, the infinitesimal generator $\mathcal{A} : \mathrm{S}_1 \to \mathcal{S}_0$ can be uniquely extended to $\tilde{\tilde{\mathcal{A}}} : \mathcal{S}_{-\varepsilon} \to \mathcal{S}_{-1-\varepsilon}$.

Corollary 1 *The semigroup* $\{\mathrm{e}^{\mathcal{A}t}\}_{t\geq 0}$ *with the generator* $\mathcal{A} : \mathrm{S}_1 \to \mathcal{S}_0$ *has a unique extension to a contraction semigroup* $\{\mathrm{e}^{\tilde{\tilde{\mathcal{A}}}t}\}_{t\geq 0}$ *on* $\mathcal{S}_{-1-\varepsilon}$ *with the generator* $\tilde{\tilde{\mathcal{A}}} : \mathcal{S}_{-\varepsilon} \to \mathcal{S}_{-1-\varepsilon}$ *for any* $0 < \varepsilon < \frac{1}{2}$.

3 Exact observability

We start with the definition of exact observability.

Definition 3 The pair $(\mathcal{A}^*, B^*)$ is exactly observable in time $T > 0$ if there exists a positive constant $C(T)$ such that for all $\varphi^0 \in \mathrm{H}$

$$\int_0^T \|B^* \mathrm{e}^{\mathcal{A}^* t} \varphi^0\|^2 \, \mathrm{d}t \geq C(T) \|\varphi^0\|^2_{\mathrm{H}}.$$

The following theorem is proved in [29].

Theorem 4 *Assume that* $\frac{\zeta_2}{\zeta_1} \in \mathbb{R} - \mathbb{Q}$. *The eigenvalues* $\{\tilde{\lambda}^{\mp}_{1j} = \frac{\mp i\sigma_j}{\zeta_1}, \ \tilde{\lambda}^{\mp}_{2m} = \frac{\mp i\sigma_m}{\zeta_2}, \ \ j, m \in \mathbb{N}\}$ *given by Theorem 3 can get arbitrarily close to each other for some choices of* j *and* m. *Therefore, the system* (29) *is not exactly observable on* H.

For the system (29), Ingham-type theorems (see, i.e., [23,42]) can not be used to obtain the observability inequality since they require a uniform gap between the eigenvalues. This type of problem is well studied for joint structures with a point mass at the joint (see [23] and references therein), or for networks of strings/beams with different lengths (see [16] and references therein). The main idea of proving observability result is based on the use of divided differences [44], the generalized Beurling's theorem, and the Diophantine's approximation. We try the idea in [23] with the following technical result to prove our main observability result.

Lemma 2 *Assume that* $\frac{\zeta_2}{\zeta_1} \in \mathbb{R} - \tilde{\mathbb{Q}}$ *where the set* $\tilde{\mathbb{Q}}$ *is defined in Theorem 10. Then there exists a number* $\tilde{\tau} > 0$ *such that if*

$$0 < |\tilde{\lambda}^{+}_{kj} - \tilde{\lambda}^{+}_{lm}| \leq \tilde{\tau}, \quad k, l = 1, 2, \quad j, m \in \mathbb{N} \tag{40}$$

then $k \neq l$ *and*

$$|\tilde{\lambda}^{+}_{1j} - \tilde{\lambda}^{+}_{2m}| \geq \frac{C_\alpha}{|\tilde{\lambda}^{+}_{1j}|^\alpha}, \quad |\tilde{\lambda}^{+}_{1j} - \tilde{\lambda}^{+}_{2m}| \geq \frac{C_\alpha}{|\tilde{\lambda}^{+}_{2m}|^\alpha}$$

for every $\alpha > 1$, *with a constant* C_α *independent of the particular choice of* $\tilde{\lambda}_{1j}$ *and* $\tilde{\lambda}_{2m}$.

Proof of Lemma 2 Since $\zeta_1, \zeta_2 \in \mathbb{R} - \tilde{\mathbb{Q}}$, we have $\tilde{\lambda}^+_{kj} \neq \tilde{\lambda}^+_{lm}$ for any $k, l = 1, 2,\quad j, m \in \mathbb{N}$. If we choose $\tilde{\tau} < \left(\frac{\pi}{L}\right)\min(\frac{1}{\zeta_1}, \frac{1}{\zeta_2})$, (40) is satisfied. This implies that $k \neq l$. By Theorem 10, there exists a sequence of odd integers $\{\tilde{p}_j\}, \{\tilde{q}_j\} \to \infty$ and $\alpha > 1$ such that

$$\left|\tilde{q}_j \frac{\zeta_2}{\zeta_1} - \tilde{p}_j\right| \geq \frac{\tilde{C}_\alpha}{(\tilde{q}_j)^\alpha}.$$

Therefore

$$\left|\tilde{\lambda}^+_{1j} - \tilde{\lambda}^+_{2m}\right| = \frac{\pi}{2L}\left|\frac{(2j+1)}{\zeta_1} - \frac{(2m+1)}{\zeta_2}\right| \geq \frac{\pi}{2L}\frac{\tilde{C}_\alpha}{(2j+1)^\alpha} \geq \frac{\tilde{C}_\alpha}{|\tilde{\lambda}_{1j}|^\alpha},$$

and there is always a rational number r such that $(2j+1) = r(2m+1)$ so that C_α can be chosen smaller to get

$$\left|\tilde{\lambda}^+_{1j} - \tilde{\lambda}^+_{2m}\right| \geq \frac{C_\alpha}{|\tilde{\lambda}^+_{2m}|^\alpha}.$$

