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Torus Orbits On Homogeneous Varieties And Kac Polynomials Of Quivers

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ABSTRACT. In this paper we prove that the counting polynomials of certain torus orbits in products of partial flag varieties coincides with the Kac polynomials of supernova quivers, which arise in the study of the moduli spaces of certain irregular meromorphic connections on trivial bundles over the projective line. We also prove that these polynomials can be expressed as a specialization of Tutte polynomials of certain graphs providing a combinatorial proof of the non-negativity of their coefficients.

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1. Introduction

1.1. **Quivers and Kac polynomials.** Given a field κ and k parabolic subgroups $P_1, \ldots, P_k$ of $\mathrm{GL}_r(\kappa)$, we form the cartesian product of partial flag varieties $\mathcal{F} := (\mathrm{GL}_r/P_1) \times \cdots \times (\mathrm{GL}_r/P_k)$ on which GL_r acts diagonally by left multiplication. To each parabolic P_i corresponds a unique partition μ^i of r (given by the size of the blocks). From the k-tuple $\boldsymbol{\mu} = (\mu^1, \ldots, \mu^k)$ we define in a natural way (see for instance [10]) a star-shaped quiver Γ with k legs whose lengths are the lengths of the partitions $\mu^1, \ldots, \mu^k$ minus 1. We also define from $\boldsymbol{\mu}$ a dimension vector $\mathbf{v}$ of Γ with coordinate r on the central vertex and coordinates $n - \mu^i_1, n - \mu^i_1 - \mu^i_2, \ldots$ on the nodes of the i-th leg. Denote by $Z_r \subset \mathrm{GL}_r$ the one dimensional subgroup of central matrices. The set $\mathcal{E}(\kappa)$ of GL_r-orbits of $\mathcal{F}$ whose stabilizer is, modulo Z_r, a unipotent group is in bijection with the isomorphism classes of absolutely indecomposable representations of $(\Gamma, \mathbf{v})$ over the field κ. Hence the size of $\mathcal{E}(\mathbb{F}_q)$ coincides with the evaluation at q of the Kac polynomial $A_{\Gamma,\mathbf{v}}(t)$ of $(\Gamma, \mathbf{v})$, see §2.1.2. Now it is known from Crawley-Boevey and van den Bergh [3] that when the dimension vector $\mathbf{v}$ is *indivisible* (i.e. the gcd of the parts of the partitions μ^i, $i = 1, \ldots, k$, is one), the polynomial $A_{\Gamma,\mathbf{v}}(t)$ coincides (up to a known power of q) with the Poincaré polynomial of some quiver varieties $\mathfrak{M}_\xi(\mathbf{v})$ attached to $(\Gamma, \mathbf{v})$. Let us give a concrete description of this quiver variety. Assume given k distinct points $a_1, \ldots, a_k \in \mathbb{C}$ and a *generic* tuple $(C_1, \ldots, C_k)$ of semisimple adjoint orbits of $\mathfrak{gl}_r(\mathbb{C})$ such that the multiplicities of the eigenvalues of C_i is given by the partition μ^i. By Crawley-Boevey [2] we can identify this quiver variety $\mathfrak{M}_\xi(\mathbf{v})$ with the moduli space of meromorphic connections

$$\nabla = d - \sum_{i=1}^{k} A_i \frac{dz}{z - a_i}$$

on the trivial rank r vector bundle over the Riemann sphere $\mathbb{P}^1$ with residues $A_i \in C_i$ for $i = 1, \ldots, k$ and with no further pole at ∞, i.e. $A_1 + \cdots + A_k = 0$.

In conclusion, when $\gcd(\mu^i_j)_{i,j} = 1$, the counting over $\mathbb{F}_q$ of the GL_r-orbits of $\mathcal{F}$ with unipotent stabilizer (modulo Z_r) gives the Poincaré polynomial of the moduli space of some regular connections (i.e connections with simple poles at punctures $a_1, \ldots, a_k$) on the trivial rank r vector bundle over $\mathbb{P}^1$. In general (i.e. without assuming $\gcd(\mu^i_j)_{i,j} = 1$), it is conjectured (see [11, Conjecture 1.3.2]) that this counting coincides with the pure part of the mixed Hodge polynomial of the moduli space of $\mathbb{C}^r$-local systems on $\mathbb{P}^1 \setminus \{a_1, \ldots, a_k\}$ with local monodromy in semisimple conjugacy classes $C_1, \ldots, C_k$ of $\mathrm{GL}_r(\mathbb{C})$ with $(C_1, \ldots, C_k)$ *generic* semisimple of type $\boldsymbol{\mu}$.

1.2. **Torus orbits on homogeneous varieties.** There is another geometric counting problem that also arises in this setup. Let $T \subset \mathrm{GL}_r$ be the maximal torus of diagonal matrices. We can consider the enumeration over $\mathbb{F}_q$ of the T-orbits in $\mathcal{F}$. In general this is a very subtle problem, even for the simplest case of a single maximal parabolic subgroup of GL_r where we would be counting torus orbits on Grassmannians. This problem is connected to matroids, configuration spaces of points in projective spaces, generalizations of the dilogarithm, hypergeometric functions, and moduli spaces of genus 0 pointed curves [5–7, 16].

In this paper we show (Theorem 3.15) that the counting function $E^T(q)$ of the T-orbits of $\mathcal{F}$ whose stabilizer is equal to Z_r coincides with $A_{\Gamma,\mathbf{v}}$, the Kac polynomial of a certain quiver Γ for a certain dimension vector $\mathbf{v}$ (see §2.2 for the definitions and a picture of Γ). As a consequence, $E^T(q)$ is a monic polynomial in q with non-negative integer coefficients whose degree is given by an explicit formula. Moreover, we obtain necessary and sufficient condition for $E^T(q)$ to be non-zero (Theorem 3.15).

The quiver Γ belongs to a class of quivers known as *supernova quivers* (the name is due to Boalch). The corresponding generic quiver varieties $\mathfrak{M}_\xi(\mathbf{v})$ have the following explicit interpretation. Given a tuple $(C_1, \ldots, C_k)$ of semisimple adjoint orbits of $\mathfrak{gl}_r(\mathbb{C})$ of type $\boldsymbol{\mu}$ as above, it follows from Boalch [1, Theorem 9.11 & Theorem 9.16] that $\mathfrak{M}_\xi(\mathbf{v})$ is isomorphic to the moduli space of meromorphic connections on the trivial rank r vector bundle over $\mathbb{P}^1$ with k simple poles at $a_1, \ldots, a_k$ with residues in $C_1, \ldots, C_k$, and with an extra pole of order 2 whose coefficient in dz/z^2 (in a local trivialization) is a semisimple regular matrix. Hence using the main result of [3] (on the connection between Kac polynomial $A_{\Gamma,\mathbf{v}}(t)$ and Poincaré polynomials of $\mathfrak{M}_\xi(\mathbf{v})$), Boalch's result and the results of this paper, we end up with an interpretation of $E^T(q)$ as the Poincaré polynomial of the moduli space of certain irregular meromorphic connections as above on the trivial rank r bundle over $\mathbb{P}^1$.

1.3. **Graph polynomials.** The second main result of this paper is a refined analysis of the coefficients of the polynomials $A_{\Gamma,\mathbf{v}}(q) = E^T(q)$. More precisely, we express $E^T(q)$ as a sum of the specialization $x = 1, y = q$ of the *Tutte polynomial* of certain associated graphs (see Theorem 3.13 and §3.3). We deduce that the coefficients of $A_{\Gamma,\mathbf{v}}(q)$ count spanning trees in these graphs of a given weight, which accounts for their nonnegativity.

Recall that Kac conjectured that the coefficients of Kac polynomials (for any finite quiver) are non-negative [14]. This conjecture was proved in in the case of an indivisible dimension vector by Crawley-Boevey and van den Bergh [3] with further case proved by Mozgovoy [17]; it was proved in full generality by Hausel-Letellier-Villegas [12]. The proofs all give a cohomological interpretation of the coefficients of the Kac polynomial. Our proof of the non-negativity for Kac polynomials of the supernova quivers is completely different relying, as mentioned, on Tutte's interpretation of the coefficients of his polynomial in terms of spanning trees. This proof is purely combinatorial and opens a new approach in understanding the Kac polynomials.

In a continuation to this paper we will discuss how, in fact, the whole Tutte polynomial of the associated graphs is related to counting T-orbits of $\mathcal{F}$.

2. Supernova complete bi-partite quivers

2.1. **Generalities on quivers.** Let Γ be a finite quiver, I its set of vertices and Ω its set of arrows. We assume that Γ has no loops. For $\gamma \in \Omega$ we denote by $h(\gamma)$ (respectively, $t(\gamma)$) the head (resp., tail) of γ. A *dimension vector* $\mathbf{v}$ of Γ is a tuple $(v_i)_{i\in I}$ of non-negative integers indexed by I.

2.1.1. *Roots.* We now recall some well known properties of roots in quivers. For more information, we refer the reader to [14].

