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A Closed Model of Careers in a Simple Hierarchy

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A vertically differentiated labor market can be viewed longitudinally as careers or in cross section as hierarchy. The equilibrium structure results from simultaneous choice of careers by individuals and hierarchy by firms. We compare steady states in a model which is simple because there are only two levels of hierarchy; simplicity enables us to close the model by permitting firms to choose how many seniors and juniors to employ, while individuals simultaneously choose how long to persist at the lower rank before quitting. The model generates testable predictions about careers and hierarchy.

1. Introduction

■ A differentiated labor market exists when there are several kinds of labor which are imperfect substitutes. This paper concerns a labor market which is differentiated vertically, as in occupations employing two kinds of workers, juniors and seniors. Each individual career is a longitudinal view of such a market, while hierarchy within the firm or profession presents the market viewed in cross section. In such markets the price for differentiated labor is the outcome of simultaneous choices by individuals, who choose their careers, and firms, which choose their internal structure. On the supply side, the individual must examine the progress of his or her career and decide whether to continue in his or her present course or quit and accept alternative employment. This is a problem of optimal stopping, whose solution is the quitting age for junior members of the firm or profession. On the demand side, the firm must decide the proportions in which to employ the different grades of labor and the speed at which to promote people to maximize its profits. This is a problem of choosing the optimal structure for the hierarchy, whose solution gives the ratio of juniors to seniors. Thus, in this paper we develop the logic of choice in

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both sides of the market. Most papers in the same general area have developed the logic of choice on only one side of the market.[1] Our model is closed by identifying conditions under which a steady state competitive equilibrium exists.

Our model postulates a stochastic process by which workers at one level in a hierarchy get promoted to the next level, but there is latitude in our formal analysis for different interpretations of this process. The reader may wish to think of a factory in which the foremen are promoted from the ranks of laborers, but many laborers quit if they work for several years without promotion. Alternatively, the reader may envision a white collar bureaucracy where clerks and typists quit unless they get promoted to posts with administrative responsibility. Or, the reader may imagine a prestigious law firm in which promotion from junior to senior partner is uncertain; a junior who gets promoted enjoys a salary increase and takes on new managerial responsibilities, but those who are not promoted eventually quit the firm and find employment outside the firms of top rank. One restriction in our analysis is that *there must be more workers at the lower level than the higher level*; thus it would fail for those law firms in which there are more partners than associates.

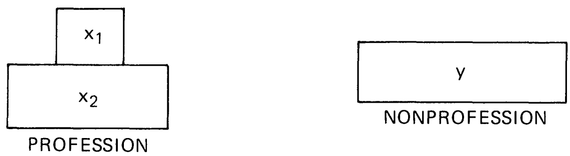
We do not examine deep questions about labor hierarchy, such as whether it is explained by transaction costs and information asymmetries or a power struggle between classes. There is no account of the problems of labor discipline and the centralization of power. Instead we offer an extensive comparative statics analysis, in which we show how the quitting age and ratio of junior to senior workers change with the system's parameters. The parameters include the probability of promotion, the rate of entry into the profession, the retirement age, demand for the profession's products, and the market power of capital and each grade of labor. In brief, we offer a descriptive theory of the structure —not the cause—of vertical differentiation in labor markets. The result is a series of testable propositions which are useful even to those interested in the deeper questions. For example, we are able to demonstrate a grave conflict of interest between owners and senior workers (or managers).

Our strategy in this paper is to develop intensively the implications of a simple model which is mathematically tractable. In Section 2 we describe the elements of the model and develop the logic of maximization on both sides of the market. In Section 3 we define a steady state competitive equilibrium and provide conditions for existence. In Section 4 we obtain various propositions about the steady state and in Section 5 we offer our comparative statics results. The comparative statics are extended in Section 6 to deal with monopoly and monopsony. An informal discussion about the connection between actual institutions and the structure of hierarchy or careers follows in Section 7. We offer a much more general model, but one with less specific conclusions, in Section 8. Our closing remarks are in Section 9. The model presented in this paper is a simplification of a more general model which we have developed elsewhere (Cooter and Restrepo, 1977).

[1] Many investigators have tried to identify the determinants of the labor contract; in particular, they have tried to explain the conditions under which a labor market will be internalized. For example, see Mirrlees (1976) or Stiglitz (1975). We attempt to identify the relations among structural variables in a stratified labor market, which is more in the spirit of Williamson (1967) or Hartog (1976).

FIGURE 1

STRUCTURE OF EMPLOYMENT



2. A simple model

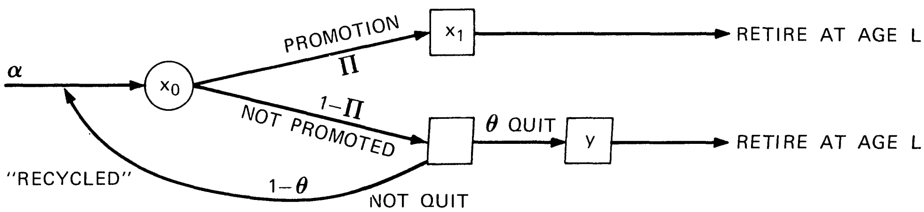
■ **Elements of the theory.** The model applies equally to a stratified profession, trade, or firm; we shall say "profession" to avoid repeating the phrase "trade, firm, or profession." The model considers a profession with two levels—junior and senior—and an outside occupation. The profession employs a total of x_0 workers at the junior level at a wage w_0, and a total of x_1 workers at the senior level at a wage w_1; the remainder y of the labor force is outside the profession, receiving a uniform wage v, which is the opportunity cost of working in the profession. It does not matter whether we think of v as a certain wage or an expected wage. (See Figure 1.) The internal structure of the firms is thus described by the ratio x_0/x_1, which we call the apprentice ratio. It could also be called the ratio of workers to bosses, or the ratio of supervised to supervisors (span of control), if supervision is in fact the task of seniors. It is assumed that workers reach the senior level only by being promoted from the junior rank.

It is easy to visualize a discrete time model in which workers are hired on an annual basis and the opportunity for promotion arises only at the end of each year when contracts expire. However, it is mathematically more convenient to allow the review period for promotion to become shorter and shorter until it approaches a process of continuous review, and to do likewise for the hiring process. Thus, we shall assume that workers enter the profession at a continuous rate α and that they are continuously reviewed for promotion. There are no demotions, and all workers retire from the labor force L periods after their initial employment. In a steady state new workers join the labor force at the same rate α at which old workers retire, so the total labor explicitly analyzed is αL.

The case that interests us is the one in which wage and promotion rates prompt juniors to quit the profession if they reach a certain age without promotion. Young workers are motivated to join the profession by the hope of someday earning the senior wage w_1, which exceeds the outside wage v; older juniors who have not been promoted are motivated to quit the profession because the outside wage v exceeds the junior wage w_0. The flow of workers is then described by Figure 2, in which a proportion Π of juniors are promoted at

FIGURE 2

FLOW OF WORKERS



each point in time, a proportion θ of those who are not promoted quit, and the remainder are "recycled" back into the junior ranks where they are joined by new workers.

To specify the simple model we shall make the following assumptions:

Assumption 1: The senior wage w_1 exceeds the outside wage v, which in turn exceeds the junior wage w_0.

Assumption 2: The outside wage v is exogenous, while workers in the firm or profession receive their marginal product.

Assumption 3: The promotion probability distribution has the uniform density function: $\Pi(t) = \Pi$.

Assumption 4: Each cohort contains exactly α workers and all workers retire at age L.

Assumption 5: Total wages paid to professionals or workers in the firm exceed what they could earn in alternative employment:

$$w_1x_1 + w_0x_0 > v(x_1 + x_0).$$

(This assumption ensures that the existence of the profession or firm is efficient; it is discussed in Section 3.)