□

We also need the following technical lemma from [23, Chap.9] which is a slightly different version of the result obtained in [44]:

Lemma 3 *Given an increasing sequence* $\{s_n\}$ *of real numbers satisfying*

$$s_{n+2} - s_n \geq 2\tau \quad \text{for all } n, \tag{41}$$

fix $0 < \tau' \leq \tau$ *arbitrarily and introduce the divided differences of* $\{e_n(t), e_{n+1}(t)\}$ *of exponential functions* $\{\mathrm{e}^{i s_n t}, \mathrm{e}^{i s_{n+1} t}\}$ *by*

$$e_n(t) = \mathrm{e}^{i s_n t}, \quad e_{n+1}(t) = \frac{\mathrm{e}^{i s_{n+1} t} - \mathrm{e}^{i s_n t}}{s_{n+1} - s_n}. \tag{42}$$

Then there exists positive constants $\tilde{c}_3(T)$ *and* $\tilde{c}_4(T)$ *such that* $T > \frac{2\pi}{\tau}$

$$\tilde{c}_3(T) \sum_{n=-\infty}^{\infty} |a_n|^2 \leq \int_0^T |f(t)|^2 \,\mathrm{d}t \leq \tilde{c}_4(T) \sum_{n=-\infty}^{\infty} |a_n|^2$$

holds for all functions given by the sum $f(t) = \sum_{n=-\infty}^{\infty} a_n e_n(t) \ :\ \sum_{n=-\infty}^{\infty} |a_n|^2 < \infty$.

Now we can prove our main observability result:

Theorem 5 *Let $\frac{\zeta_2}{\zeta_1} \in \mathbb{R} - \tilde{\mathbb{Q}}$ and $T > 2L(\zeta_1 + \zeta_2)$. Then there exists a constant $C = C(T) > 0$ such that solutions φ of the problem* (29) *satisfy the following observability estimate:*

$$\int_0^T |B^*\varphi|^2 \, \mathrm{d}t \geq C(T)\|\varphi^0\|^2_{\mathcal{S}_{-1-\varepsilon}}. \tag{43}$$

where $\mathcal{S}_{-1-\varepsilon}$ is defined by (39).

Proof Let $s_{1j} = \frac{\sigma_j}{\zeta_1} = \frac{(2j-1)\pi}{2L\zeta_1}$ and $s_{2j} = \frac{\sigma_j}{\zeta_2} = \frac{(2j-1)\pi}{2L\zeta_2}$ for $j \in \mathbb{N}$. The set of eigenvalues (32) can be rewritten as

$$\tilde{\lambda}^{\mp}_{kj} = \mp i s_{kj}, \quad k = 1, 2, \quad j \in \mathbb{N}. \tag{44}$$

Since $\mathcal{A}^* = -\mathcal{A}$, the function $\varphi = \mathrm{e}^{\mathcal{A}^* t}\varphi^0$, given explicitly by (34), solves (29), and by (25) and (32)–(37)

$$B^*\varphi = \sum_{k=1,2} \sum_{j\in\mathbb{N}} b_k(-1)^j \left(c_{kj}\mathrm{e}^{is^+_{kj}t} + d_{kj}\mathrm{e}^{-is^+_{kj}t} \right).$$

By (39), showing (43) is equivalent to showing

$$\int_0^T |B^*\varphi|^2 \, \mathrm{d}t = \int_0^T \left| \sum_{k=1,2} \sum_{j\in\mathbb{N}} b_k(-1)^j \left(c_{kj}\mathrm{e}^{is^+_{kj}t} + d_{kj}\mathrm{e}^{-is^+_{kj}t} \right) \right|^2 \mathrm{d}t$$
$$\geq C(T) \sum_{k=1,2} \sum_{j\in\mathbb{N}} \frac{\left|c_{kj}\right|^2 + \left|d_{kj}\right|^2}{|\tilde{\lambda}_{kj}|^{2+2\varepsilon}}. \tag{45}$$

Let's rearrange $\{\mp s^+_{kj} : k = 1, 2, \ j \in \mathbb{N}\}$ into an increasing sequence of $\{s_n, \ n \in \mathbb{N}\}$. Denote the coefficients $\{(-1)^j b_k c_{kj}, (-1)^j b_k d_{kj}\}$ by g_n (recall that $b_1, b_2 \in \mathbb{R} - \{0\}$). Then showing (45) is equivalent to showing

$$\int_0^T |B^*\varphi|^2 \, \mathrm{d}t = \int_0^T \left| \sum_{n\in\mathbb{N}} g_n \mathrm{e}^{is_n t} \right|^2 \mathrm{d}t \geq C(T) \sum_{n\in\mathbb{N}} \frac{|g_n|^2}{|s_n|^{2+2\varepsilon}}. \tag{46}$$