For $i \in I$, let $\mathbf{e}_i \in \mathbb{Z}^I$ be the tuple with coordinate i equal 1 and all other coordinates 0. Let $\mathbf{C} = (c_{ij})_{i,j}$ be the Cartan matrix of Γ, namely

$$c_{ij} = \begin{cases} 2 & \text{if } i = j \\ -n_{ij} & \text{otherwise,} \end{cases}$$

where n_{ij} is the number of edges joining vertex i to vertex j. The Cartan matrix determines a symmetric bilinear form $(\ ,\ )$ on $\mathbb{Z}^I$ by

$$(\mathbf{e}_i, \mathbf{e}_j) = c_{ij}.$$

For $i \in I$, define the fundamental reflection $s_i \colon \mathbb{Z}^I \to \mathbb{Z}^I$ by

$$s_i(\lambda) = \lambda - (\lambda, \mathbf{e}_i)\,\mathbf{e}_i, \quad \lambda \in \mathbb{Z}^I.$$

The Weyl group $W = W_\Gamma$ of Γ is defined as the subgroup of automorphisms $\mathbb{Z}^I \to \mathbb{Z}^I$ generated by the fundamental reflections $\{s_i \mid i \in I\}$. A vector $\mathbf{v} \in \mathbb{Z}^I$ is called a *real root* if $\mathbf{v} = w(\mathbf{e}_i)$ for some $w \in W$ and $i \in I$. Let $M = M_\Gamma$ be the set of vectors $\mathbf{u} \in \mathbb{Z}^I_{\geq 0} - \{0\}$ with connected support such that for all $i \in I$, we have

$$(\mathbf{e}_i, \mathbf{u}) \leq 0.$$

Then a vector $\mathbf{v} \in \mathbb{Z}^I$ is said to be an *imaginary root* if $\mathbf{v} = w(\delta)$ or $\mathbf{v} = w(-\delta)$ for some $\delta \in M$ and $w \in W$. Elements of M are called *fundamental imaginary roots*. We denote by $\Phi = \Phi(\Gamma) \subset \mathbb{Z}^I$ the set of all roots of Γ (real and imaginary).

A root is said to be *positive* if its coordinates are all non-negative. One can show that an imaginary root is positive if and only if it is of the form $w(\delta)$ with $\delta \in M$. In particular the Weyl group W preserves the set of positive imaginary roots.

For any vector $\mathbf{u} \in \mathbb{Z}^I$ put

$$\Delta(\mathbf{u}) := -\frac{1}{2}(\mathbf{u}, \mathbf{u}).$$

We have the following characterization of the imaginary roots [15, Proposition 5.2]:

Lemma 2.1. *Assume that* $\mathbf{v} \in \Phi$. *Then* $\mathbf{v}$ *is imaginary if and only if* $\Delta(\mathbf{v}) \geq 0$. □

2.1.2. *Representations.* Let κ be a field. A *representation* $\boldsymbol{\varphi}$ of Γ over κ is a finite-dimensional graded κ-vector space $V^{\boldsymbol{\varphi}} := \bigoplus_{i\in I} V_i^{\boldsymbol{\varphi}}$ and a collection $(\varphi_\gamma)_{\gamma\in\Omega}$ of linear maps $\varphi_\gamma \colon V^{\boldsymbol{\varphi}}_{t(\gamma)} \to V^{\boldsymbol{\varphi}}_{h(\gamma)}$. The vector $\mathbf{v} = (\dim V_i)_{i\in I}$ is called the *dimension vector* of $\boldsymbol{\varphi}$. We denote by $\mathrm{Rep}_{\Gamma,\mathbf{v}}(\kappa)$ the κ-vector space of representations of Γ of dimension vector $\mathbf{v}$ over κ.

For $\boldsymbol{\varphi} \in \mathrm{Rep}_{\Gamma,\mathbf{v}}(\kappa)$ and $\boldsymbol{\varphi}' \in \mathrm{Rep}_{\Gamma,\mathbf{v}'}(\kappa)$, we have the obvious notions of morphism $\boldsymbol{\varphi} \to \boldsymbol{\varphi}'$ and direct sum $\boldsymbol{\varphi} \oplus \boldsymbol{\varphi}' \in \mathrm{Rep}_{\Gamma,\mathbf{v}+\mathbf{v}'}(\kappa)$. We say that a representation of Γ over κ is *indecomposable* if it is not isomorphic to a direct sum of two non-zero representations of Γ over κ. An indecomposable representation of Γ over κ that remains indecomposable over any finite field extension of κ is called an *absolutely indecomposable* representation of Γ over κ.

Recall [14] that there exists a polynomial $A_{\Gamma,\mathbf{v}}(t) \in \mathbb{Z}[t]$ such that for any finite field $\mathbb{F}_q$, the evaluation $A_{\Gamma,\mathbf{v}}(q)$ counts the number of isomorphism classes of absolutely indecomposable representations of Γ of dimension $\mathbf{v}$ over $\mathbb{F}_q$. We call $A_{\Gamma,\mathbf{v}}(t)$ the *Kac polynomial* of Γ with dimension vector $\mathbf{v}$.[1]

Theorem 2.2. *The polynomial $A_{\Gamma,\mathbf{v}}(t)$ satisfies the following properties [14]:*

(i) *The polynomial $A_{\Gamma,\mathbf{v}}(t)$ does not depend on the orientation of the underlying graph of Γ.*
(ii) *The polynomial $A_{\Gamma,\mathbf{v}}(t)$ is non zero if and only if $\mathbf{v} \in \Phi(\Gamma)$. Moreover $A_{\Gamma,\mathbf{v}}(t) = 1$ if and only if $\mathbf{v}$ is a real root.*
(iii) *If non-zero, the polynomial $A_{\Gamma,\mathbf{v}}(t)$ is monic of degree $\Delta(\mathbf{v}) + 1$.*
(iv) *For all $w \in W$, we have $A_{\Gamma,w(\mathbf{v})}(t) = A_{\Gamma,\mathbf{v}}(t)$.*

We have also the following theorem (see Hausel-Letellier-Villegas [12]), which was conjectured by Kac [14]:

Theorem 2.3. *The polynomial $A_{\Gamma,\mathbf{v}}(t)$ has non-negative integer coefficients.*

For $\mathbf{v} = (v_i)_{i\in I}$ a dimension vector, put

$$G_{\mathbf{v}} := \prod_{i\in I} \mathrm{GL}_{v_i}(\kappa),$$

and identify $\mathrm{Rep}_{\Gamma,\mathbf{v}}(\kappa)$ with $\bigoplus_{\gamma\in\Omega} \mathrm{Mat}_{v_{h(\gamma)},v_{t(\gamma)}}(\kappa)$. Under this identification the group $G_{\mathbf{v}}$ acts on $\mathrm{Rep}_{\Gamma,\mathbf{v}}(\kappa)$ by simultaneous conjugation:

$$g \cdot \boldsymbol{\varphi} = (g_{v_{h(\gamma)}} \varphi_\gamma g^{-1}_{v_{t(\gamma)}})_{\gamma\in\Omega}.$$

[1] In the literature this polynomial is sometimes called the *A-polynomial*.

Then two representations are isomorphic if and only if they are $G_{\mathbf{v}}$-conjugate. Put

$$Z_{\mathbf{v}} = \{(\lambda \cdot \mathrm{Id}_{v_i})_{i\in I} \in G_{\mathbf{v}} \mid \lambda \in \kappa^{\times}\}.$$

The group $Z_{\mathbf{v}}$ acts trivially on $\mathrm{Rep}_{\Gamma,\mathbf{v}}(\kappa)$. We have the following characterization of absolute indecomposibility in terms of $G_{\mathbf{v}}$ and $Z_{\mathbf{v}}$:

Proposition 2.4. *[14, §1.8] A representation in* $\mathrm{Rep}_{\Gamma,\mathbf{v}}(\kappa)$ *is absolutely indecomposable if and only if the quotient of its stabilizer in* $G_{\mathbf{v}}$ *by* $Z_{\mathbf{v}}$ *is a unipotent group.*

2.2. **Complete bipartite supernova quivers.** We now introduce the main objects of this paper. For fixed non-negative integers $r, k, s_1, \dots, s_k$ consider the quiver Γ with underlying graph as in Figure 1. The subgraph with vertices $(1), \dots, (r), (1;0), \dots, (k;0)$ is the complete bipartite graph of type (r, k), i.e. there is an edge between any two vertices of the form (i) and $(j;0)$. We orient all edges toward the vertices $(1;0), \dots, (k;0)$, and denote by I the set of vertices of Γ and by Ω the set of its arrows. We call paths of the form $(j;s_j), (j;s_j-1), \dots, (j;0)$ the *long legs* of the graph, and the edges of the complete bipartice subgraph the *short legs*.

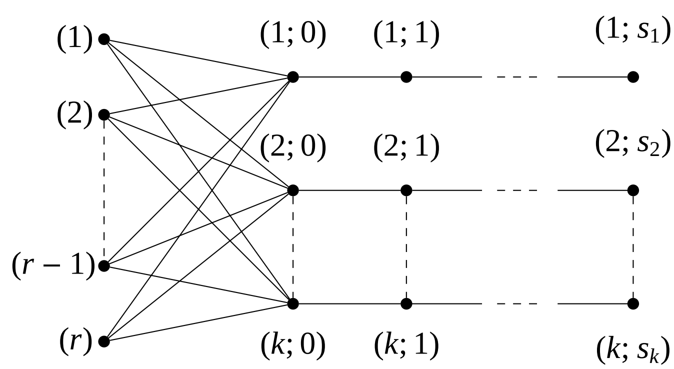


FIGURE 1. The complete bipartite supernova graph

For $\mathbf{v} \in \mathbb{Z}^I_{\geq 0}$ and $i = 1, \dots, k$, define

$$\delta_i(\mathbf{v}) := -(\mathbf{e}_{(i;0)}, \mathbf{v}) = -2v_{(i;0)} + v_{(i;1)} + \sum_{j=1}^{r} v_{(j)}.$$

Lemma 2.5. *Let* $\mathbf{v} \in \mathbb{Z}^I_{\geq 0}$. *Then* $\mathbf{v}$ *is in* M_Γ *if and only if the following three conditions are satisfied*

(i) for all $i = 1, \dots, k$ we have $\delta_i(\mathbf{v}) \geq 0$,
(ii) for all $l = 1, \dots, r$,

$$\sum_{j=1}^{k} v_{(i;0)} \geq 2v_{(l)},$$

(iii) for all $i = 1, \dots, k$ and all $j = 0, \dots, s_i - 1$,

$$v_{(i;j)} - v_{(i;j+1)} \geq v_{(i;j+1)} - v_{(i;j+2)} \tag{2.2.1}$$

with the convention that $v_{(i;s_i+1)} = 0$.