□ **Supply of labor.** With the constant entrance rate α, the supply of labor to the firm or profession is determined by the quitting policy adopted by workers remaining in the junior level. A person who is a junior enjoys a wage which is less than he could earn outside the profession; he is kept in it by the prospect of promotion. If he stays in the profession at the junior level, he is taking a gamble on promotion; strategy consists in deciding how long to gamble.

We shall calculate now the optimal age for a junior to stop gambling and quit the profession. For simplicity we shall calculate first the quitting age when earnings are not discounted for their futurity, and no one can influence his promotion prospects. An individual who decides upon entering the profession to quit if not promoted by age q faces two possibilities: First, promotion may occur at some age $t < q$, in which case his total earnings will be tw_0 before age t and $(L - t)w_1$ after age t; the expectation of these earnings is

$$\int_0^q [w_0t + w_1(L - t)]\Pi(t)dt,$$

where $\Pi(t)$ is the probability density of promotion. The second possibility is that this same individual will not be promoted before his quitting age q, in which case his total earnings will be w_0q before quitting and $v(L - q)$ after quitting; the expectation of these earnings is

$$[w_0q + v(L - q)]\left[1 - \int_0^q \Pi(t)dt\right].$$

With such a quitting policy, the expected lifetime earnings will be the sum of the expected earnings from the two preceding possibilities, and the optimal quitting policy is the one which maximizes this sum. Thus the first-order condition for the optimal quitting age, under the assumption that $\Pi(t) = \Pi$, is that

$$(w_1 - v)(L - q)\Pi = (v - w_0)(1 - \Pi q). \tag{1}$$

Equation (1) has a simple interpretation in terms of the gamble taken by the individual worker while he stays in the profession at the junior level. If this worker changes his quitting policy from age q to age $q + dq$, he stands the chance to increase his lifetime earnings by the amount $(w_1 - v)(L - q)$, provided that he is promoted during the interval between q and $q + dq$, this promotion's occurring with probability $\Pi(q)dq$. But if he is not promoted during the interval in question or earlier, which occurs with probability $(1 - \Pi q - \Pi dq)$, then his loss of income by not quitting at q is $(v - w_0)dq$. At the optimal quitting time this expected gain and loss are equal, yielding condition (1). These expected gains and losses are illustrated in Figure 3. A more general optimization problem is described in Section 8, where we allow each individual to influence his promotion prospects Π.

□ **Demand for labor.** The profession is organized into firms which combine the labor of its members and sell their services to consumers. The firms continuously review the performance of juniors to decide who should be promoted. At each interval in time some workers are promoted and some are not; those who advance assume new tasks, including the supervision of juniors. Part of the meaning of hierarchy is that juniors and seniors do different kinds of work and have different responsibilities. A person's value in the labor market is assumed to increase immediately when he or she becomes a senior. The production technology takes into account the fact that seniors, juniors, and capital C are imperfect substitutes:

$$\text{output} = h(x_1, x_0, C).$$

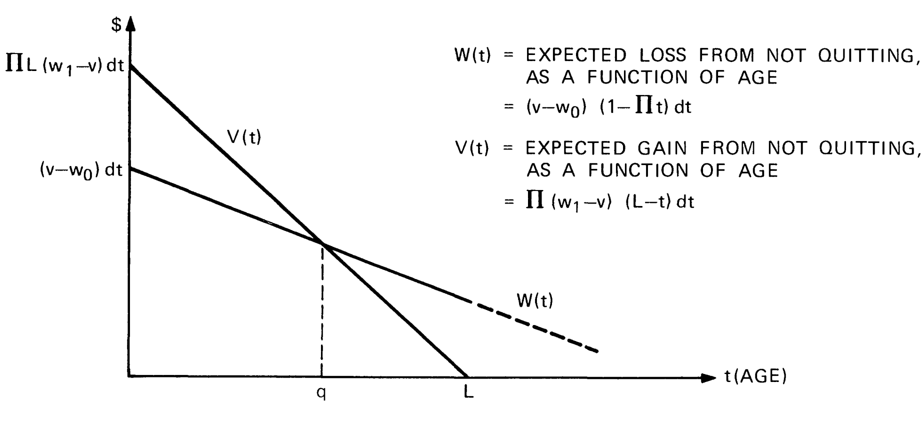
We shall treat C as a constant and rewrite $h(\cdot)$ in the form

$$\text{output} = f(x_1, x_0).$$

We have postulated that there is a hierarchy and a stochastic process by which one distinct kind of labor gets transformed into another. There is latitude in our formal analysis for different interpretations of this stochastic process. One interpretation is that juniors are being taught new skills and screened for ability to learn and accept responsibility. The individual junior

FIGURE 3

OPTIMAL QUITTING AGE



is uncertain about his promotion prospects because he does not know how accurate the firm's assessment of his abilities will be; furthermore, he may not know exactly what abilities he possesses. In each period that passes without promotion he acquires more skill, which makes him more worthy of promotion, but he also reveals that he is not the quickest learner, which makes him less worthy of promotion. In the simple model we proceed as if these influences *cancelled each other*, so that the promotion probability Π is uniform. In the more general model Π varies with the individual's age, his investment in learning, and the firm's investment in teaching.

As in the preceding section, we first formulate the model which is simplest and most tractable. The profits of a representative firm are described in the usual way:

$$\text{profit} = f(x_1, x_0) - w_0 x_0 - w_1 x_1,$$

where the firm's output is chosen as the *numeraire*. We shall make various assumptions about the motives which guide the demand for labor by firms. The simplest assumption is perfect competition, by which we mean that the firms take wage rates as given and choose the combination of juniors and seniors which maximizes their profit:

$$\max_{x_0 x_1} f(x_1, x_0) - w_1 x_1 - w_0 x_0. \tag{2}$$

We follow the standard practice of assuming that the production function is linear homogeneous, so that (2) requires factors to receive their marginal product:

$$\begin{aligned} w_1 &= f_1(\cdot) \\ w_2 &= f_2(\cdot). \end{aligned} \tag{3}$$

Linear homogeneity permits us to write production as a function of the apprentice ratio:

$$f(x_1, x_0) = x_1 g(s),$$

where $s = x_0/x_1$. We can rewrite (3) in the form

$$w_1 = g - g's \tag{3a}$$

$$w_0 = g'. \tag{3b}$$

These equations are useful in the steady-state analysis which follows.

We shall treat the nonprofession parametrically by assuming that the number of people who enter it by quitting the profession does not influence the outside wage (Assumption 2). This suggests that quitters are few relative to total outside employment; there is a large stream of entrants directly into outside employment which does not figure in our analysis. In Section 6 we shall extend this simple model to deal with monopoly power and in Section 8 we generalize further to allow the firm to influence promotion probabilities.

3. Steady state and equilibrium

■ The endogenous variables which characterize the labor market in the simple model are the apprentice ratio s and the quitting age q. The labor market is in a steady state when s and q do not change. If exogenous variables are constant, then an increase in q causes the stock of juniors to increase; more juniors mean more promotions *ceteris paribus*, so the stock of seniors also increases. If q is held constant at its new value, then s will reach a constant value after an amount of time not exceeding L has elapsed. Each value of the stock variable s will be sustained in the long run by some constant value of the quitting age q. We indicate this relation by the function $q = Q(s)$, which we call the "steady-state function," because it describes the combinations of q and s which are consistent in the long run. However, not all consistent values of q and s represent an optimal quitting strategy for the workers and a profit-maximizing apprentice ratio for the employers. The combinations of q and s at which buyers and sellers of labor are both at a maximum will be denoted by $q = q(s)$, and we shall call it the "maximizing function." A steady-state competitive equilibrium consists of values q^* and s^* which satisfy $q^* = Q(s^*)$ and $q^* = q(s^*)$ simultaneously. Our immediate aim is to elucidate the functions $q(\cdot)$ and $Q(\cdot)$ to find conditions under which both are satisfied.[2]

To compute the function $Q(\cdot)$, we shall determine first the numbers x_0 ana x_1 of juniors and seniors in the labor force. To be a junior a worker must have been employed for a period $u < q$ and must not have been promoted during that period. Thus,

$$x_0 = \alpha \int_0^q \left[1 - \int_0^u \Pi(t)dt \right] du. \tag{4}$$

Similarly, x_1 consists of workers of age $u < L$ who have been promoted:

$$x_1 = \alpha \int_0^q \int_0^u \Pi(t)dtau + \alpha(L - q) \int_0^q \Pi(t)dt. \tag{4a}$$

To obtain an explicit relationship between the apprentice ratio and the quitting age in a steady state we need to integrate these equations. Integration is possible if we select $\Pi(t) = \Pi$, as in Assumption 3. Carrying out this integration, forming the ratio, and solving for q, we obtain

$$q = \frac{2(\Pi L s - 1)}{\Pi(s - 1)} \equiv Q(s), \qquad \text{where} \qquad s \neq 1. \tag{5}$$

Thus the steady-state function is given by (5), with the parameters Π and L fixed. An implication of (5) and $L > q > 0$ is that $L\Pi < 1$ for $s > 1$, and $L\Pi > 1$ for $s < 1$. We shall concentrate on the case where $s > 1$, so that the labor hierarchy is a simple, two-level pyramid.