Let $n^+(r)$ denotes the largest number of terms of the sequence $\{s_n, \ n \in \mathbb{N}\}$ contained in an interval of length r. Then

$$\frac{L(\zeta_1 + \zeta_2)\, r}{\pi} - 1 \leq n^+(r) \leq \frac{L(\zeta_1 + \zeta_2)\, r}{\pi} + 1$$

Therefore $D^+ = \lim_{r\to\infty} \frac{n^+(r)}{r} = \frac{L(\zeta_1+\zeta_2)}{\pi}$. Now let $\tau = \frac{\pi}{2L}\min\left(\frac{1}{\zeta_1}, \frac{1}{\zeta_2}\right)$ so that

$$s_{n+2} - s_n \geq 2\tau, \quad \text{for all } \ n. \tag{47}$$

Note that the condition $T > \frac{2\pi}{\tau}$ can be replaced by $T > 2\pi D^+ = 2L(\zeta_1 + \zeta_2)$ (See Prop. 9.3 in [23]). Now we fix $0 < \tau' < \tau$ and define sets A_1 and A_2 of integers by

$$A_1 := \{m, m+1 \in \mathbb{N} \ : \ s_{m+1} - s_m < \tau'\}$$
$$A_2 := \{k \in \mathbb{N} \ : \ |s_k - s_m| \geq \tau', \ \ m \in A_1\}.$$

Observe that index n of the eigenvalues $\{s_n\}$ belongs to either A_1 or A_2. For $m \in A_1$, the exponents $\{s_m, s_{m+1}\}$ form a *chain of close exponents* for τ' and there is no chain of close exponents longer than two elements. For $m \in A_1$, the divided differences $e_m(t), e_{m+1}(t)$ of the exponential functions are defined by (42).

Therefore, by Lemma 3 for all $T > 2L(\zeta_1 + \zeta_2)$ we have

$$\int_0^T \left| \sum_{n\in\mathbb{N}} a_n e_n(t) \right|^2 \, \mathrm{d}t \asymp \sum_{n\in\mathbb{N}} |a_n|^2. \tag{48}$$

If $m \in A_1$, we rewrite the sums as

$$\sum_{n=m}^{m+1} g_n \mathrm{e}^{i s_n t} = \sum_{n=m}^{m+1} a_n e_n(t)$$

where $a_m = g_m + \frac{a_{m+1}}{s_{m+1} - s_m}, \quad a_{m+1} = g_{m+1}(s_{m+1} - s_m)$. Then there exists a constant $C > 0$ independent of m such that

$$\sum_{n=m}^{m+1} |g_n|^2 |s_{m+1} - s_m|^2 \leq C \sum_{n=m}^{m+1} |a_n|^2. \tag{49}$$

By Lemma 2, there exists a constant $C_\alpha > 0$ such that

$$|s_{m+1} - s_m|^2 \geq \frac{C_\alpha}{|s_m|^{2\alpha}}, \quad \text{and} \quad |s_{m+1} - s_m|^2 \geq \frac{C_\alpha}{|s_{m+1}|^{2\alpha}}$$

where $\alpha > 1$. Therefore by (49) for all $\alpha = 1 + \varepsilon$

$$\sum_{n=m}^{m+1} \frac{|g_n|^2}{|s_n|^{2+2\varepsilon}} \leq \frac{C}{C_{1+\varepsilon}} \sum_{n=m}^{m+1} |a_n|^2.$$

On the other hand, if $n \in A_2$, with the choice of a smaller $C_{1+\varepsilon}$ (if necessary) we get

$$\frac{|g_n|^2}{|s_n|^{2+2\varepsilon}} \leq \frac{C}{C_{1+\varepsilon}} |g_n|^2, \tag{50}$$

and (48), (49), and (50) imply that for $T > 2\pi D^+ = 2L(\zeta_1 + \zeta_2)$

$$\sum_{n\in\mathbb{N}} \frac{|g_n|^2}{|s_n|^{2+2\varepsilon}} \le \frac{C}{C_{1+\varepsilon}} \sum_{n\in\mathbb{N}} |a_n|^2.$$

This together with (48) implies (46), and therefore (43) holds. □

Corollary 2 *Let* $\frac{\zeta_2}{\zeta_1} \in \tilde{\tilde{\mathbb{Q}}}$ *and* $T > 2L(\zeta_1 + \zeta_2)$. *Then there exists a constant* $C = C(T) > 0$ *such that solutions of the problem* (29) *satisfy the following observability estimate:*

$$\int_0^T |B^*\varphi|^2 \, \mathrm{d}t \ge C(T)\|\varphi^0\|^2_{\mathcal{S}_{-1}}. \tag{51}$$

where $\mathcal{S}_{-1}$ *is defined by (38).*

Proof If we replace the inequality of (66) by (67), then the proofs of Lemma 2 and Theorem 5 can be adapted for $\varepsilon = 0$. This implies that the observability inequality (43) holds as $\mathcal{S}_{-1+\varepsilon}$ is replaced by $\mathcal{S}_{-1}$. □

Remark 2 Note that the lower bound of the control time $T = 2L(\zeta_1 + \zeta_2)$ obtained in Theorem 5 and Corollary 2 is optimal. The optimality of the control time can be obtained by using the theory (i.e., see [7,8]). However, since the main scope of the paper is proving the polynomial stability and investigating the decay rates, we plan to use their idea in the upcoming research of exact controllability of the elastic beam/patch system.