Consider a k-tuple of non-zero partitions $\boldsymbol{\mu} = (\mu^1, \dots, \mu^k)$, where μ^i has parts $\mu^i_1 \geq \mu^i_2 \geq \cdots \geq \mu^i_{s_i+1}$ with μ^i_j possibly equal to 0. This tuple defines a dimension vector $\mathbf{v}_{\boldsymbol{\mu}} = (v_i)_{i\in I} \in \mathbb{Z}^I_{\geq 0}$ as follows. Put $v_{(l)} = 1$ for $l = 1, \dots, r$, $v_{(i;0)} = |\mu^i|$ and $v_{(i;j)} = |\mu^i| - \sum_{f=1}^{j} \mu^i_f$ for $j = 1, \dots, s_i$. Thus the long leg attached to the node $(i; 0)$ (i.e., the type A_{s_i+1} graph with nodes $(i; 0), (i; 1), \dots, (i; s_i)$) is labelled with a strictly decreasing sequence of numbers, and the tips of the short leg are labelled with 1.

Notice that for all $i = 1, \dots, k$, we have

$$\delta_i(\mathbf{v}_{\boldsymbol{\mu}}) = r - |\mu^i| - \mu^i_1 =: \delta(\mu^i),$$

and that $\mathbf{v}_\mu$ satisfies already the condition Lemma 2.5 (iii). The condition (ii) is always satisfied unless $k = 1$ and $v_{(1;0)} = 1$, in which case $\mathbf{v}_{\boldsymbol{\mu}}$ is a real root. This implies the following lemma:

Lemma 2.6. *Assume $k > 1$ or $v_{(1;0)} > 1$. Then $\mathbf{v}_\mu \in M_\Gamma$ if and only if for all $i = 1, \dots, k$ we have $r \geq |\mu^i| + \mu^i_1$.*

Recall [11, Lemma 3.2.1] that if $f = (f_\gamma)_{\gamma\in\Omega}$ is an indecomposable representation (over an algebraically closed field) of Γ of dimension vector $\mathbf{v}$ and if $v_{(i;0)} > 0$ then the linear maps f_γ, where γ runs over the arrows of the long leg attached to the node $(i; 0)$ are all injective. Recall also (see §2.1.2) that a dimension vector $\mathbf{v} \in \mathbb{Z}^I_{\geq 0} \setminus \{0\}$ is a root of Γ if and only if there exists an indecomposable representation of Γ with dimension vector $\mathbf{v}$. We deduce the following fact:

Lemma 2.7. *Let $\mathbf{v} \in \mathbb{Z}^I_{\geq 0}$. If $\mathbf{v} \in \Phi(\Gamma)$ and $v_{(i;0)} > 0$ then $v_{(i;0)} \geq v_{(i;1)} \geq v_{(i;2)} \geq \cdots \geq v_{(i;s_i)}$.*

Corollary 2.8. *Assume that $\mathbf{v}_{\boldsymbol{\mu}}$ is an imaginary root. Then $r \geq |\mu^i|$ for all $i = 1, \dots, k$.*

Proof. Since $\mathbf{v} = \mathbf{v}_\mu$ is a positive imaginary root, $\mathbf{v}' = s_{(i;0)}(\mathbf{v})$ is also a positive imaginary root. In particular $v'_{(i;0)} = r - \mu^i_1 > 0$ and so by Lemma 2.7 we must have $r - \mu^i_1 = v'_0 \geq v'_{(i;1)} = v_{(i;1)} = |\mu^i| - \mu^i_1$, i.e. $r \geq |\mu^i|$. □

Remark 2.9. Corollary 2.8 is false for real roots. For instance assume $k = 1$, $\mu = (3, 1)$ and $r = 3$. Then clearly $s_{(1;0)}(\mathbf{v}_\mu)$ is a real root with coordinate 0 at the vertex $(1; 0)$ and with coordinate 1 at the edge vertices. Thus $\mathbf{v}_\mu$ is also a real root, but note that $r < |\mu|$.

3. Kac polynomial of complete bipartite supernova quivers

3.1. **Preliminaries.**

3.1.1. *Row echelon forms.* Recall that κ denotes an arbitrary field. Denote by B the subgroup of $\mathrm{GL}_n(\kappa)$ of **lower** triangular matrices. Let $r \geq n$ be an integer. Given a sequence of non-negative integers $\mathbf{s} = (s_1, s_2, \ldots, s_d)$ such that $\sum_i s_i = n$ we denote by $P_\mathbf{s}$ the unique parabolic subgroup of GL_n containing B and having $L_\mathbf{s} = \mathrm{GL}_{s_d} \times \cdots \times \mathrm{GL}_{s_1}$ as a Levi factor. Consider a matrix $A \in \mathrm{Mat}_{n,r}(\kappa)$ and decompose its set of rows into d blocks; the first block consists of the first s_d rows of A, the second block consists of the following s_{d-1} rows, and so on.

Definition 3.1. We say that A is in *row echelon form* with respect to $\mathbf{s}$ if the following hold:

(i) The rightmost non-zero entry in each row (called a *pivot*) equals 1.
(ii) All entries beneath any pivot vanish.
(iii) If a block contains two pivots with coordinates (i, j) and (i', j'), then $i < i'$ if and only if $j < j'$.

We have the following easy proposition, whose proof we leave to the reader:

Proposition 3.2. *For any matrix $A \in \mathrm{Mat}_{n,r}(\kappa)$ of rank n there exists a unique $g \in P_\mathbf{s}$ such that gA is in row echelon form with respect to* $\mathbf{s}$.

3.1.2. *Bruhat decomposition.* We identify the symmetric group S_r with permutation matrices in GL_r (if $w \in S_r$, the corresponding permutation matrix $(a(w)_{ij})_{i,j}$ is defined by $a(w)_{ij} = \delta_{i,w(j)}$). Then S_r acts on the maximal torus $T \simeq (\kappa^\times)^r$ of diagonal matrices as $w \cdot (t_1, \ldots, t_r) = (t_{w^{-1}(1)}, \ldots, t_{w^{-1}(r)})$. Consider a parabolic $P_\mathbf{s}$ of GL_r for some sequence $\mathbf{s} = (s_1, \ldots, s_d)$ with $\sum_i s_i = r$ and denote by $S_{r,\mathbf{s}}$ the subset of S_r of permutation matrices which are in row echelon form with respect to $\mathbf{s}$. Equivalently, if we form the partition

$$\{1,\ldots,r\} = \{1,\ldots,s_d\} \cup \{s_d+1,\ldots,s_{d-1}\} \cup \cdots \cup \{r-s_1+1,\ldots,r\} \tag{3.1.1}$$

corresponding to our partition of the rows, then we have $w^{-1}(i) < w^{-1}(j)$ for any $i < j$ in the same block. Then we have the following generalized version of the Bruhat decomposition:

$$\mathrm{GL}_r = \coprod_{w\in S_{r,\mathbf{s}}} P_{\mathbf{s}} wB.$$

Denote by R the root system of GL_r with respect to T. Recall that it is the set $\{\alpha_{i,j} \mid 1 \le i \ne j \le r\}$ of group homomorphisms $\alpha_{i,j} \colon T \to \kappa^\times$ given by

$$\alpha_{i,j}(t_1,\ldots,t_r) = t_i/t_j.$$

We have $\alpha_{i,j} = \alpha_{j,i}^{-1}$ for all $i \ne j$. The symmetric group S_r acts on R by $w\cdot\alpha \colon T \to \kappa^\times$, $(t_1,\ldots,t_r) \mapsto \alpha(t_{w(1)},\ldots,t_{w(r)})$. In particular $w\cdot\alpha_{i,j} = \alpha_{w(i),w(j)}$ for all $i \ne j$. Let

$$R^+ := \{\alpha_{i,j} \mid 1 \le j < i \le r\}$$

be the set of positive roots with respect to B, and let $R^- = R \smallsetminus R^+$.

For $\alpha \in R$, denote by U_α the unique closed one dimensional unipotent subgroup of GL_r such that for all $t \in T$ and $g \in U_\alpha$, we have $t(g-1)t^{-1} = \alpha(t)\cdot(g-1)$. Explicitly, if $\alpha = \alpha_{i,j}$, then the group U_α consists of matrices of the form $I + xE_{i,j}$, where $x \in \kappa$ and $E_{i,j}$ is the matrix whose only non-zero entry is 1 in position (i,j). We denote by $R_{\mathbf{s}} \subset R$ the set of roots α such that U_α is contained in the Levi factor $L_{\mathbf{s}}$. For $w \in S_{r,\mathbf{s}}$, put

$$U_w := \prod_{\{\alpha\in R^+ \mid w(\alpha)\in R^-\smallsetminus R_{\mathbf{s}}\}} U_\alpha.$$

One can show that U_w is a subgroup of GL_r (see for instance [20, 10.1.4]). We have the following lemma, whose proof we omit:

Lemma 3.3.