Next we seek an explicit form for the maximization equation $q(s)$. Workers are at a maximum when they adopt the optimal quitting strategy given by (1); firms are at a maximum when the seniority ratio satisfies (3a) and (3b). We combine these equations and solve for q:

$$q = \frac{\Pi L(g - g's - v) - v + g'}{\Pi(g + g'(1 - s) - 2v)} \equiv q(s). \tag{6}$$

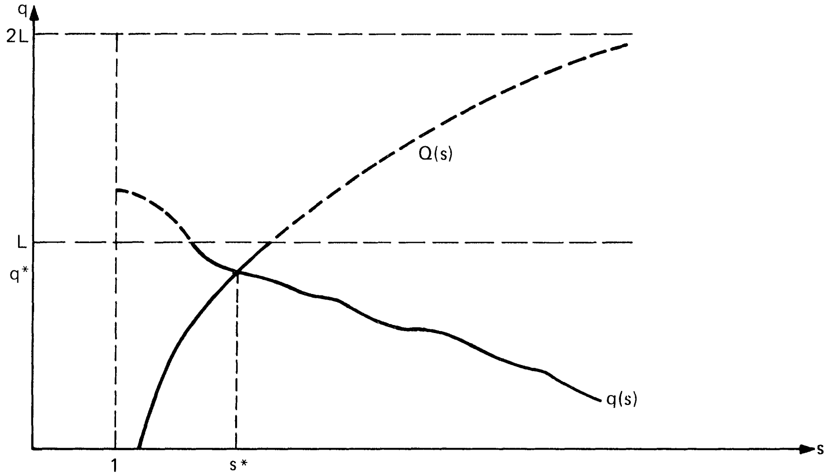
The maximization function is (6) with the parameters Π, L, and v fixed.

A steady state will be sustained as a competitive equilibrium if the maximization function is satisfied. We seek a quitting age q^* and an apprentice

[2] A more general theory would offer an account of the formation of expectations. Various authors have shown that lagged perception in vertical labor markets can lead to cycles. In these models the "junior" grade in the hierarchy is that of a student. For example, see Arrow and Capron (1959) or Freeman (1971). We avoid these difficulties by dealing with steady states in which expectations are fulfilled.

FIGURE 4

EXISTENCE OF STEADY STATE



ratio s^* which satisfy (5) and (6). From Figure 4 it is apparent that the two functions must intersect if their derivatives q' and Q' are opposite in sign and their origins are placed appropriately. We shall find conditions under which this will occur. Differentiate (6) to obtain

$$q' = \left(\frac{-\Pi g''}{[\Pi(w_1 + w_0 - 2v)]^2}\right)(L\Pi - 1)(w_1 - v + w_0 s - vs).$$

Then $g'' < 0$ by concavity, and we have seen that (5) implies $L\Pi < 1$ for $s > 1$; so a sufficient condition for the derivative to be negative is $w_1 - v > s(w_0 - v)$. Now differentiate the steady state function $Q(s)$:

$$Q' = \frac{2(1 - L\Pi)}{\Pi(s - 1)^2}.$$

The derivative is positive if $L\Pi < 1$, which is implied by (5) whenever $s > 1$, as already noted. We have found that the derivatives are opposite in sign when $w_1 - v > s(w_0 - v)$. Sufficient conditions for a steady-state equilibrium in the pyramidal firm are:

$$w_1 - v > s(v - w_0) \qquad \text{and} \qquad q(1) > Q(1).$$

The first of these two conditions is true by Assumption 5; the second is also true since $Q(1) = -\infty$, while $q(1)$ is finite because its denominator is positive by Assumption 5.

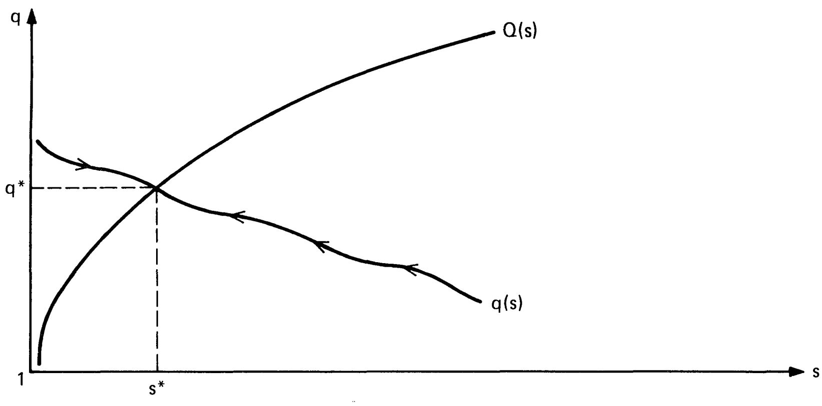
We have proved that a steady-state competitive equilibrium exists in the market for professional labor, given Assumptions 1–5. In fact we have also proved that the equilibrium is stable, as we now show. Our existence proof established that $q'(\cdot) < 0$ and $Q'(\cdot) > 0$, which implies

$$\frac{d}{ds}(q(s) - Q(s)) < 0.$$

This is the stability condition in the labor market for professionals when $s > 1$ and there is no lag in the perception of price changes. If $q(s) > Q(s)$, then the quitting age is too high to sustain the existing apprentice ratio; the stock of

FIGURE 5

STABILITY



juniors is rising, thereby causing the apprentice ratio to rise. If $q(s) < Q(s)$, then the quitting age is too low to sustain the existing apprentice ratio; the stock of juniors is falling, which causes the apprentice ratio to fall. We may think of $q(s)$ as indicating points of temporary equilibrium and $q(s) - Q(s)$ an indicating the sign of the excess of additions to stocks over subtractions from stocks, as depicted in Figure 5.

Our proof of existence and stability assumes $\Pi L < 1$, which is guaranteed by equation (5) for a firm whose labor force is a pyramid ($s > 1$). What happens if there are more seniors than juniors, so that $s < 1$? It is easy to prove that in this case the optimal quitting age is the upper limit: $q = L$.[3] If the firm is an inverted pyramid, then no juniors choose to quit; they prefer to gamble on promotion right up to retirement age. With q constant at the upper limit, the system will always reach a steady state in no less than L periods. The steady state exists and it is stable,[4] provided that the second-order condition for optimal quitting is always satisfied.[5]

The fact that $q = L$ when $s < 1$ indicates that our simple model is not a good description of those professions which are inverted pyramids. The problem is our assumption that the promotion probability Π is uniform. If we relax this assumption and let Π approach zero at an age well before L, then juniors will quit well before retirement age, even though $s < 1$. We shall discuss such a model in Section 8.

4. Steady-state results

■ Our first proposition is an ergodic theorem, which we have proved elsewhere in a more general setting (Cooter and Restrepo, 1977):

Proposition 1: The proportion of the labor force in each of the three job categories at any epoch in time equals the proportion of time a new worker will expect to spend in each job.