4 Stabilization

The signal $\dot{p}(L, t)$ is the observation dual to the control operator B in (24), and so we choose the feedback $V(t) = -\frac{1}{2}B^*z = \frac{1}{2h}\dot{p}(L, t)$ in (24). Also, since $\dot{p}(L, t)$ is the total current at the electrodes, this variable can be measured easier than the velocity of the beam at one end. The system (5) can be put in the following form

$$\begin{cases} \dot{z}(t) = \mathcal{A}_d z(t) = \begin{pmatrix} 0 & I_{2\times 2} \\ -A & -\frac{1}{2}B_0B_0^* \end{pmatrix} z, & \text{(52a)} \\ z(x, 0) = z^0, & \text{(52b)} \\ y(t) = -B^*z(t) & \text{(52c)} \end{cases}$$

where $\mathcal{A}_d : \mathrm{Dom}(\mathcal{A}_d) \subset \mathrm{H} \to \mathrm{H}$ and $\mathrm{Dom}(\mathcal{A}_d)$ is defined by

$$\begin{aligned}\mathrm{Dom}(\mathcal{A}_d) = \big\{ & z \in (H^2(0,L))^2 \times (H^1_L(0,L))^2 \ : \ z_1(0) = z_2(0) = 0, \\ & \alpha z_{1x}(L) - \gamma\beta z_{2x}(L) = 0, \ \ \beta z_{2x}(L) - \gamma\beta z_{1x}(L) = -\tfrac{z_4(L)}{2h^2} \big\}. \end{aligned} \tag{53}$$

Note that the system above is equivalent to the system studied in [6].

Definition 4 The semigroup $\{\mathrm{e}^{\mathcal{A}_d t}\}_{t\geq 0}$ with the generator $\mathcal{A}_d$ is exponentially stable on H if there exists constants $M, \mu > 0$ such that $\|\mathrm{e}^{\mathcal{A}_d t}\|_{\mathrm{H}} \leq M\mathrm{e}^{-\mu t}$ for all $t \geq 0$.

Theorem 6 [28,29]

(i) $\mathcal{A}_d : \mathrm{Dom}(\mathcal{A}_d) \to \mathrm{H}$ *is the infinitesimal generator of a* C_0*-semigroup of contractions. Therefore for every* $T \geq 0$*, and* $z^0 \in \mathrm{H}$*,* z *solves (52) with* $z \in C([0,T];\mathrm{H})$.
(ii) *The spectrum* $\sigma(\mathcal{A}_d)$ *of* $\mathcal{A}_d$ *has all isolated eigenvalues. The semigroup* $\{\mathrm{e}^{\mathcal{A}_d t}\}_{t\geq 0}$ *is strongly stable on* H *if and only if* $\frac{\zeta_1}{\zeta_2} \neq \frac{2n+1}{2m+1}$*, for some* $n, m \in \mathbb{N}$*. where* ζ_1 *and* ζ_2 *are defined by* (13) *and* (14)*, respectively.*
(iii) *Assume that* $\frac{\zeta_2}{\zeta_1} \in \mathbb{R} - \mathbb{Q}$*. The semigroup* $\{\mathrm{e}^{\mathcal{A}_d t}\}_{t\geq 0}$ *is not exponentially stable on* H.

Decay estimates

We need the following results to prove our main stabilization results given by Theorem 7.

Lemma 4 [6, Lemma 4.4] *Let* $\{\mathcal{E}_k\}_{k\in\mathbb{N}}$ *be a sequence of real numbers satisfying*

$$\mathcal{E}_{k+1} \leq \mathcal{E}_k - C\mathcal{E}_{k+1}^{2+\alpha}$$

where $C > 0$ *and* $\alpha > -1$ *are constants. Then there exists a positive constant* $M = M(\alpha, C)$ *such that*

$$\mathcal{E}_k \leq \frac{M}{(k+1)^{\frac{1}{1+\alpha}}}, \quad k \in \mathbb{N}.$$

Lemma 5 [12, Theorem 2.2] *Let* $m \in \mathbb{R}^+$*,* $\omega_1 : (m,\infty) \to (0, \omega_1(m))$ *and* $\omega_2 : (m,\infty) \to (0,\infty)$ *be convex and increasing, and convex and decreasing functions, respectively, with* $\omega_1(\infty) = 0$ *and* $\omega_2(\infty) = \infty$*. Let* $\Phi : (0, \omega_1(m)) \to (0,\infty)$ *and* $\Psi : (0, \omega_2(m)) \to (0,\infty)$ *be concave and increasing functions with* $\Phi(0) = 0$*,* $\Psi(\infty) = \infty$*, and for all* $t > m$

$$1 \leq \Phi(\omega_1(t))\Psi(\omega_2(t)).$$

Then for $j \in \mathbb{N}$, $j \geq m$, and any $0 \neq f = \{f_j\}_{j\in\mathbb{N}} \in l_1(\mathbb{N};\mathbb{R})$, we have