(i) *Any element g in the cell $P_{\mathbf{s}}wB$ can be written uniquely as pwu with $p \in P_{\mathbf{s}}$ and $u \in U_w$.*

(ii) *Any element of the form wu with $u \in U_w$ is in its row echelon form.*

For any $u \in U_w$ let u_α be its image under the projection $U_w \to U_\alpha$. The group $H_{\mathbf{s}} := P_{\mathbf{s}} \times T$ acts on wU_w via $(p,t)\cdot wu = pwut^{-1}$. For any group H acting on a set X and any point $x \in X$, let $C_H(x) \subset H$ denote the stabilizer of x.

Lemma 3.4. *For $u \in U_w$ we have*

$$C_{H_{\mathbf{s}}}(wu) \simeq C_T(u) = \bigcap_{\{\alpha\in R^+ | w\cdot\alpha\in R^-\smallsetminus R_{\mathbf{s}},\, u_\alpha\neq 1\}} \operatorname{Ker}\alpha.$$

Proof. Let $(p,t) \in H_{\mathbf{s}}$ such that $pwut^{-1} = wu$. Then

$$(pwt^{-1}w^{-1})\, w\,(tut^{-1}) = wu.$$

By Lemma 3.3 this identity is equivalent to $p = wtw^{-1}$ and $tut^{-1} = u$ as T normalizes U_w. But $tut^{-1} = u$ holds if and only if for all α we have $tu_\alpha t^{-1} = u_\alpha$ which identity is equivalent to $t \in \operatorname{Ker}\alpha$ when $u_\alpha \neq 1$. □

Lemma 3.5. *For any $w \in S_{r,\mathbf{s}}$ we have*

$$\begin{aligned}\{\alpha \in R^+ \mid w\cdot\alpha \in R^- \smallsetminus R_{\mathbf{s}}\} &= \{\alpha \in R^+ \mid w\cdot\alpha \in R^-\}\\ &= \{\alpha_{i,j} \mid j < i,\ w(j) > w(i)\}.\end{aligned}$$

Proof. Only the first equality requires proof. If $\alpha_{i,j} \in R^+$ and $w\cdot\alpha_{i,j} \in R^-$, i.e., $j < i$ and $w(i) < w(j)$, then by definition of $S_{r,\mathbf{s}}$ we cannot have $w(i)$ and $w(j)$ in the same block of the partition (3.1.1), i.e., $U_{w\cdot\alpha_{i,j}} = wU_{\alpha_{i,j}}w^{-1}$ is not contained in $L_{\mathbf{s}}$. We have thus proved that the right hand side of the first equality is contained in the left hand side. The reverse inclusion is easy. □

To simplify notation, we put $U_{i,j} = U_{\alpha_{i,j}}$, so that

$$U_w = \prod_{j<i,\, w(i)<w(j)} U_{i,j}.$$

Definition 3.6. For any k-tuple $\mathbf{w} = (w_1,\dots,w_k) \in (S_r)^k$, we denote by $K_{\mathbf{w}}$ the *inversion graph* of $\mathbf{w}$. Namely, the vertices of $K_{\mathbf{w}}$ are labelled by $1,2,\dots,r$ and for any two vertices i and j such that $j < i$, put an edge from i to j for each w_t in $\mathbf{w}$ such that $w_t(i) < w_t(j)$. Thus $K_{\mathbf{w}}$ can have multiple edges. We can think of each edge as having one of k possible colors.

For $\mathbf{u} = (u_1,\dots,u_k) \in U_{\mathbf{w}} := U_{w_1}\times\cdots\times U_{w_k}$, we denote by $K_{\mathbf{w},\mathbf{u}}$ the subgraph of $K_{\mathbf{w}}$ that for any pair i,j includes the edge colored t between vertices i and j if $(u_t)_{i,j} \neq 1$.

Denote by Z_r the center of GL_r and let T act diagonally by conjugation on $U_{\mathbf{w}}$.

Proposition 3.7. *For $\mathbf{u} \in U_{\mathbf{w}}$ we have $C_T(\mathbf{u}) = Z_r$ if and only if the graph $K_{\mathbf{w},\mathbf{u}}$ is connected.*

Proof. This is clear since $\operatorname{Ker}\alpha_{i,j}$ is the subtorus of elements $(t_1,\dots,t_r)$ such that $t_i = t_j$. □

3.2. **Computing the Kac polynomials.** Now Γ is as in §2.2. We want to investigate the polynomial $A_{\Gamma,\mathbf{v}}(t)$. Recall (see Theorem 2.2) that $A_{\Gamma,\mathbf{v}}(t) = 1$ if $\mathbf{v}$ is a real root and $A_{\Gamma,\mathbf{v}}(t) = 0$ if $\mathbf{v}$ is not a root. Moroever $A_{\Gamma,\mathbf{v}}(t)$ is invariant under the Weyl group action. We are reduced to study the polynomials $A_{\Gamma,\mathbf{v}}(t)$ with $\mathbf{v}$ is in the fundamental domain M_Γ. Here we restrict our study to the case where $\mathbf{v}\in M_\Gamma$ is of the form $\mathbf{v} = \mathbf{v}_{\boldsymbol{\mu}}$ for some partition $\boldsymbol{\mu}$. The important thing for our approach is that the coordinates of $\mathbf{v}_{\boldsymbol{\mu}}$ at the vertices (j), $j = 1,\dots,r$, equal 1.

Fix once for all a multi-partition $\boldsymbol{\mu} = (\mu^1,\dots,\mu^k)$ as in §2.2, and to aleviate the notation put $n_i := |\mu^i|$. We assume that $\mathbf{v}_{\boldsymbol{\mu}}$ is in M_Γ, and so that $r \geq n_i + \mu^i_1$ for all $i = 1,\dots,k$ (see Lemma 2.6).

For a partition $\mu = (\mu_1,\dots,\mu_s)$, we denote by P_μ the parabolic subgroup of $\mathrm{GL}_{|\mu|}$ as defined in §3.1.1 and we denote simply by S_μ the subset $S_{|\mu|,\mu}$ of the symmetric group $S_{|\mu|}$ as defined in §3.1.2.

Proposition 3.8. *Assume $\boldsymbol{\varphi}\in \mathrm{Rep}_{\Gamma,\mathbf{v}_\mu}(\kappa)$ is indecomposable. Then*

(i) *the maps φ_γ, where γ runs over the arrows on the k long legs, are all injective, and*
(ii) *for each $i = 1,\dots,k$, the images of $\varphi_{(j)\to(i;0)}$, with $j = 1,\dots,r$, span $V^{\boldsymbol{\varphi}}_{(i;0)}$.*

Proof. Let us prove (ii). Let $W_{(i;0)}$ be the subspace generated by the images of the maps $\varphi_{(j)\to(i;0)}$ with $j = 1,\dots,r$. If $W_{(i;0)} \subsetneq V^{\varphi}_{(i;0)}$ we define subspaces $U_{(i;1)}, U_{(i;2)},\dots,U_{(i;s_i)}$ by $U_{(i;1)} := \varphi^{-1}_{(i;1)\to(i;0)}(W_{(i;0)})$, $U_{(i;p)} := \varphi^{-1}_{(i;p)\to(i;p-1)}(U_{(i;p-1)})$. Let $\boldsymbol{\varphi}'$ be the restriction of $\boldsymbol{\varphi}$ to

$$W_{(i;0)}\oplus\bigoplus_{j=1}^{r} V^{\boldsymbol{\varphi}}_{(j)}\oplus\bigoplus_{p=1}^{s_i} U_{(i;p)}\oplus\bigoplus_{f\neq i}\bigoplus_{j=1}^{s_f} W^{\boldsymbol{\varphi}}_{(f;j)}.$$

Let $W'_{(i;0)}$ be any subspace such that $V^{\boldsymbol{\varphi}}_{(i;0)} = W_{(i;0)}\oplus W'_{(i;0)}$ and define subspaces $U'_{(i;j)}\subset V^{\boldsymbol{\varphi}}_{(i;j)}$ by taking the inverse images of $W'_{(i;0)}$. Then define $\boldsymbol{\varphi}''$ as the restriction of $\boldsymbol{\varphi}$ to

$$W'_{(i;0)}\oplus\bigoplus_{p=1}^{s_i} U'_{(i;p)}.$$

Clearly $\boldsymbol{\varphi} = \boldsymbol{\varphi}'\oplus\boldsymbol{\varphi}''$. Hence we must have $W_{(i;0)} = V^{\boldsymbol{\varphi}}_{(i;0)}$. □

We denote by $\mathbb{X}_{\boldsymbol{\mu}} = \mathbb{X}_{\boldsymbol{\mu}}(\kappa)$ the subset of representations $\boldsymbol{\varphi} = (\varphi_\gamma)_{\gamma\in\Omega} \in \mathrm{Rep}_{\Gamma,\mathbf{v}_\mu}(\kappa)$ that satisfy the conditions (i) and (ii) in Proposition 3.8. As in §2.1.2 we identify $\mathrm{Rep}_{\Gamma,\mathbf{v}_\mu}(\kappa)$ with spaces of matrices and so for each $i = 1,\dots,k$, the coordinates $\varphi_{(1)\to(i;0)},\dots,\varphi_{(r)\to(i;0)}$ of any $\boldsymbol{\varphi}\in\mathbb{X}_{\boldsymbol{\mu}}$ are identified with non-zero vectors in κ^{n_i} which form the columns of a matrix in $\mathrm{Mat}_{n_i,r}$ of rank n_i. For a partition $\mu = (\mu_1,\dots,\mu_s)$ of n, denote by G_μ the group $\mathrm{GL}_n\times\mathrm{GL}_{n-\mu_1}\times\mathrm{GL}_{n-\mu_1-\mu_2}\times\cdots\times\mathrm{GL}_{\mu_s}$. Let $G_{\boldsymbol{\mu}}$ be the subgroup $\prod_{i=1}^k G_{\mu^i}$ of $G_{\mathbf{v}_\mu}$ and denote by T the r-dimensional torus $(\mathrm{GL}_1)^r$. Note that $G_{\mathbf{v}_\mu}\simeq G_{\boldsymbol{\mu}}\times T$.