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[3] From (1) we have the optimality condition $(w_1 - v)(L - q)\Pi + (w_0 - v)(1 - \Pi L) \geq 0$. If $s < 1$, then $L\Pi > 1$; hence at $L = q$ the inequality is strict.

[4] When $s < 1$, we have $Q' < 0$; in this case stability requires $q'(s) - Q'(s) > 0$, which is the opposite of the case where $s > 1$.

[5] The second-order condition is $\partial^2 J/\partial q^2 = \Pi(-w_1 - w_0 + 2v) < 0$; this condition is always satisfied by Assumption 5 when $s > 1$, but it may be unsatisfied when $s < 1$.

We prove Proposition 1 by direct calculation of proportions and expectations. First, the expected proportion of time spent outside the firm or profession is the number of working years remaining at the quitting age times the probability of quitting, divided by the total active lifetime L; that is,

$$E_y = (L - q)(1 - \Pi q)/L.$$

Next, the time spent as a junior will be $t < q$ if promotion occurs at working age $t < q$; and this time will be q if promotion does not come by age q. Thus, the expected proportion of time spent at the junior level will be

$$E_0 = \left[\int_0^q t\Pi dt + q(1 - \Pi q)\right]\bigg/L = \frac{q(1 - \Pi q/2)}{L}.$$

Finally, the expected proportion of time spent as a senior equals 1 minus the expected proportion of active life spent elsewhere. Thus,

$$E_1 = 1 - E_0 - E_y = [(L - q)\Pi q + \Pi q^2/2]/L.$$

Similarly, the proportion of the labor force found in each category in a steady state may be found by integrating equations (4) and (4a); one obtains

$$\begin{aligned} x_0/\alpha L &= q(1 - \Pi q/2)/L = E_0 \\ x_1/\alpha L &= [(L - q)\Pi q + \Pi q^2/2]/L = E_1 \\ y/\alpha L &= (L - q)(1 - \Pi q)/L = E_y. \end{aligned}$$

The ergodic theorem is the precise statement that a career is hierarchy viewed longitudinally and hierarchy is careers viewed in cross section. It permits us to pass from proportions to expectations. The proposition's significance is that the members of a profession or firm whose shape is a pyramid ($s > 1$) will expect to spend a decreasing proportion of their working lives in successive levels of the hierarchy. Of course, the proportions to which we refer in Proposition 1 concern the labor force which enters the firm or profession in question; the proportions do not refer to members of the labor force who are outside of our analysis by virtue of the fact that they begin their careers in outside firms or professions.

Our next proposition is an implication of the ergodic theorem:

Proposition 2: The derivative with respect to retirement age of expected time spent as a senior (or nonprofessional) equals the cumulative probability of promotion (or nonpromotion).

We prove this proposition by taking derivatives of the equalities in Proposition 1:

$$E_1 L = \frac{x_1}{\alpha} = \left[(L - q)\Pi q + \frac{\Pi q^2}{2}\right] \Rightarrow \frac{d[\cdot]}{dL} = \Pi q$$

$$E_y L = \frac{y}{\alpha} = [(L - q)(1 - \Pi q)] \Rightarrow \frac{d[\cdot]}{dL} = 1 - \Pi q.$$

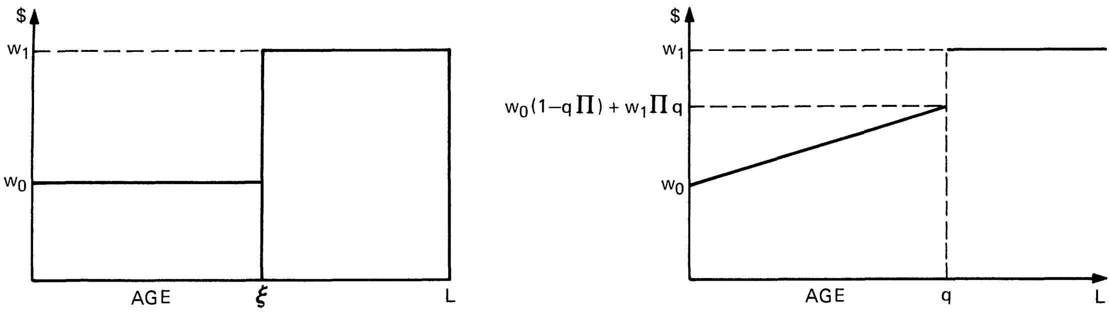
We have proved Proposition 2 in a model with n levels of hierarchy.[6] Its significance is the fact that we can pass from proportions to cumulative probabilities.

Workers who are promoted leave the junior ranks at an age which is

[6] This is Theorem 2 in Cooter and Restrepo (1977).

FIGURE 6

(a) AGE/EARNINGS PROFILE, LONGITUDINAL (b) AGE/EARNINGS PROFILE FROM CROSS SECTION



typically less than q, but workers who quit leave at exactly age q. The expected time spent as a junior is an average of these two values; consequently, the expected age of promotion—assuming promotion occurs—is less than the expected time spent as a junior. We formulate the relationship precisely in the following equation, where ξ is expected age of promotion:

$$\begin{aligned}\xi &= \int_0^q \Pi t\,dt \\ &= E_0 - q(1 - \Pi q).\end{aligned}$$

According to this equation the expected age of promotion is the expected time spent as a junior less the product of the probability of quitting and the time quitters spend as juniors.

A familiar exercise in labor economics is to derive the age-earnings profile, which expresses the wage as a function of the age of a representative worker. There are two ways to make the calculation. If we make the calculation longitudinally, by following the careers of individual workers and observing their age of promotion, then we graph the wage against the average age of promotion ξ. The longitudinal method yields the bar graph shown in Figure 6a. If we make the calculation in cross section, then we calculate the average wage for a professional of given age. Consider the tth cohort, in which every worker's age is t. If $t > q$, then every worker in that cohort who is still in the profession or firm enjoys a wage w_1. If $t \leq q$, then the average wage in the cohort is

$$W(t) = w_1\Pi t + w_0(1 - \Pi t).$$

The cross section calculation of the age earnings profile is piecewise linear with a discontinuity at q, as shown in Figure 6b.[7] The slope of this profile is $\Pi(w_1 - w_0)$ for all ages $t < q$; from (3a) and (3b) we know that $(w_1 - w_0)$ is increasing in s, which leads to our next proposition:

Proposition 3: The age earnings profile is steeper for industries or firms with a high apprentice ratio s or promotion probability Π.

[7] The standard empirical finding is that age-earnings profiles are concave and increasing. For example, see Becker (1964), Mincer (1974), or Haley (1976). Our Figure 6b is consistent with these findings if the discontinuity is small, as it presumably would be if we had many more than two levels of hierarchy.

TABLE 1

PARAMETRIC SHIFTS IN q(s) AND Q(s)

MAXIMIZATION: $q = q(s)$	STEADY STATE: $q = Q(s)$
$q = \dfrac{\Pi L(w_1 - v) - v + w_o}{\Pi(w_1 + w_o - 2v)}$	$s = \dfrac{2 - \Pi q}{\Pi(2L - q)}$
$\dfrac{\partial q}{\partial \Pi} = \dfrac{v - w_o}{\Pi^2(w_1 + w_o - 2v)}$ > 0 IF $(w_1 - v) > (v - w_o)$	$\dfrac{\partial s}{\partial \Pi} = \dfrac{-2}{\Pi^2(2L - q)} < 0$
$\dfrac{\partial q}{\partial L} = \dfrac{(w_1 - v)}{(w_1 + w_o - 2v)}$ > 0 IF $(w_1 - v) > (v - w_o)$	$\dfrac{\partial s}{\partial L} = \dfrac{(\Pi q - 2)^2}{\Pi(2L - q)^2}$
$\dfrac{\partial q}{\partial v} = \dfrac{(w_1 - w_o)(L\Pi - 1)}{\Pi(w_1 + w_o - 2v)^2}$ < 0 IF $L\Pi < 1$	$\dfrac{\partial s}{\partial v} = 0$

Proposition 3 suggests definite signs for the coefficients in a regression equation. In Section 8 we argue that democratic unions and monopoly employers will lower s and autocratic unions will raise s relative to its competitive value; we also argue that democratic unions will raise Π and autocratic unions will lower Π. These effects are offsetting by Proposition 3; we do not know how market power will influence the slope of the age earnings profile without determining which effect is stronger.