$$1 \leq \Phi\left(\frac{\sum_{j\in\mathbb{N}} |f_j|\omega_1(j)}{\sum_{j\in\mathbb{N}} |f_j|}\right)\Psi\left(\frac{\sum_{j\in\mathbb{N}} |f_j|\omega_2(j)}{\sum_{j\in\mathbb{N}} |f_j|}\right)$$

where $\{f_j\omega_1 j\}_{j\in\mathbb{N}}, \{f_j\omega_2(j)\}_{j\in\mathbb{N}} \in l^1(\mathbb{N};\mathbb{R})$, and therefore

$$\sum_{j\in\mathbb{N}} |f_j|\omega_1(j) \geq \mathcal{G}^{-1}_{\Phi,\Psi}\left(\frac{\sum_{j\in\mathbb{N}} |f_j|\omega_2(j)}{\sum_{j\in\mathbb{N}} |f_j|}\right), \quad \mathcal{G}_{\Phi,\Psi}(j) = \frac{1}{\Psi^{-1}\left(\frac{1}{\Phi(j)}\right)}. \tag{54}$$

Lemma 5 is the discrete version of the Hölder's inequality originally proved in [12]. That is, we use the discrete measure μ with the measurable weights ω_1 and ω_2. For instance $\int_m^\infty |f(x)|\mathrm{d}\mu(x) = \sum_{(m\leq)j\in\mathbb{N}} |f(x_j)|\omega_1(j)$ where $f = \{f_j\}_{(m\leq)j\in\mathbb{N}} \in l^1(\mathbb{N};\mathbb{R})$.

Now we are ready to prove our main stabilization result:

Theorem 7 (I) *Suppose that $\frac{\zeta_2}{\zeta_1} \in \tilde{\mathbb{Q}}$ where $\tilde{\mathbb{Q}}$ is defined in Theorem 10. Then for all $t \geq 0$, there exists a positive constant M_1 such that*

$$\|z(t)\|^2_{\mathrm{H}} \leq \frac{M_1}{(t+1)^{\frac{1}{1+\varepsilon}}}\|z^0\|^2_{\mathcal{S}_{1+\varepsilon}}. \tag{55}$$

(II) *Suppose that $\frac{\zeta_2}{\zeta_1} \in \tilde{\tilde{\mathbb{Q}}}$ where $\tilde{\tilde{\mathbb{Q}}}$ is defined in (67). Then for all $t \geq 0$, there exists a positive constant M_2 such that*

$$\|z(t)\|^2_{\mathrm{H}} \leq \frac{M_2}{t+1}\|z^0\|^2_{\mathcal{S}_1}. \tag{56}$$

Proof Assume that ψ and φ solve (24) and (29) with the initial data $\psi^0 = 0$, $\varphi^0 = z^0$, and with $V(t) = B^*z$ so that $z = \varphi + \psi$ solves (52). By (43) we have

$$\int_0^T |B^*\varphi|^2 \,\mathrm{d}t \geq C(T)\|z^0\|^2_{\mathcal{S}_{-1-\varepsilon}}. \tag{57}$$

On the other hand since $B^*z = B^*\varphi + B^*\psi$, we can write

$$|B^*\varphi| \leq |B^*z| + |B^*\psi|.$$

By (30) with $V(t) = B^*z$, $\psi^0 = 0$, and $y(t) = B^*\psi$

$$|B^*\psi| \leq |B^*z|,$$

and by (57) we obtain

$$\int_0^T |B^* z|^2 \, \mathrm{d}t \geq C(T) \|z^0\|^2_{\mathcal{S}_{-1-\varepsilon}}. \tag{58}$$

This proves the observability result for (52).

To apply Lemma 5, we choose

$$m = 1/2, \quad \omega_1(j) = \frac{1}{(2j-1)^{2+2\varepsilon}}, \quad \omega_2(j) = (2j-1)^2,$$

and two functions $\Phi(t)$ and $\Psi(t)$:

$$\Phi(t) = \frac{1}{\omega_1^{-1}(t)} = \frac{2}{\left(\frac{1}{t}\right)^{\frac{1}{2+2\varepsilon}} + 1}, \quad \Psi(t) = \omega_2^{-1} = \frac{\sqrt{t}+1}{2}.$$

Then $\mathcal{G}_{\Phi,\Psi}(t) = \frac{1}{t^{\frac{1}{1+\varepsilon}}}$ and $\mathcal{G}^{-1}_{\Phi,\Psi}(t) = \frac{1}{t^{1+\varepsilon}}$. Denoting $\{|f_j|\}$ and $\{\tilde{\lambda}_{kj}\}$ by $\{|c_{kj}|^2 + |d_{kj}|^2\}$ and $\{\tilde{\lambda}_j\}$, respectively,

$$\sum_{j\in\mathbb{N}} |f_j| \, \omega_1(j) = \|f\|^2_{\mathcal{S}_{-1-\varepsilon}}, \quad \sum_{j\in\mathbb{N}} |f_j| = \|f\|^2_{\mathrm{H}}, \quad \sum_{j\in\mathbb{N}} |f_j| \, \omega_2(j) = \|f\|^2_{\mathcal{S}_1} \tag{59}$$