Denote by $\mathbb{X}_{\boldsymbol{\mu}}/G_{\boldsymbol{\mu}}$ the set of $G_{\boldsymbol{\mu}}$-orbits of $\mathbb{X}_{\boldsymbol{\mu}}$. Since the actions of $G_{\boldsymbol{\mu}}$ and T on $\mathrm{Rep}_{\Gamma,\mathbf{v}_\mu}$ commute, we have an action of T on $\mathbb{X}_{\boldsymbol{\mu}}/G_{\boldsymbol{\mu}}$.

For $i = 1,\dots,k$, put $\mu_0^i := r - n_i$. Note that $\tilde{\mu}^i := (\mu_0^i,\mu_1^i,\dots,\mu_{s_i}^i)$ is a partition of r, i.e., $\mu_0^i \geq \mu_1^i$. Consider

$$S_{\tilde{\boldsymbol{\mu}}} := S_{\tilde{\mu}^1}\times\cdots\times S_{\tilde{\mu}^k}\subset (S_r)^k,$$

where S_μ is defined as in the paragraph preceding Proposition 3.8.

Proposition 3.9. *We have a T-equivariant bijection*

$$\mathbb{X}_{\boldsymbol{\mu}}/G_{\boldsymbol{\mu}} \xrightarrow{\sim} \coprod_{\mathbf{w}\in S_{\tilde{\boldsymbol{\mu}}}} \mathbf{w}U_{\mathbf{w}}, \tag{3.2.1}$$

where T acts on $\mathbf{w}U_{\mathbf{w}}$ as $t\cdot(w_1u_1,\dots,w_ku_k) = (w_1tu_1t^{-1},\dots,w_ktu_kt^{-1})$.

Remark 3.10. By Lemma 3.3 the right hand side of (3.2.1) is isomorphic to $\prod_{i=1}^k \mathrm{GL}_r/P_{\tilde{\mu}^i}$ on which T-acts diagonally by left multiplication.

Proof. We first explain how to construct the bijection (3.2.1). For each $i = 1,\dots,k$, denote by $\mathcal{F}_{\mu^i}$ the set of partial flags of κ-vector spaces

$$\{0\}\subset E^{s_i}\subset\cdots\subset E^1\subset E^0 = \kappa^{n_i}$$

such that $\dim E^j = n_i - \sum_{f=1}^j \mu_f^i$. Let $G'_{\mu^i}\subset G_{\mu^i}$ be the subgroup $\mathrm{GL}_{n_i-\mu_1^i}\times\cdots\times\mathrm{GL}_{\mu^i_{s_i+1}}$ and put $G'_{\boldsymbol{\mu}} = \prod_{i=1}^k G'_{\mu^i}$. Let $\mathrm{Mat}'_{n_i,r}\subset\mathrm{Mat}_{n_i,r}$ be the subset of matrices of rank n_i. Then we have a natural $\mathrm{GL}_{n_1}\times\cdots\times\mathrm{GL}_{n_k}$-equivariant bijection

$$\mathbb{X}_{\boldsymbol{\mu}}/G'_{\boldsymbol{\mu}} \simeq \prod_{i=1}^k\left(\mathcal{F}_{\mu^i}\times\mathrm{Mat}'_{n_i,r}\right) \tag{3.2.2}$$

that takes a representation $\boldsymbol{\varphi}\in\mathbb{X}_{\boldsymbol{\mu}}$ to $(F^i_{\boldsymbol{\varphi}},\varphi_{(1)\to(i;0)},\dots,\varphi_{(r)\to(i;0)})$; here $F^i_{\boldsymbol{\varphi}}$ is the partial flag obtained by taking the images of the compositions of the φ_γ, where γ runs over the arrows of the i-th long leg.

Now fix an element $\boldsymbol{\varphi} \in \mathbb{X}_{\boldsymbol{\mu}}$, and denote by $(F_{\boldsymbol{\varphi}}, M_{\boldsymbol{\varphi}})$ its image in

$$\left(\prod_i \mathcal{F}_{\mu^i}\right) \times \left(\prod_i \mathrm{Mat}'_{n_i,r}\right)$$

via (3.2.2). Since we are only interested in the $G_{\boldsymbol{\mu}}$-orbit of $\boldsymbol{\varphi}$, after taking a $G_{\boldsymbol{\mu}}$-conjugate of $\boldsymbol{\varphi}$ if necessary we may assume that the stabilizer of $F^i_{\boldsymbol{\varphi}}$ is the parabolic subgroup P_{μ^i} of GL_{n_i}. By Lemma 3.2 we may further assume that for all $i = 1, \ldots, k$, the i-th coordinate $M^i_{\boldsymbol{\varphi}}$ of $M_{\boldsymbol{\varphi}}$ is in its row echelon form with respect to $(\mu^i_1, \mu^i_2, \ldots, \mu^i_{s_i+1})$, this time taking a conjugate $p \cdot M^i_{\boldsymbol{\varphi}}$ with $p \in P_{\mu^i}$ if necessary. It is easy to see that there is a unique way to complete the matrix $M^i_{\boldsymbol{\varphi}}$ to a matrix $\tilde{M}^i_{\boldsymbol{\varphi}} \in \mathrm{GL}_r$ that is in row echelon form with respect to $(\mu^i_0, \mu^i_1, \ldots, \mu^i_{s_i+1})$. (cf. Example 3.11).

Now the pivots of $\tilde{M}^i_{\boldsymbol{\varphi}}$ form a permutation matrix $w^i_{\boldsymbol{\varphi}} \in S_{\tilde{\mu}^i}$ and $\tilde{M}^i_{\boldsymbol{\varphi}} \in w^i_{\boldsymbol{\varphi}} U_{w^i_{\boldsymbol{\varphi}}}$. We thus defined a map $X_{\boldsymbol{\mu}}/G_{\boldsymbol{\mu}} \to \prod_{i=1}^k \left(\coprod_{w \in S_{\tilde{\mu}^i}} wU_w\right)$. The inverse map is obtained by truncating the last μ^i_0 rows in each coordinate. The fact that the inverse map is T-equivariant is easy to see from the relation $wtut^{-1} = (wtw^{-1}) \cdot wu \cdot t^{-1}$. □

Example 3.11. For example, suppose $\mathbf{s} = (1, 1)$ and

$$A = \begin{pmatrix} * & * & 1 & 0 & 0 \\ * & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Then

$$\tilde{A} = \begin{pmatrix} * & * & 1 & 0 & 0 \\ * & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

is the completion of A to the corresponding echelon form with respect to $(3, 1, 1)$.

Proposition 3.12. *Let $\boldsymbol{\varphi} \in \mathbb{X}_{\boldsymbol{\mu}}$ and let $\mathbf{w} \in S_{\tilde{\boldsymbol{\mu}}}$, $\mathbf{u} \in U_{\mathbf{w}}$ such that the image of $\boldsymbol{\varphi}$ under (3.2.1) is $\mathbf{wu}$. Then the following assertions are equivalent.*

(i) $\boldsymbol{\varphi}$ is absolutely indecomposable,

(ii) $C_{G_{\mathbf{v}_\mu}}(\boldsymbol{\varphi}) = Z_{\mathbf{v}_\mu}$,

(iii) the graph $K_{\mathbf{w},\mathbf{u}}$ is connected.

By Proposition 3.8, the absolutely indecomposable representations of $(\Gamma, \mathbf{v}_\mu)$ over κ are all in $\mathbb{X}_\mu$.

Proof of Proposition 3.12. First assume $\boldsymbol{\varphi}$ is absolutely indecomposable. Then $C_{G_{\mathbf{v}_\mu}}(\boldsymbol{\varphi})/Z_{\mathbf{v}_\mu}$ is unipotent, see Proposition 2.4. Therefore $C_T(\mathbf{u})$ must reduce to Z_r. Indeed if $t \in C_T(\mathbf{u})$, then there exists $g \in G_\mu$ such that $(g,t) \in C_{G_{\mathbf{v}_\mu}}(\boldsymbol{\varphi})$ and so we must have $t \in Z_r$ for (g,t) to be unipotent modulo $Z_{\mathbf{v}_\mu}$. By Proposition 3.7, the graph $K_{\mathbf{w},\mathbf{u}}$ is connected.

Now assume that the graph $K_{\mathbf{w},\mathbf{u}}$ is connected. By Proposition 2.4 the representation $\boldsymbol{\varphi}$ is absolutely indecomposable if and only if the group $C_{G_\mu \times T}(\boldsymbol{\varphi})/Z_{\mathbf{v}_\mu}$ is unipotent. Taking a conjugate of $\boldsymbol{\varphi}$ if necessary we may assume that the image $(F_{\boldsymbol{\varphi}}, M_{\boldsymbol{\varphi}})$ under (3.2.2) is such that the stabilizer of

$$F^i_{\boldsymbol{\varphi}} = (E_i^{s_i} \subset \cdots \subset E_i^1 \subset E_i^0 = \kappa^{n_i})$$

in GL_{n_i} is the parabolic subgroup P_{μ^i} and $M^i_{\boldsymbol{\varphi}}$ is in its row echelon form with respect to $(\mu^i_{s_i+1}, \mu^i_{s_i}, \ldots, \mu^i_1)$. Let $(g,t) \in G_\mu \times T$ be such that

$$(g,t)\cdot \boldsymbol{\varphi} = \boldsymbol{\varphi}. \tag{3.2.3}$$

Then $g = (g^{(i;t)})_{i,t} \in G_\mu$ must satisfy $g^{(i;0)} \in P_{\mu^i}$ and $g^{(i;t)} = g^{(i;0)}|_{E_i^t}$ for all $i = 1, \ldots, k$ and $t = 1, \ldots, s_i$. Taking the image of $(g,t)\cdot\boldsymbol{\varphi} = \boldsymbol{\varphi}$ by (3.2.1) we find that $t \cdot (\mathbf{wu}) = \mathbf{wu}$. Therefore $t \in C_T(\mathbf{u})$.