5. Comparing steady states: perfect competition

■ We may investigate the comparative statics of our model by observing how $q(s)$ and $Q(s)$ shift when there are changes in the parameters. The direction of the shift is determined by taking the appropriate derivative of (5) and (6), which are given in the accompanying table. It turns out that reasonable values of the variables imply that $Q(s)$ has a steep slope, whereas $q(s)$ is nearly flat.[8] This observation is sometimes necessary to determine the sign of changes, so we state it explicitly:

Assumption 6: $|Q'(s)| \gg |q'(s)|$ whenever $s > 1$.

The propositions which follow are immediate consequences of the signs of the derivatives shown in Table 1 and our assumptions.

Proposition 4: If Assumptions 1–5 are true, then an increase in the probability of promotion Π causes an increase in the quitting age q for a pyramidal firm ($s < 1$); $q = L$ for an inverted pyramid.

[8] For example, the following values of the parameters give $q' = +.0022g''$ and $Q' = 8.75$:

$s = 5$	$w_1 = \$35{,}000$
$v = \$15{,}000$	$L = 30$
$w_0 = \$13{,}000$	$\Pi = .01$.

There is nothing surprising about the conclusion that juniors will postpone quitting if there is an increase in the probability of promotion. If $q = L$, then an increase in the promotion probability has no effect, because it is impossible to postpone quitting beyond retirement.

Proposition 4a: If Assumptions 1–6 are true, then an increase in the probability of promotion Π causes a decrease in the apprentice ratio s for all firms.

An increase in Π causes both $Q(s)$ and $q(s)$ to shift up, and these two shifts represent offsetting effects: more promotions with no change in quitting age tend to decrease the apprentice ratio, but later quitting with no more promotions tends to increase it. The first effect dominates with reasonable values of the parameters, as assumed in Assumption 6; the second effect is nil for inverted pyramids.

Proposition 5: If Assumptions 1–5 are true, then an increase in the retirement age L causes an increase in the quitting age q for a pyramidal firm ($s > 1$); $q = L$ for an inverted pyramid.

Proposition 5a: If Assumptions 1–6 are true, then an increase in the retirement age L causes a decrease in the apprentice ratio s for a pyramidal firm ($s > 1$); s decreases for an inverted pyramid if $q\Pi < 2$.

We see that an increase in retirement age has the same effect upon quitting age and apprentice ratio as an increase in the promotion probability. These two events both serve to increase the expected winnings of the gamble taken by juniors. Juniors respond by gambling longer. When juniors gamble longer, their numbers increase, but there is also an increase in the number of seniors. The first effect dominates the second, given Assumption 6, which implies an increase in the apprentice ratio.

Proposition 6: If Assumptions 1–5 are true, then an increase in the outside wage v causes a fall in the quitting age q, and the apprentice ratio s of a pyramidal firm ($s > 1$); both are constant for an inverted pyramid.

An increase in the outside wage increases the cost of the gamble taken by juniors, with no offsetting increase in the benefits. The quitting age falls, which means that there are fewer juniors, so the apprentice ratio falls. The reader should recall that we chose the output of the profession for the *numeraire* good, so an increase in v is equivalent to a fall in the price of the profession's output.

Entry into the profession is a constant α. It is not unusual for professional associations or unions to gain control over the profession's portals, in which case α may be lower by deliberate policy than in a competitive situation. The consequence of lowering α is to reduce total supply of the industry's output which results in a rise in its relative price. After this change a given amount of the industry's output will exchange for more labor in the nonprofessional occupation. We have chosen the industry's output as *numeraire*, so a decrease in α must cause a fall in v, which gives us our next proposition:

Proposition 6a: If Assumptions 1 and 3–5 are true, then a decrease in cohort size results in a rise in the quitting age q and the apprentice ratio s of a pyramidal firm ($s > 1$); both values are constant for an inverted pyramid.

6. Steady state with market power

■ Textbook treatments of monopoly power deal with a world of homogeneous capital and labor, which tells us nothing about labor hierarchies or career choices. We may extend the analysis in the preceding section to work out the effects of monopoly power upon a labor market which is differentiated vertically. The effect of market power upon the steady-state competitive equilibrium can be described by a small revision in our wage equations. A convenient way to proceed is to treat market power as if it were a tax levied upon one party and paid to the other. Rather than assuming that juniors receive their marginal product, we assume that juniors receive a fraction τ_0 of their marginal product:

$$w_0 = g'\tau_0. \tag{8a}$$

If $\tau_0 = 1$, then we have a competitive situation, whereas $\tau_0 > 1$ indicates that juniors enjoy a subsidy. Similarly, there is a tax-subsidy coefficient for seniors, τ_1:

$$w_1 = g - g's\tau_1. \tag{8b}$$

Equations (8a) and (8b) are generalizations of (3a) and (3b). We now rewrite the function $q(s)$ by combining (1) with (8a) and (8b):

$$q = \frac{\Pi L(g - g's\tau_1 - v) - v + g'\tau_0}{\Pi(g - g's\tau_1 + g'\tau_0 - 2v)}.$$

Our steady-state analysis is unchanged except that we now have an additional pair of parameters, τ_0 and τ_1.

When there is cross subsidization, the subsidy to juniors equals the tax for seniors:[9]

$$\tau_0 = \tau_1 \equiv \tau.$$

What happens to the steady-state competitive equilibrium (q^*,s^*) if there is an increase in the rate of cross subsidy τ? The steady-state function is unchanged:

$$\frac{\partial Q}{\partial \tau} = 0.$$

The derivative of the maximization function with respect to τ is positive by Assumption 5:

$$\frac{\partial q}{\partial \tau} = \left[\frac{(\Pi L - 1)g'}{\Pi(w_1 + w_0 - 2v)^2}\right][(v - w_0)s - (w_1 - v)] > 0 \text{ by Assumption 5}$$
$$\text{for } \Pi L < 1.$$

The effect is an upward shift in $q(s)$.

We state this result as our next proposition:

Proposition 7: If Assumptions 1–5 are true, then an increase in the rate of

[9] The cross subsidy term is written to insure that receipts equal payments of subsidies. For example, with $\tau > 1$, we have

$$\textit{receipts}: (\tau - 1)g'x_0$$

$$\textit{payments}: x_1(g - g's) - x_1(g - g's\tau) = (\tau - 1)g'x_0.$$

subsidy paid to juniors by seniors will result in an increase in the quitting age q and apprentice ratio s for a pyramidal firm.

The preceding proposition gives the effect of cross subsidization, which occurs when juniors have power over seniors or *vice versa*. Another possibility is that a monopsonistic buyer has power over both kinds of professional labor. A monopsonist recognizes that wage rates will vary with the quantity of labor which he hires; his maximization problem is

$$\max_{x_1 x_0} f(x_1,x_0) - w_1(x_1,x_0)x_1 - w_0(x_1,x_0)x_0.$$

The first-order conditions are given in the following equations, where $\eta_{ij} = d\log x_i | d\log w_j$ is the elasticity of supply:

$$w_1 = \frac{g - g's}{1 + 1/\eta_{11}} - \left(\frac{\eta_{11} s w_0}{\eta_{10}(1 + \eta_{11})}\right) \tag{9a}$$

$$w_0 = \frac{g'}{1 + 1/\eta_{00}} - \left(\frac{\eta_{00} w_1}{\eta_{01}(1 + \eta_{00})s}\right). \tag{9b}$$

It is convenient to express the wage equations (9a) and (9b) in terms of a monopoly tax τ_1 and τ_0 on the marginal product of labor; we obtain this expression by combining (8a) and (8b) with (9a) and (9b):

$$\tau_1 = \frac{g's\eta_{11} + g}{g's(\eta_{11} + 1)} + \left(\frac{\eta_{11}\tau_{00}}{\eta_{10}(1 + \eta_{11})}\right) \tag{10a}$$

$$\tau_0 = \frac{\eta_{00}}{1 + \eta_{00}} - \left(\frac{\eta_{00}(g - g's\tau_1)}{\eta_{01} g's(1 + \eta_{00})}\right). \tag{10b}$$

The first term in (10a) and (10b) is the effect upon the monopoly tax of the own price elasticity, and the second term (bracketed) is the effect of the other price elasticity.