where we used (38) and Remark 1. By using (54) together with (59) we obtain

$$\|z^0\|^2_{\mathcal{S}_{-1-\varepsilon}} \geq \|z^0\|^2_{\mathrm{H}} \, \mathcal{G}^{-1}_{\Phi,\Psi}\left(\frac{\|z^0\|^2_{\mathcal{S}_1}}{\|z^0\|^2_{\mathrm{H}}}\right) = \|z^0\|^2_{\mathrm{H}}\left(\frac{\|z^0\|^2_{\mathrm{H}}}{\|z^0\|^2_{\mathcal{S}_1}}\right)^{1+\varepsilon} = \frac{\|z^0\|^{4+2\varepsilon}_{\mathrm{H}}}{\|z^0\|^{2+2\varepsilon}_{\mathcal{S}_1}}. \tag{60}$$

By Theorem 7, (58), (60), and the fact that the function $t \mapsto \|z(t)\|_{\mathrm{H}}$ is nonincreasing, we obtain

$$\begin{aligned}
\|z(T)\|^2_{\mathrm{H}} &= \|z^0\|^2_{\mathrm{H}} - \int_0^T |B^* z|^2 \, \mathrm{d}t \\
&\leq \|z^0\|^2_{\mathrm{H}} - C(T)\|z^0\|^2_{\mathcal{S}_{-1-\varepsilon}} \\
&\leq \|z^0\|^2_{\mathrm{H}} - C(T)\frac{\|z^0\|^{4+2\varepsilon}_{\mathrm{H}}}{\|z^0\|^{2+2\varepsilon}_{\mathcal{S}_1}} \\
&\leq \|z^0\|^2_{\mathrm{H}} - C(T)\frac{\|z(T)\|^{4+2\varepsilon}_{\mathrm{H}}}{\|z^0\|^{2+2\varepsilon}_{\mathcal{S}_1}}.
\end{aligned} \tag{61}$$

The estimate (61) remains valid in successive intervals $[mT, (m+1)T]$. So, for all $m \geq 0$, we have

$$\|z((m+1)T)\|_{\mathrm{H}}^2 \le \|z(mT)\|_{\mathrm{H}}^2 - C(T)\frac{\|z((m+1)T)\|_{\mathrm{H}}^{4+2\varepsilon}}{\|z^0\|_{\mathcal{S}_1}^{2+2\varepsilon}}. \tag{62}$$

By letting $\mathcal{E}_m = \frac{\|z(mT)\|_{\mathrm{H}}^2}{\|z^0\|_{\mathcal{S}_1}^2}$, (62) gives

$$\mathcal{E}_{m+1} \le \mathcal{E}_m - C(T)(\mathcal{E}_{m+1})^{2+\varepsilon}, \quad m \in \mathbb{N}.$$

Hence, by Lemma 4, there exists a constant $M_1 = M_1(C(T)) > 0$ such that (56) follows.

The proof of (II) is similar to the proof of (I) modulo a few simple changes. We take $\varepsilon = 0$, and use the observability inequality (51) instead of (43). □

5 Conclusion and future research

The main result of this paper is to show that magnetic effects in piezoelectric beams, even though small, have a dramatic effect on exact observability and stabilizability. The piezoelectric beam model, without magnetic effects, is exactly observable and exponentially stabilizable, by a B^*-type feedback However, when magnetic effects are included, the beam is not exactly observable or exponentially stabilizable. By the B^*-type feedback, the beam can be exactly observable and polynomially stabilizable for the initial data z^0 in $\mathcal{S}_1$ and $\mathcal{S}_{1+\varepsilon}$ when the ratio $\frac{\xi_2}{\xi_1}$ is in the sets $\tilde{\mathrm{Q}}$ or $\tilde{\tilde{\mathrm{Q}}}$, respectively. These sets are of Lebesgue measure zero even though they are uncountable.

A single piezoelectric beam model using the Euler–Bernoulli or Mindlin–Timoshenko small displacement assumptions is assumed to contract/extend only (by the linear theory). The voltage control does not even affect the bending motions [32]. A related and more interesting problem is to find the decay rates of the elastic beam/patch system (see Fig. 2a). Once the magnetic effects are included [32], the behavior of the system differs substantially from the classical counterparts [10,20,39,40] which use electrostatic or quasi-static assumptions. In this model, the stretching equations (2) are coupled to the bending (and rotation) equations, and it is similar in nature to the transmission problem proposed by Lions [24]. The beam domain is divided into three sub-domains; first and the third for the pure elastic and the second for piezo-elastic coupling. Previous research on controllability of elastic beam/plate with piezoelectric patches without magnetic effects showed that the location of the patch(es) on the beam/plate strongly determines the controllability and stabilizability. This paper, [28,29,32] suggest that the controllability and stabilizability depends on not only the location of the patches but also the system parameters. This is currently under investigation.