Since (by assumption) $K_{\mathbf{w},\mathbf{u}}$ is connected, Proposition 3.7 implies $C_T(\mathbf{u}) = Z_r$. Thus (3.2.3) reduces to

$$(\lambda^{-1}\cdot g^{(i;0)})\cdot M^i_{\boldsymbol{\varphi}} = M^i_{\boldsymbol{\varphi}}$$

for all $i = 1, \ldots, k$, with $t = \lambda \cdot \mathrm{I}_r \in Z_r$ for some $\lambda \in \kappa$. By Proposition 3.2, we find that $g^{(i;0)} = \lambda \cdot \mathrm{I}_n$, i.e., $(g,t) \in Z_{\mathbf{v}_\mu}$. Hence $C_{G_\mu \times T}(\boldsymbol{\varphi}) = Z_{\mathbf{v}_\mu}$ and therefore $\boldsymbol{\varphi}$ is absolutely indecomposable. This completes the proof. $\square$

For $\mathbf{w} \in S_{\tilde{\mu}}$ we put

$$R_{\mathbf{w}}(q) := \sum_{K \subset K_{\mathbf{w}}} (q-1)^{b_1(K)}, \tag{3.2.4}$$

where the sum is over the connected subgraphs of $K_{\mathbf{w}}$; here $b_1(K) = e(K) - r + 1$ is the first Betti number and $e(K)$ is the number of edges of K. If the graph $K_{\mathbf{w}}$ is not connected then we put $R_{\mathbf{w}}(q) = 0$.

Denote by $\mathbb{X}_\mu^{\mathbf{w}} \subset \mathbb{X}_\mu$ the subset of representations corresponding to $\mathbf{w}U_{\mathbf{w}}$ in the bijection (3.2.1).

Theorem 3.13. *The polynomial $R_{\mathbf{w}}(q)$ counts the number of isomorphism classes of absolutely indecomposable representations in $\mathbb{X}_\mu^{\mathbf{w}}(\mathbb{F}_q)$.*

Proof. The T-equivariant bijection (3.2.1) induces an isomorphism between the isomorphism classes of $\mathbb{X}_\mu^{\mathbf{w}}$ with the T-orbits of $\mathbf{w}U_{\mathbf{w}}$. By Proposition 3.12 the isomorphism classes of absolutely indecomposable representations in $\mathbb{X}_\mu^{\mathbf{w}}$ corresponds to the T-orbits of $C = \{\mathbf{wu} \in \mathbf{w}U_{\mathbf{w}} \,|\, K_{\mathbf{w},\mathbf{u}} \text{ is connected}\}$. Now for a given subgraph K of $K_{\mathbf{w}}$, the number of elements $\mathbf{u} \in U_{\mathbf{w}}(\mathbb{F}_q)$ such that $K = K_{\mathbf{w},\mathbf{u}}$ equals $(q-1)^{e(K)}$. Moroever by Proposition 3.7, the group T/Z_r acts trivially on C and so the number of T-orbits of C over $\mathbb{F}_q$ equals $R_{\mathbf{w}}(q)$. □

We can now state the main result of our paper:

Theorem 3.14. *We have*

$$A_{\Gamma,\mathbf{v}_\mu}(q) = \sum_{\mathbf{w}\in S_{\tilde{\mu}}} R_{\mathbf{w}}(q).$$

3.3. **Tutte polynomial of graphs.** The above polynomials $R_{\mathbf{w}}(q)$ are related to classical graph polynomials. Recall (cf. [8, 18]) that the Tutte polynomial $T_K(x,y) \in \mathbb{Z}[x,y]$ for a graph K with edge set E and vertex set V can be defined by

$$T_K(x,y) = \sum_{A\subseteq E} (x-1)^{k(A)-k(E)}(y-1)^{k(A)+|A|-|V|},$$

where $k(A)$ is the number of connected components of the subgraph with edge set A. Tutte proved that for a connected graph K we also have

$$T_K(x,y) = \sum_T x^{i(T)} y^{e(T)},$$

where the sum is over all spanning trees T of K and $i(T), e(T)$ are respectively their *internal* and *external activity* (for some fixed but arbitrary ordering of the edges of K). In particular, the coefficients of $T(x,y)$ are nonnegative integers.

In this paper we will only be concerned with the specialization (for K a connected graph)

$$R_K(q) := T_K(1,q) = \sum_T q^{e(T)},$$

which we will call the *external activity polynomial* of K. Up to a variable change and renormalization, $R_K(q)$ coincides with the reliability polynomial

$$(1-p)^{|V|-k(K)} p^{|E|-|V|+k(K)} T_K(1, 1/p),$$

which computes the probability that a connected graph K remains connected when each edge is independently deleted with fixed probability p.

A result of Hausel and Sturmfels [13] implies that the Kac polynomial of a quiver with dimension vector consisting of all 1's equals the external activity polynomial of the underlying graph.

It is clear that if $K = K_{\mathbf{w}}$ is connected then

$$R_{\mathbf{w}}(q) = R_{K_{\mathbf{w}}}.$$

Hence Theorem 3.14 together with Tutte's result provide an alternative proof of the non-negativity of the coefficients of the Kac polynomials $A_{\Gamma,\mathbf{v}_\mu}(q)$ (see Theorem 2.3).

3.4. **Counting T-orbits on flag varieties.** Let $P_1, \dots, P_k$ be parabolic subgroups of GL_r containing the lower triangular matrices (this is only for convenience). Recall that T denotes the maximal torus of GL_r of diagonal matrices. To each parabolic P_i corresponds a unique partition $\tilde{\mu}^i = (\tilde{\mu}^i_1, \tilde{\mu}^i_2, \dots)$ given by the size of the blocks. Denote by $E^T_{\tilde{\mu}}(q)$ the number over $\mathbb{F}_q$ of T-orbits in $\prod_{i=1}^k \mathrm{GL}_r/P_i$ whose stabilizers equal Z_r. For $i = 1, \dots, k$, put $n_i := r - \tilde{\mu}^i_1$, and denote by μ^i the partition $(\tilde{\mu}^i_2, \tilde{\mu}^i_3, \dots)$ of n_i. From the tuple $\boldsymbol{\mu} = (\mu^1, \dots, \mu^k)$ and r we consider the associated quiver Γ equipped with dimension vector $\mathbf{v}_\mu$ as in §2.2.

In view of Remark 3.10, we deduce from Proposition 3.12 the following result, which relates Kac polynomials of complete bipartite supernova quivers to counting T-orbits:

Theorem 3.15. *We have*

$$E^T_{\tilde{\mu}}(q) = A_{\Gamma,\mathbf{v}_\mu}(q).$$

In particular, $E^T_{\tilde{\mu}}(q)$ is non zero if and only if $\mathbf{v}_\mu \in \Phi(\Gamma)$. Moreover $E^T_{\tilde{\mu}}(q) = 1$ if and only if $\mathbf{v}_\mu$ is a real root.

Remark 3.16. According to Theorem 3.15, Theorem 3.14 and §3.3 we can count certain T-orbits on homogeneous varieties over $\mathbb{F}_q$ in terms of specializations of Tutte polynomials of certain graphs. Work of Fink and Speyer [4,19] provides a geometric interpretation of the Tutte polynomial of realizable matroids and the T-equivariant K-theory of torus orbits. It would be interesting to understand the relationship between our work and theirs.

4. Examples

4.1. **Notation.** In this section we present examples to illustrate Theorems 3.14 and 3.15. We first consider the special case when k, the number of long legs of the supernova, equals 1. We call such quivers *dandelion quivers* (cf. Figure 2). In these examples the tuple of permutations $\mathbf{w}$ consists of a single element w, so we lighten notation and write K_w for $K_{\mathbf{w}}$, etc. We represent permutations $w \in S_r$ by giving the sequence of their values, using square brackets to avoid conflict with cycle notation. Thus $[3, 2, 4, 1] \in S_4$ means the permutation taking $1 \mapsto 3, 2 \mapsto 2, 3 \mapsto 4, 4 \mapsto 1$. When possible we omit brackets and commas and write e.g. 3241 for $[3, 2, 4, 1]$.

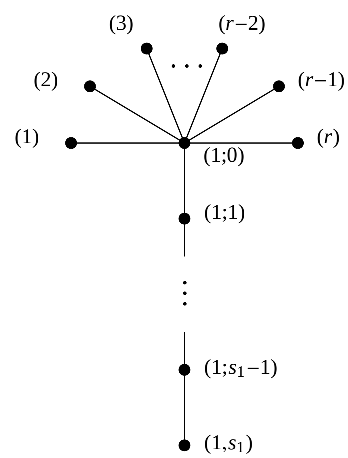


FIGURE 2. The dandelion quiver

4.2. **Projective space.** Consider the dandelion quiver with no long leg, and with central node labelled with n. In this example we consider the two cases $r = n$ and $r = n + 1$. It is not hard to see that the corresponding root is real. Indeed, apply a reflection at the central node. If $r = n$ we get all leaf nodes labelled with 1 and with the central node labelled with 0. If $r = n + 1$, the central node is labelled with 1. We can further apply reflections along the leaves to make every leaf have label 0. Thus in these cases the root is real and we should have $A = 1$.