Competitive labor markets are characterized by infinite own and other price elasticities faced by the individual firms, in which case it is easy to show that τ_0 and τ_1 approach unity (zero monopoly tax). If own-price effects dominate other-price effects, then the monopoly tax decreases as the own-supply elasticities increase: $\partial\tau_1/\partial\eta_{11} < 0$ and $\partial\tau_0/\partial\eta_{00} > 0$. Similarly, the monopoly tax decreases when the cross-price elasticities increase: $\partial\tau_1/\partial\eta_{10} < 0$ and $\partial\tau_0/\partial\eta_{01} > 0$. So an increase in the competitiveness of the labor markets will cause firms to lower the monopoly tax on labor. This statement concerns the supply elasticities observed by individual firms, which depend upon market structure, as distinct from the elasticities of the labor market as a whole, which depend upon how much workers adjust their quitting time in response to wage changes.

How will an increase in the competitiveness of the labor market influence the steady state values of q and s? We are contemplating a change in market structure which causes firms to observe higher values of η. We have seen that firms will normally respond by reducing their monopoly tax on wages. It is easy to calculate from the optimal quitting equation (1) that workers quit later if their wage-tax is reduced: $\partial q/\partial\tau_1 < 0$ and $\partial q/\partial\tau_0 > 0$. This change in q will cause a change in the supply elasticity of labor to the whole industry; let us suppose that the impact of this second round change upon an individual firm is negligible. By combining the derivatives we find that the maximization equation shifts up when the supply elasticity increases:

$$\frac{\partial q}{\partial \tau_0}\frac{\partial \tau_0}{\partial \eta_{00}} > 0 \quad \text{and} \quad \frac{\partial q}{\partial \tau_1}\frac{\partial \tau_1}{\partial \eta_{11}} > 0. \tag{11}$$

The implications of (11) for a steady state are stated as a formal proposition:

Proposition 8: If Assumptions 1–5 are true, then a monopsony will lower the quitting age q and apprentice ratio s relative to a competitive industry consisting of pyramidal firms ($s > 1$); both are constant for an inverted pyramid.

7. Market power and performance: discussion

■ In this section we relate the various institutional forms of markets for professional labor to our comparative statics results. The institutional forms which we shall consider are the guild or labor aristocracy, the democratic union, the monopsonist, and the professional school. These institutions may have power to interfere with the competitive mechanism in the labor market. In each case we ask what will be the consequences of institutional power over the parameters of our system. The parameters have equilibrium values in a competitive system; we determine the direction in which they would be altered by noncompetitive institutions. Our conclusions rest upon Proposition 4–8, in some cases supplemented by informal arguments.

We define a guild as a monopoly on the supply of labor to a firm or profession which maximizes the income of seniors. We have in mind professional associations and unions dominated by senior members. The senior wage rises when there is an increase in the apprentice ratio. A guild which held power over the parameters of the system would choose a low promotion probability (Proposition 4a) and a small cohort size (Proposition 6a) to increase the seniority ratio. In the case of retirement age, an increase in working life increases senior income if wages are held constant, but an increase in L decreases the apprentice ratio (Proposition 5a). We cannot be certain which effect is stronger, but a good guess is that senior income increases with retirement age L. A guild would seek a wage structure which taxes juniors and subsidizes seniors, although this tendency will be moderated by the resulting fall in the apprentice ratio (Proposition 7). These conclusions about the guild are summarized in Table 2.

TABLE 2

MARKET POWER AND PERFORMANCE

	PROMOTION PROBABILITY Π	RETIREMENT AGE L	COHORT SIZE χ	JUNIOR SUBSIDY τ_0	SENIOR SUBSIDY $-\tau_1$	APPRENTICE RATIO s
COMPETITIVE	o	o	o	o	o	o
GUILD	–	+	–	–	+	+
DEMOCRATIC UNION	+	+	?	+	–	–
MONOPSONY	?	?	?	–	–	–
PROFESSIONAL SCHOOL	+	+	+	+	–	–

+ GREATER THAN COMPETITIVE VALUE
– SMALLER THAN COMPETITIVE VALUE
o SAME AS THE COMPETITIVE VALUE

Contemporary theories of the firm have stressed the potential conflict of interest between owners and managers. In the preceding paragraph we catalogued distortions in the market for professional labor which are sought by seniors. These distortions are against the interests of capital because they produce an excess burden which will be born in part by capital.[10] There is a conflict of interest between owners and managers concerning choice of the parameters Π, L, and τ. The possible exception to this observation concerns choice of the cohort size α. By restricting the industry's size, the price of its services will rise. This tends to increase the marginal value product of factors in the industry, including capital. However, restriction on the entry of labor, without any restriction on capital, will increase the capital/labor ratio, which decreases the marginal value product of capital. We cannot know *a priori* which effect is stronger.

We define a democratic union as a monopoly on the supply of professional labor which maximizes the income of juniors. We have in mind professional associations or unions which juniors dominate because of their numerical superiority. More precisely, a democratic union attempts to maximize the expected future income of the professional of median age t_{med}:[11]

$$\frac{\Pi}{1 - \Pi t_{med}} \int_{t_{med}}^{q} [w_1(L - t)t + w_0(t - t_{med})]dt + [(L - q)v + w_0(q - t_{med})] \frac{1 - \Pi q}{1 - \Pi t_{med}} .$$

It is a reasonable assumption that this value will increase when the apprentice ratio decreases. In that case a democratic union will choose a high promotion probability (Proposition 4a), a late retirement age (Proposition 5a), and a wage structure which subsidizes juniors and taxes seniors (Proposition 7). Juniors benefit directly from a decrease in cohort size, because their marginal value product increases with the relative price of the profession's output, but this effect is moderated by the resulting rise in the apprentice ratio (Proposition 6a). We cannot know *a priori* which effect will dominate.

Our conclusion that a democratic union will try to raise the promotion probability agrees with the observation that unions base promotion upon seniority. If promotion is based upon seniority alone, then the future wage of juniors is certain. The optimal stopping problem has a corner solution when uncertainty is eliminated: either no one wants to quit the profession or no one wants to enter it. Democratic unions have an incentive to acquire control over the portals of entry into the junior ranks to make promotion by seniority a viable policy. The guild avoids this problem, since it increases the uncertainty associated with promotion.

If market power resides with capital rather than labor, then capital will tax labor by paying a wage which is less than the marginal product. The size of the tax decreases with the elasticity of supply, as demonstrated in (10a) and

[10] Seniors exercise power in a way which increases the ratio of capital to total labor in the short run. We expect the profit rate to fall in the usual cases, although we must be cautious about this inference because labor is heterogeneous. Our generalization is no substitute for a detailed theory of incidence.

[11] We are assuming that each professional has single peaked preferences over the model's parameters; furthermore, we assume that among the professionals of median age is a junior whose preferred value on a given parameter is the median.

(10b). The supply of labor by seniors is inelastic in the short run, whereas the supply of labor by juniors is elastic. If the monopolist is myopic, then he will exploit this difference in supply elasticities by taxing seniors at a higher rate than he taxes juniors. In Proposition 8 we proved that monopsony lowers the apprentice ratio, which will cause a further reduction in the senior wage. Seniors have an incentive to resist the accumulation of market power by capital, which is a conflict between owners and managers.