Our results in this paper also have strong implications on the controllability of smart sandwich structures such as Active Constrained Layer (ACL) damped structures (see Fig. 2b). The classical sandwich beam or plate is an engineering model for a multi-layer beam consisting of "face" plates and "core" layers that are orders of magnitude are more compliant than the face plates. ACL damped beams are sandwich structures of elastic, viscoelastic, and piezoelectric layers. These structures are being successfully

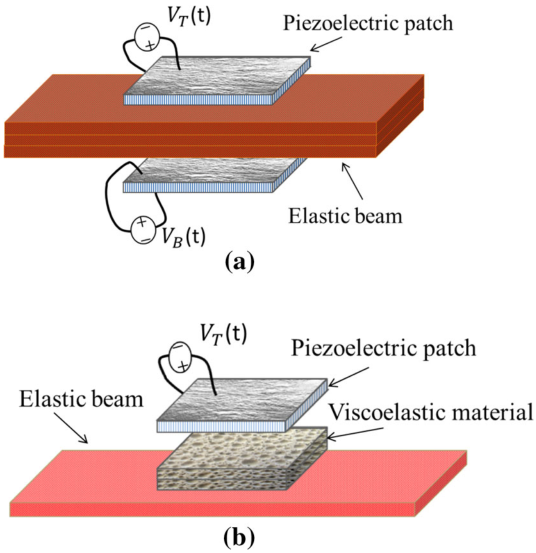

Fig. 2 **a** An elastic beam with piezoelectric patches where the voltages $V_T(t)$ and $V_B(t)$ are applied to the *top* and *bottom* piezoelectric patches, respectively. **b** Active constrained layer (ACL) damped beam/plate where the voltage V_T is applied to the piezoelectric patch

used for a variety of applications such as spacecraft, aircraft, train and car structures, wind turbine blades, boat/ship superstructures; i.e., see [11]. The modeling and control strategies developed in [18,19,27,29–31] play a key role in accurate analysis of these structures. The controllability/stabilization problems in the case of voltage actuation is still an open problem. This is currently under investigation.

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Appendix: Some results in number theory

In this section, we briefly mention some fundamental results of Diophantine's approximation. The theorem of Khintchine (Theorem 8) plays an important role to determine the Lebesgue measure of sets investigated in this paper.

Let $f : \mathbb{N} \to \mathbb{R}^+$ be called an approximation function if

$$\lim_{\tilde{q}\to\infty} f(\tilde{q}) = 0.$$

A real number ζ is f-approximable if ζ satisfies

$$\left| \zeta - \frac{\tilde{p}}{\tilde{q}} \right| < f(\tilde{q}) \tag{63}$$

for infinitely many rational numbers $\frac{\tilde{p}}{\tilde{q}}$. Let $P(f)$ be the set of all f-approximable numbers. We recall the following theorem to find the measure of sets of type $P(f)$.

Theorem 8 (Khintchine's theorem) [13, Page4] *Let μ be the Lebesgue measure. Then*

$$\mu(P(f)) = \begin{cases} 0, & \text{if } \sum_{\tilde{q}\in\mathbb{N}} \tilde{q} f(\tilde{q}) < \infty, \\ \text{full}, & \text{if } \tilde{q} f(\tilde{q}) \text{ is nonincreasing and } \sum_{\tilde{q}\in\mathbb{N}} \tilde{q} f(\tilde{q}) = \infty. \end{cases} \tag{64}$$

Dirichlet's theorem [14] states that every irrational number can be approximated to the order 2. The following theorem from [35] is a special case of Dirichlet's theorem:

Theorem 9 *Let $\zeta \in \mathbb{R} - \mathbb{Q}$. Then there exists a constant $C \geq 1$, and increasing sequences of coprime odd integers $\{\tilde{p}_j\}$, $\{\tilde{q}_j\}$ satisfying the asymptotic relation*

$$\left| \zeta - \frac{\tilde{p}_j}{\tilde{q}_j} \right| \leq \frac{C}{\tilde{q}_j^2}, \quad j \to \infty. \tag{65}$$

It obvious by Theorem 8 that the set $\mathbb{R} - \mathbb{Q}$ is uncountable and it has a full Lebesgue measure.

Definition 5 A real number ζ is a Liouville's number if for every $m \in \mathbb{N}$ there exists $\frac{\tilde{p}_m}{\tilde{q}_m}$ with $p_m, q_m \in \mathbb{Z}$ such that

$$\left| \zeta - \frac{\tilde{p}_m}{\tilde{q}_m} \right| < \frac{1}{\tilde{q}_m^m}.$$

It is proved that any Liouville's number is transcendental. Theorem 8 implies that the set of Liouville's numbers is of Lebesgue measure zero.

Definition 6 A real number ζ is an algebraic number if it is a root of a polynomial equation

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$$

with each $a_i \in \mathbb{Z}$, and at least one of a_i is non-zero. A number which is not algebraic is called transcendental.

Now we give the following results of Diophantine's approximations:

Theorem 10 *There exists a set $\tilde{\mathbb{Q}}$ such that if $\zeta \in \mathbb{R} - \tilde{\mathbb{Q}}$, then for every $\varepsilon > 0$ there are infinitely many $\frac{\tilde{p}}{\tilde{q}} \in \mathbb{Q}$ and a constant $C_\zeta > 0$ such that*

$$\left| \zeta - \frac{\tilde{p}}{\tilde{q}} \right| \geq \frac{C_\zeta}{\tilde{q}^{2+\varepsilon}}. \tag{66}$$

Moroever, $\mu(\tilde{\mathbb{Q}}) = 0$.