If $r = n$, then the homogeneous variety is that of n-planes in κ^n, i.e. is a single point. There is one inversion graph, which is itself a point, and Theorem 3.14 implies that the Kac polynomial equals 1.

On the other hand, if $r = n + 1$, then our homogeneous variety is that of n-planes in κ^{n+1}, i.e. is a projective space. This time the only connected inversion graph corresponds to the permutation $w = [n + 1, 1, 2, \ldots, n]$, which indexes the open Schubert cell. The graph K_w is a tree, and again the Kac polynomial equals 1.

4.3. **A grassmannian.** Now we consider a more complicated example. Let Γ be the quiver in Figure 3, with the indicated dimension vector $\mathbf{v}_\mu$. One can check using Lemma 2.5 that this vector gives an imaginary root. The homogeneous variety is $Gr(2,5)$, the grassmannian of 2-planes in κ^5. This variety is 6-dimensional and can be paved by 10 Schubert cells $U_w = P_s wB$, where w ranges over the minimal length elements in the 10 cosets of $S_2 \times S_3$ in S_5. Hence there are 10 graphs K_w of order 5 that we need for $A_{\Gamma,\mathbf{v}_\mu}(q)$. Of these graphs, only 4 are connected. In fact, the number of edges of K_w equals the dimension of the Schubert cell U_w, and since we must have at least four edges for a graph of order 5 to be connected, only the cells of dimensions ≥ 4 need to be considered. These are labelled by the permutations 31452, 34125, 34152, and 34512.

Figures 11–5 show these four graphs. We consider each in turn:

- The graph K_{34125} is not connected, so $R_{34125} = 0$.
- The graph K_{31452} is a connected tree, which implies $R_{31452} = 1$.
- The graph K_{34152} is a 4-cycle with an extra edge. There are 4 spanning trees contributing 1 each, and the full graph contributes $q - 1$. Thus $R_{34152} = q + 3$.
- The last graph K_{34512} is a complete bipartite graph of type $(2,3)$. There are 12 spanning trees; each contributes 1 to R_{34512}. Deleting any single edge yields a graph isomorphic to K_{34152}, each of which contributes $q - 1$. Finally, the full graph itself has betti number 2 and thus contributes $(q-1)^2$. Altogether we find $R_{34512} = q^2+4q+7$.

Thus

$$A_{\Gamma,\mathbf{v}_\mu}(q) = R_{31452} + R_{34152} + R_{34512} = q^2 + 5q + 11. \tag{4.3.1}$$

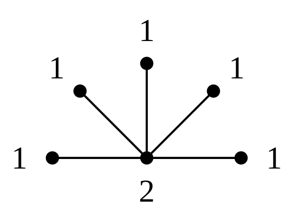
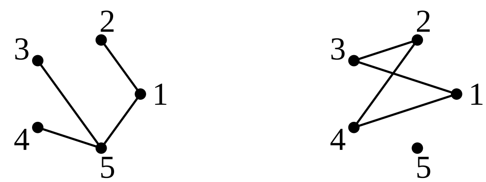


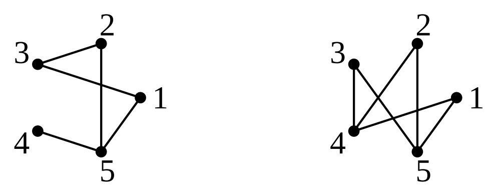
FIGURE 3.

4.4. **A two-step flag variety.** Now consider the dandelion quiver in Figure 6, with the indicated dimension vector. This is of course the same example we just treated, except that now we regard one of the short legs as being the



(a) $w = 31452$ (b) $w = 34125$

FIGURE 4.



(a) $w = 34152$ (b) $w = 34512$

FIGURE 5.

long leg. The corresponding homogeneous variety is no longer a a grassmannian; instead we have the partial flag variety of two-step flags $E^3 \subset E^2$ in κ^4. This time the inversion graphs have 4 vertices, so we need at least 3 edges in any K_w for it be connected, and there are 6 permutations with at least three inversions. The graphs are show in Figures 7–9. We leave it to the reader to check the following:

- $R_{3142} = 1$
- $R_{3214} = 0$
- $R_{3412} = q + 3$
- $R_{2341} = 1$
- $R_{3241} = q + 2$
- $R_{3421} = q^2 + 3q + 4$

Thus

$$A_{\Gamma,\mathbf{v}_\mu}(q) = q^2 + 5q + 11, \tag{4.4.1}$$

which agrees with (4.3.1).

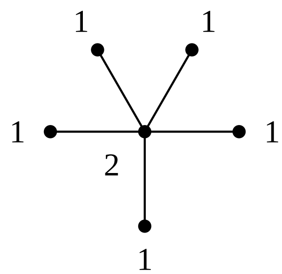
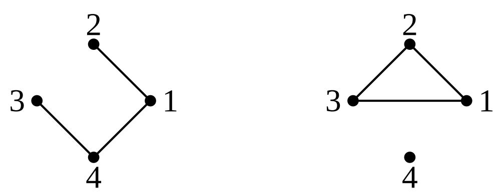
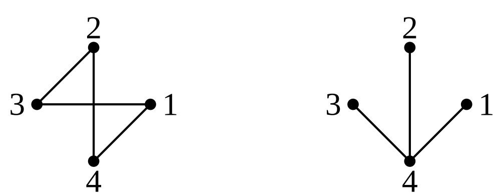


FIGURE 6.



(a) $w = 3142$ (b) $w = 3214$

FIGURE 7.



(a) $w = 3412$ (b) $w = 2341$

FIGURE 8.

4.5. **A product of projective planes.** Now we consider a more general supernova quiver. We take $r = 3$ and $(n_1, n_2) = (1, 1)$. Thus the quiver is the complete bipartite graph of type $(3, 2)$, and the dimension vector assigns 1 to each vertex. In terms of T-orbits, we are counting the orbits of dimension 2 on a product of two projective planes with a 2-dimensional torus acting diagonally.

The inversion graphs are labelled by pairs of permutations $(w_1, w_2) \in (S_3)^2$. There are five connected inversion graphs; they are characterized by

(a) $w = 3241$ (b) $w = 3421$

FIGURE 9.

having at least one w_i equal to 312, the longest permutation for this Bruhat decomposition. We show the graphs in Figures 10–12 (edges curving in correspond to the first permutation, and those curving out to the second). We find

- $R_{123,312} = R_{312,123} = 1$
- $R_{132,312} = R_{312,132} = q + 2$
- $R_{312,312} = q^2 + 2q + 1$

Altogether we obtain

$$A_{\Gamma,\mathbf{v}_\mu} = q^2 + 4q + 7. \tag{4.5.1}$$

We remark that (4.5.1) is in fact the external activity polynomial of the underlying graph of the quiver thanks to the result of Hausel and Sturmfels (see §3.3). Indeed, the Tutte polynomial of the complete bipartite graph of type $(3, 2)$ is

$$x^4 + 2x^3 + 3x^2 + x + y^2 + 4y.$$

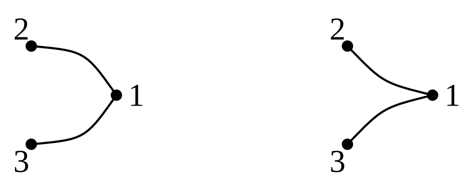
We can also recover (4.5.1) by counting 2-dimensional torus orbits in $\mathcal{F} = \mathbf{P}^2 \times \mathbf{P}^2$, following Theorem 3.15. Let $\pi\colon \mathcal{F} \to \mathbf{P}^2$ be the projection onto the first factor. The action of the torus T commutes with π.

- Choose a point p_0 in the image of π with trivial stabilizer. Any point in the inverse image of p_0 determines a unique 2-dimensional orbit, and thus this accounts for $q^2 + q + 1$ orbits.
- Now choose a point p_0 in the image of π with 1-dimensional stabilizer. We claim the inverse image of p_0 determines $q + 1$ orbits. Indeed, after we have fixed p_0, have one dimension of T left. This can move points along the lines in T-fixed point not contained in the closure of the orbit of p_0. There are $q + 1$ such lines, and hence $q + 1$ orbits. Since there are 3 choices for p_0 (corresponding to the

three 1-dimensional T orbits in $\mathbf{P}^2$ we obtain $3q + 3$ orbits altogether.

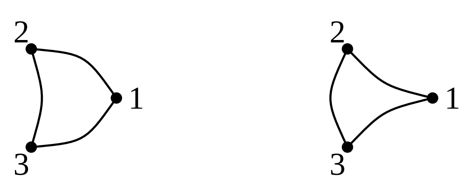
- Finally we can choose a point p_0 fixed by T. There is one 2-dimensional T-orbit in the inverse image of p_0. Since there are 3 choices of p_0 we get 3 orbits this way.

Hence altogether we find $q^2 + 4q + 7$ torus orbits of dimension 2, which coincides with (4.5.1).



(a) $(123, 312)$ (b) $(312, 123)$

FIGURE 10.



(a) $(132, 312)$ (b) $(312, 132)$

FIGURE 11.

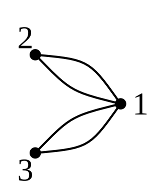


FIGURE 12. $(312, 312)$

4.6. **Counting T-orbits.** We conclude by illustrating Theorem 3.15 for the grassmannian $Gr(2,5)$ from section 4.3. The main tool we use is the *Gel′fand–MacPherson correspondence*, which we state in Theorem 4.1. We refer to [5–7, 16,] for more details.