We may think of juniors as students in a professional school and seniors as graduates. A professional school has monopoly power when no other institution can transform juniors into seniors. There is no definitive theory about the objectives which such a school would pursue. One possibility is that it would maximize fees collected from juniors. Juniors enjoy a surplus which equals the difference between the expected future income of a junior and outside income. At age zero a junior's surplus is:

$$S_0 = \int_0^q [w_1(L - t) + w_0 t]\Pi dt + \left(1 - \int_0^q \Pi dt\right)[(L - q)v + w_0 q] - Lv.$$

Our postulated objective is to maximize the total surplus of persons entering the junior ranks:

$$\max \alpha S_0.$$

This objective differs little from the aim of a democratic union. A professional school will act like a democratic union in basing graduation upon seniority. No professional school has power over the retirement age of its graduates, but, if it did, it would increase L. The cohort size would be increased until the elasticity of the total surplus with respect to cohort size was unity. A professional school with the power to cross subsidize would tax graduates and subsidize students.

Our institutional analysis is summarized in Table 2, which shows how each institution would alter the competitive values of the parameters of the system if it possessed sufficient power.

8. Towards generalization

■ Perhaps the most objectionable feature of the simple model is the assumption of uniform promotion probabilities; no role was allowed for differential learning by individuals or teaching by firms, or for transmitting and receiving signals about the ability of individual workers. In this section we make the simplest changes which will remedy this defect, but even so, it is no longer possible to close the model and carry out comparative statics with determinate signs. The signs depend upon substitution possibilities whose magnitudes cannot be determined by pure theory. If it turns out that the direct effects described in the simple model are stronger than the indirect effects which vitiate them in the general model, then our simple model will be a good empirical theory and our predictions will be confirmed; otherwise it will be necessary to resort to more general and less conclusive models. The point of this section is to sketch those general models and to show why they are inconclusive.

□ **Supply.** The most general formulation of the supply problem allows the individual to affect his promotion prospects as well as to decide whether to quit. We assume that individuals have control over a variable k which improves their promotion prospects; for example k might be knowledge. We also assume that firms control a variable κ which influences promotion rates; for example, κ might be on-the-job training. We define the cumulative values of k and κ: $K(t) = \int_0^t k(m)dm$ and $\chi(t) = \int_0^t \kappa(m)dm$. Promotion prospects for the ith individual depend upon time and the stock variables: $\Pi_i = \Pi(t, K_i(t), \chi(t))$. Some individuals are able to obtain k at a lower price than others; for example, some people may learn much faster than others, so that a smaller sacrifice is required for them to learn. We indicate this individual price by n; thus a junior's net earnings at any point in time are $w_0 - nk(t)$. The optimal q and $k(t)$ for anyone will depend upon his n. In the general model we have a distribution of persons in any cohort, each with a different quitting age $q(n)$ and an investment schedule $k(t,n)$.

This characterization of the choice variables leads to a control formulation of the individual's maximization problem. The promotion probability varies across time for each individual and across individuals at any point in time. We have already seen that we need nonuniform Π across time to avoid a corner solution on quitting age for professions which are inverted pyramids, so this general analysis is particularly valuable when $s < 1$. However, the development of such a theory would require a separate paper, so we shall merely formulate the control problem in a footnote.[12]

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[12] Define the cumulative discounted values of k and κ: $\bar{K} = \int_0^t k(m)e^{-rm}dm$ and $\bar{\chi}(t) = \int_0^t \kappa(m)e^{-rm}dm$. To avoid double integrals in the control problem it is necessary to proceed in the assumption that Π can be written as a function of $\bar{K}$ and $\bar{\chi}$, rather than K and χ. Define the discount factor $\delta(t) = (1/t)(\int_0^t e^{-rm}dm)$ and the cumulative probability $\Pi(q) = \int_0^q \Pi(t, \bar{K}(t), \bar{\chi}(t))dt$. The individual's expected lifetime earnings are

$$
\begin{aligned}
J &= \int_0^q [w_0 t\delta(t) - n\bar{K}(t) + w_1(L\delta(L) - t\delta(t))]\Pi(t, \bar{K}, \bar{\chi})dt \\
&\quad + [w_0 q\delta(q) - n\bar{K}(\mathrm{d}q) + v(L\delta(L) - q\delta(q))][1 - \int_0^q \Pi(t, \bar{K}, \bar{\chi})dt] \\
&= \phi(q, \bar{K}(q), \bar{\Pi}(q)) + \int_0^q L(t, \bar{K}, \bar{\chi})dt,
\end{aligned}
$$

where

$$
\phi(q, \bar{K}(q), \bar{\Pi}(q)) = vL\delta(L) + [(w_0 - v)q\delta(q) - n\bar{K}(q)][1 - \bar{\Pi}(q)]
$$

$$
L(t, \bar{K}(t), \bar{\chi}(t)) = [(w_0 - w_1)t\delta(t) - n\bar{K}(t) + (w_1 - v)L\delta(L)]\Pi(t, \bar{K}, \bar{\chi}).
$$

The individual's problem is to choose q and $k(t)$ to maximize this expression, subject to the differential constraints implied in the definitions of $\bar{K}$, $\bar{\Pi}$. This is an optimal control problem with control variable k, state variables $\bar{K}$ and $\bar{\Pi}$, and variable terminal time q. The Hamiltonian function for this problem is

$$
H = \lambda_0 L + \lambda_1 \frac{d\bar{K}}{dt} + \lambda_2 \frac{d\bar{\Pi}}{dt} = \lambda_0 L + \lambda_1 k e^{-rt} + \lambda_2 \Pi(t, \bar{K}, \bar{\chi}),
$$

with $\lambda_0 = 0$ or 1. Since no constraints (other than their dynamic equations) are imposed on the state variables at the terminal time q, the present problem is a "normal" problem so that one may take $\lambda_0 = 1$. The necessary conditions for this problem in addition to the maximum principle are the Hamiltonian equations

$$
\frac{d\bar{K}}{dt} = ke^{-rt}, \qquad \frac{d\bar{\Pi}}{dt} = \Pi(t, \bar{K}, \bar{\chi}), \qquad -\mathring{\lambda}_1 = \frac{\partial H}{\partial \bar{K}}, \qquad -\mathring{\lambda}_2 = \frac{\partial H}{\partial \bar{\Pi}}
$$

together with the transversality conditions

$$
\lambda_1(q) = \frac{\partial \phi}{\partial \bar{K}}, \qquad \lambda_2(q) = \frac{\partial \phi}{\partial \bar{\Pi}},
$$

A simpler approach retains the assumption of uniform Π across time for each person, but allows the individual to influence his Π by a once-for-all investment in training K. Thus we write i's promotion probability as a function of K and χ, but not time: $\Pi_i = \Pi(K_i,\chi)$. In formulating this optimization problem it is convenient to define a discount factor $\delta(t)$ such that $t\delta(t)$ is the present value of a dollar earned continuously and discounted at a rate r over the time interval 0 to t:

$$\delta(t) = \frac{1}{t}\int_0^t e^{-rm}dm.$$

It is straightforward to obtain the following expression for discounted expected lifetime earnings:

$$J = w_0 q\delta(q) - nK + v[L\delta(L) - q\delta(q)] + \int_0^q \{w_0[t\delta(t) - q\delta(q)] + w_1[L\delta(L) - t\delta(t)] + v[q\delta(q) - L\delta(L)]\}\Pi(K,\chi)dt.$$

J reduces to the expression for undiscounted expected lifetime earnings obtained in Section 2 by eliminating the terms δ, K, and χ. The individual's problem is to choose q and K to maximize J, with first-order conditions

$$\begin{aligned} 0 &\geq \partial J/\partial q \\ 0 &\geq \partial J/\partial K. \end{aligned} \tag{12}$$