Proof We know that the irrational algebraic numbers belong to $\tilde{\mathbb{Q}}$ by Roth's theorem (Page 103, [14]). Therefore $\tilde{\mathbb{Q}}$ is not empty. We proceed to the second part of the lemma. The first part of the theorem implies that if $\zeta \in \tilde{\mathbb{Q}}$ then for all $C_\zeta > 0$, the inequality $\left|\zeta - \frac{\tilde{p}}{\tilde{q}}\right| < \frac{C_\zeta}{\tilde{q}^{2+\varepsilon}}$ holds for some $\frac{\tilde{p}}{\tilde{q}} \in \mathbb{Q}$. Now define the set

$$\tilde{\mathbb{Q}}_\varepsilon = \left\{\zeta \in \mathbb{R} \ : \ \left|\zeta - \frac{\tilde{p}}{\tilde{q}}\right| < \frac{C_\zeta}{\tilde{q}^{2+\varepsilon}} \text{ for infinitely many } \frac{\tilde{p}}{\tilde{q}} \in \mathbb{Q}\right\}.$$

By the notation of Theorem 8, choose $f(\tilde{q}) = \frac{C_\zeta}{\tilde{q}^{2+\varepsilon}}$ so that $\tilde{q} f(\tilde{q})$ is nonincreasing and $\sum_{\tilde{q}\in\mathbb{N}} \frac{C_\zeta}{\tilde{q}^{1+\varepsilon}} < \infty$. By Theorem 8, $\mu(\tilde{\mathbb{Q}}_\varepsilon) = 0$. Now we prove $\tilde{\mathbb{Q}} \subset \tilde{\mathbb{Q}}_\varepsilon$ by contradiction. Assume that $\zeta \notin \tilde{\mathbb{Q}}_\varepsilon$, i.e., there are finitely many rationals $\left\{\frac{p_i}{q_i}\right\}_{i=1,\ldots,N}$ such that

$$\left|\zeta - \frac{p_i}{q_i}\right| < \frac{C_\zeta}{q_i^{2+\varepsilon}} \quad \text{for } i = 1, \ldots, N, \quad \text{and} \quad \left|\zeta - \frac{\tilde{p}}{\tilde{q}}\right| \geq \frac{C_\zeta}{q^{2+\varepsilon}} \quad \text{for } \frac{\tilde{p}}{\tilde{q}} \notin \bigcup_{i=1}^{N}\left\{\frac{\tilde{p}_i}{\tilde{q}_i}\right\}.$$

The last inequality implies that $\zeta \in \mathbb{R} - \mathbb{Q}$. This implies that the set $\mathbb{R} - \tilde{\mathbb{Q}}$ has a full Lebesgue measure. □

Now define the set $\tilde{\tilde{\mathbb{Q}}}$ by

$$\tilde{\tilde{\mathbb{Q}}} = \left\{\zeta \in \mathbb{R} \ : \ \exists C > 0, \quad \left|\zeta - \frac{\tilde{p}}{\tilde{q}}\right| \geq \frac{C}{\tilde{q}^2} \text{ for infinitely many } \frac{\tilde{p}}{\tilde{q}} \in \mathbb{Q}\right\}. \tag{67}$$

If we consider numbers $\zeta \in \mathbb{R}$ whose the partial quotients satisfy $|a_k| < C(\zeta)$ for all $k \in \mathbb{N}$ in its continued fraction expansion

$$\zeta = [a_0; a_1, a_2, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}},$$

then $\zeta \in \tilde{\tilde{\mathbb{Q}}}$. By Liouville's theorem (Page 128, [26]), $\tilde{\tilde{\mathbb{Q}}}$ also contains all quadratic irrational numbers (the roots of an algebraic polynomial of degree 2). Therefore the set is uncountable.

Lemma 6 *The set* $\tilde{\tilde{\mathbb{Q}}}$ *has a Lebesgue measure zero.*

Proof Define the set F_m by

$$F_m = \left\{\zeta \in \mathbb{R} \ : \ \left|\zeta - \frac{\tilde{p}}{\tilde{q}}\right| < \frac{C}{m\tilde{q}^2} \text{ for infinitely many } \frac{\tilde{p}}{\tilde{q}} \in \mathbb{Q}\right\}.$$

Then F_m has a full Lebesgue measure by Theorem 8, i.e., $f(\tilde{q}) = \frac{C}{m\tilde{q}^2}$, and $\sum_{\tilde{q}\in\mathbb{N}} \tilde{q} f(\tilde{q}) = \infty$. Now consider the set $\bigcap_{m\in\mathbb{N}} F_m$. This set is the countable intersection of sets F_m, and each F_m has full Lebesgue measure. Therefore $\mu\left(\bigcap_{m\in\mathbb{N}} F_m\right)$ has full Lebesgue measure. Since $\tilde{\tilde{\mathbb{Q}}} = \mathbb{R} - \bigcap_{m\in\mathbb{N}} F_m$, then $\mu(\tilde{\tilde{\mathbb{Q}}}) = 0$. □

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