We can determine the individual's response to changes in the parameters of his maximization problem by differentiating the first-order conditions and solving simultaneously in the usual way. For example, the response to an increase in w_0 is found by solving

$$\begin{pmatrix} J_{qq} & J_{qK} \\ J_{Kq} & J_{KK} \end{pmatrix}\begin{pmatrix} dq \\ dK \end{pmatrix} = \begin{pmatrix} -e^{-rq}(1 - \Pi q)dw_0 \\ -\dfrac{\partial \Pi}{\partial K}\displaystyle\int_0^q w_0(t\delta(t) - q\delta(q))dt dw_0 \end{pmatrix}.$$

The sign of the change is not generally determinate; for example,

$$\frac{\partial q}{\partial w_0} = \frac{1}{det(\cdot)}\left\{-e^{-rq}(1 - \Pi q)J_{KK} + J_{qK}\frac{\partial \Pi}{\partial K}\int_0^q w_0(t\delta(t) - q\delta(q))dt\right\}.$$

For a pyramidal firm the first term in the sum is positive and the second is negative. Notice that the second term increases with the strength of the interaction J_{qK} between q and k, so it may be interpreted as the effect of substituting more investment k for later quitting q. If this substitution effect is small, then juniors respond to an increase in their wage by quitting later. The signs are not ambiguous for all parametric shifts; for example, if there is an increase in w_1, or a decrease in n or v_1, then juniors respond by quitting later and investing more in promotions. The complete set of signs for parametric changes appears in Table 3.

---

and the free terminal time condition

$$\frac{\partial J}{\partial t} = 0 \quad \text{at} \quad t = q.$$

A similar control approach to human capital theory is found in Ben-Porath (1967, p. 352).

TABLE 3

RESPONSE OF JUNIORS TO ECONOMIC PROSPECTS

$\partial K/\partial w_0$?		$\partial q/\partial w_0 > 0$ IF J_{qK} IS SMALL
$\partial K/\partial w_1 > 0$	ASSUMING $\partial \Pi/\partial K > 0$	$\partial q/\partial w_1 > 1$
$\partial K/\partial v < 0$	ASSUMING INTEREST RATE r IS NOT TOO HIGH	$\partial q/\partial v < 0$ ASSUMING r IS NOT TOO HIGH
$\partial K/\partial n < 0$		$\partial q/\partial n < 0$
$\partial K/\partial \chi$?		$\partial q/\partial \chi$?

□ **Demand.** In the general setting, the firm with monopoly power must choose wage rates and expenditure $\kappa(t)$ on training juniors. The present value of the firm can be written

$$\int_0^\infty \{f(x_1,x_0) - w_1x_1 - w_0x_0 - \kappa x_0\}e^{-rt}dt. \tag{13}$$

There is a complicated relationship between the firm's choices and the labor supply.[13] We can calculate x_0 and x_1 by tracing the promotion and quitting behavior in each cohort. Individuals in each cohort differ according to the price n which they face; we index the individuals $i = 1, 2, \ldots, N$. There is an optimal q_i and $k_i(t)$ for each individual. The juniors are all the workers who have not been promoted or quit:

$$x_0 = \sum_{i=1}^{N} \alpha_i \int_0^{q_i} \left[1 - \int_0^u \Pi_i(t)dt\right]du. \tag{14}$$

Similarly, the seniors consist of all the workers who have been promoted:

$$x_1 = \sum_{i=1}^{N} \alpha_i \left\{\int_0^{q_i} \left[\int_0^u \Pi_i(t)dt\right]du + (L - q) \int_0^{q_i} \Pi_i(t)dt\right\}. \tag{14}$$

These expressions reduce to equations (4) and (4a) in the simple model when everyone faces the same price n. The monopolistic firm chooses w_0, w_1, and κ to maximize (13) subject to (14a), (14b), and the supply equations (12).

In a steady state where the firm's value is constant from one period to another, the first-order conditions can be expressed as linear equations in w_1, w_0, and κ, with η representing supply elasticities as in the preceding section:[14]

$$\begin{pmatrix} \eta_{11} + 1 & s\eta_{01} & s\eta_{01} \\ \eta_{10} & s(\eta_{00} + 1) & s\eta_{00} \\ \eta_{1\kappa} & s\eta_{0\kappa} & s(1 + \eta_{0\kappa}) \end{pmatrix} \begin{pmatrix} w_1 \\ w_0 \\ \kappa \end{pmatrix} \geq \begin{pmatrix} (g - g's)\eta_{11} + g's\eta_{01} \\ (g - g's)\eta_{10} + g's\eta_{00} \\ (g - g's)\eta_{1\kappa} + g's\eta_{0\kappa} \end{pmatrix}. \tag{15}$$

The solution gives the optimal wages and educational expenditure as a function of the apprentice ratio and the vector of elasticities:

$$(w_1,w_0,\kappa) = w(s,\eta).$$

[13] Labor turnover costs receive a formal treatment in Salop (1973).

[14] The actual η depends upon market structure and the individual supply curves for q and K. We simplify somewhat by proceeding as if the η observed by firms depends only upon variables of market structure. See the discussion in the preceding section.

□ **Towards comparative statics.** In the simple model we proved the existence of a steady state equilibrium by finding values of the control variables at which individuals and firms were both at a maximum—the maximization function—and the stock of seniors and juniors was not changing—the steady-state function. The intersection of these two functions gave the steady-state competitive values of q and s. The comparative statics concerned parametric shifts in this point of intersection. In the general model the maximization function is obtained by substituting equation (12) into (15), thus combining the first-order conditions for the firm and the individual. The result expresses the quitting age $q(n)$ and investment $K(n)$ as a function of the apprentice ratio s and the elasticities η observed by firms. The steady-state function is found by integrating the ratio of equation (14) and (14a):

$$s = \frac{\Sigma \kappa_i q_i(1 - \Pi_i q_i/2)}{\Sigma \kappa_i q_i \Pi_i(L - q_i/2)} .$$

It is possible at least in principle to describe the steady-state equilibrium as the intersection or fixed point for these two functions.

We were able to carry out a comparative statics analysis in the simple model because the derivatives of the steady-state and maximization functions were determinate in sign. The comparative statics relied upon a simple fact: If income prospects in the profession improve relatively, then juniors gamble longer and the apprentice ratio rises for pyramidal firms. This simple fact is vitiated in the general model because juniors may respond to improved income prospects by gambling longer but investing less in promotion, or *vice versa*, as we saw in Table 3. The problem is that any change in s causes a change in w_0, but the effect upon q and K of a change in w_0 is ambiguous. For example, we see from the table that an increase in the price of knowledge n will cause juniors to buy less of it and to quit sooner. Firms will then find that the apprentice ratio has changed and they will alter wages accordingly. But we have already noted that the sign of the response of juniors to changes in w_0 is indeterminate. As a consequence we shall be unable to determine how a change in n affects q and K after individuals and firms adjust. In brief, we cannot predict how the maximization function will shift because of the possibilities for substitution between q and K. The cost of generality is a closed model which is analytically intractable.

9. Conclusion

■ Stratified labor markets may be viewed longitudinally, from the viewpoint of an individual making career decisions, or in cross section, from the viewpoint of a firm inspecting its hierarchy. The individual must decide how long to persist in pursuing a particular career, which may be described in terms of promotions in a firm or a profession, and the firm must decide how many workers to employ in each grade. We have presented the first closed model in which firms and individuals are maximizing these choice variables in a simple two-level hierarchy. A flexible interpretation of the variables in the model permits its application to most hierarchies in which there is quitting or an "up or out" rule. Our comparative statics results permit us to analyze the effect of unions, professional associations, technical schools, and monopsonistic employers upon the structure of the labor market. This represents progress over analyzing power in labor markets under the crude assumption that labor is homogeneous. We achieve our results by not addressing some of the deeper questions about the cause of hierarchy, but we hope our model will be useful to those who do.

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