Let $E \subset \mathbb{C}^r$ be a subspace of dimension k. Assume that E does not lie in any of the coordinate hyperplanes $H_i = \{z_i = 0\} \subset \mathbb{C}^r$. The intersections $E \cap H_i$ determine a collection of r hyperplanes in E and thus a point in $(\mathbf{P}^{k-1})^r$, i.e. a projective configuration. (Here we think of $\mathbf{P}^{k-1}$ as being $\mathbf{P}(E^*)$). If E' is a T-translate of E, then the configuration corresponding to E' is equivalent to E an element of PGL_k acting diagonally on $(\mathbf{P}^{k-1})^r$.

Hence we can study T-orbits on $G(k,r)$ in terms of certain configurations of r points in $\mathbf{P}^{k-1}$. The precise statement of this fact is the Gel′fand–MacPherson correspondence. We will only need to understand what happens when the the T-orbits have maximal dimension $r-1$.

Theorem 4.1. *Let $G_\circ(k,r) \subset G(k,r)$ be the subset of all L such that $T \cdot L$ has dimension $r-1$. Let $(\mathbf{P}^{k-1})^r_\circ$ be the subset of configurations $p = (p_1, \dots, p_r)$ such that $\mathrm{PGL}_k \cdot p$ has dimension k^2-1. Then the assigment $L \mapsto p$, where $p_i = E \cap H_i$, defines a bijection of orbit spaces*

$$\Phi\colon G_\circ(k,r)/T \longrightarrow (\mathbf{P}^{k-1})^r_\circ/\mathrm{PGL}_k.$$

Remark 4.2. The bijection Φ can be extended to all of $G(k,r)$ [5, Proposition 1.5].

In general it is very difficult to determine the configurations in the image of Φ, but there is one case that is easy: the grassmannians $G(2,r)$. When $k = 2$ the configurations are sets of points in the projective line, and the only degenerations that can occur are multiple points. To make this precise, let us say that a collection of distinct points $p_1, \dots, p_m$ is r-labelled if it is equipped with a surjective map $\{1, \dots, r\} \to \{p_1, \dots, p_m\}$. We have the following characterization of the T-orbits (cf. [16, Section 1.3]).

Proposition 4.3. *Torus orbits in $G(2,r)$ of maximal dimension are in bijection with r-labelled sets of m points in $\mathbf{P}^1$ up to PGL_2-equivalence, where $3 \le m \le r$.*

Now we consider configurations over $\mathbb{F}_q$. Let $C_m(q)$ be the number of configurations of m distinct points up to equivalence. Fix three points in $\mathbf{P}^1(\mathbb{F}_q)$ and call them 0, 1, and ∞. Given m unlabelled points in $\mathbf{P}^1$, we can use PGL_2 to carry three of them to $0, 1, \infty$. This uses up all the automorphisms, which gives the following:

$$C_m(q) = \begin{cases} (q-2)(q-3)(q-(m-2)) & \text{if } m > 3, \\ 1 & \text{if } m = 3. \end{cases}$$

To complete the count we need to incorporate the labellings. An r-labelling is determined by a sujective map $\{1, \ldots, r\} \to \{p_1, \ldots, p_m\}$, in other words an equivalence relation on $\{1, \ldots, r\}$ with m classes. These are counted by $S(r, m)$, the Stirling number of the second kind. Letting $E_r^T(q)$ denote the number of T-orbits, we have

$$E_r^T(q) = \sum_{m=3}^{r} S(r, m) C_m(q).$$

For instance, when $r = 5$, we have

$$E_5^T(q) = 1 \cdot (q-2)(q-3) + 10 \cdot (q-2) + 25 = q^2 + 5q + 11,$$

in agreement with (4.3.1).

Comparing Figures 3 and 6, one sees that over $\mathbb{F}_q$ the number of $(r-1)$-dimensional torus orbits in $Gr(2, r)$ equals the number of $(r-2)$-dimensional torus orbits in the flag variety of $\{\text{point} \subset \text{line}\}$ in $\mathbf{P}^{r-2}$ (the tori have different dimensions, of course). This suggests that there should be a bijection between the sets of torus orbits for these two homogeneous varieties. This is true, and we leave the reader the pleasure of finding it.

5. Generating Functions

We will use the series [10, (1.4)] to obtain a generating function for the Kac polynomials of the supernova quivers of §2.2. The series [10, (1.4)] in the case where the quiver is the complete (k, r) bipartite graph with $k + r$ vertices is the following

(5.0.1)

$$\mathbb{H}(\mathbf{X}, \mathbf{Y}; q) := (q-1)\mathrm{Log}\left(\sum_{\lambda^i, \mu^j} \frac{q^{\sum_{i,j}\langle\lambda^i, \mu^j\rangle} \prod_i \tilde{H}_{\lambda^i}(\mathbf{x}_i; q) \prod_j \tilde{H}_{\mu^j}(\mathbf{y}_j; q)}{\prod_i q^{\langle\lambda^i, \lambda^i\rangle} b_{\lambda^i}(q^{-1}) \prod_j q^{\langle\mu^j, \mu^j\rangle} b_{\mu^j}(q^{-1})}\right),$$

where $i = 1, \ldots, k$, $j = 1, \ldots, r$

$$b_\lambda(q) := \prod_{i \geq 1} \prod_{j=1}^{m_i(\lambda)} (1 - q^j),$$

with $m_i(\lambda)$ the multiplicity of i in λ and $\mathbf{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_r)$; $\mathbf{Y} = (\mathbf{y}_1, \ldots, \mathbf{y}_k)$.

Since we are interested in a dimension vector where the r vertices have value 1 we can restrict the $\mathbf{y}$ variables to $\mathbf{y}_i = (u_i, 0, \dots)$ for some independent variables $u_1, \dots, u_r$. Furthermore, we only need to work modulo the ideal $I := \langle u_1^2, \dots, u_r^2 \rangle$.

We have $\tilde{H}_\lambda(u, 0, \dots) = u^{|\lambda|}$. It follows that the right hand side of (5.0.1) becomes

$$(q-1)\mathrm{Log}\left(\sum_{\lambda^i}\sum_{s=0}^{r} \frac{q^{s\sum_i l(\lambda^i)} e_s(u) \prod_i \tilde{H}_{\lambda^i}(\mathbf{x}_i; q)}{(q-1)^s \prod_i q^{\langle \lambda^i, \lambda^i \rangle} b_{\lambda^i}(q^{-1})}\right) \bmod I,$$

where $e_s(u) = e_s(u_1, \dots, u_r)$ is the elementary symmetric function in the u_i's. Interchanging summations this equals

$$(q-1)\mathrm{Log}\left(\sum_{s=0}^{r}\prod_i c_s(\mathbf{x}_i)\frac{e_s(u)}{(q-1)^s}\right) \bmod I,$$

where

$$c_s(\mathbf{x}) := \sum_\lambda \frac{q^{sl(\lambda)}\tilde{H}_\lambda(\mathbf{x}; q)}{q^{\langle \lambda, \lambda \rangle} b_\lambda(q^{-1})}, \qquad \mathbf{x} = (x_1, x_2, \dots).$$

Note that

$$e_{s_1}(u)\cdots e_{s_l}(u) \equiv \frac{(s_1 + \cdots s_l)!}{s_1! \cdots s_l!} e_{s_1+\cdots+s_l}(u) \bmod I\,.$$

Therefore we may replace $e_s(u)$ by a single term $U^s/s!$ and let r be arbitrary. Except for the constant term in U the values of Log and log agree since we are working modulo I. Hence we get

$$(q-1)\mathrm{Log}\left(\prod_i c_0(\mathbf{x}_i)\right) + (q-1)\log\left(\sum_{s\geq 0}\prod_i \frac{c_s(\mathbf{x}_i)}{c_0(\mathbf{x}_i)}\frac{(U/(q-1))^s}{s!}\right)$$

Define the *Rogers-Szëgo symmetric functions* as

$$R_s(\mathbf{x}) := \sum_{|\lambda|=s}\begin{bmatrix} s \\ \lambda_1, \lambda_2, \cdots \end{bmatrix} m_\lambda(\mathbf{x}), \qquad \mathbf{x} := (x_1, x_2, \dots),$$

where m_λ is the monomial symmetric function and

$$\begin{bmatrix} s \\ \lambda_1, \lambda_2, \cdots \end{bmatrix} := \frac{[s]!}{[\lambda_1]![\lambda_2]!\cdots}, \qquad [n]! := (1-q)\cdots(1-q^n),$$

is the q-multinomial and q-factorial respectively.

Proposition 5.1. *The following identity holds*

$$\frac{c_s(\mathbf{x}; q)}{c_0(\mathbf{x}; q)} = R_s(1, x_1, x_2, \dots), \qquad \mathbf{x} := (x_1, x_2, \dots).$$

Let $\mathcal{A}_s(\mathbf{x}_1, \ldots, \mathbf{x}_k; q)$ be defined by the generating function

(5.0.2)
$$\sum_{s\geq 1} \mathcal{A}_s(\mathbf{x}_1, \ldots, \mathbf{x}_k; q)\, \frac{U^s}{s!} = (q-1)\log \sum_s R_s(\mathbf{x}_1)\cdots R_s(\mathbf{x}_k)\frac{(U/(q-1))^s}{s!}.$$

Proof. It follows from the main formula proved in [9]. □

A priori $\mathcal{A}_s(\mathbf{x}_1, \ldots, \mathbf{x}_k; q)$ are symmetric functions with coefficients in $\mathbb{Q}(q)$. In fact, the coefficients are in $\mathbb{Z}[q]$ as we now see. Combining the above discussion with [10][Prop. (1.3) (i)] we finally obtain the following.

Theorem 5.2. *With the notation of §3.2 the Kac polynomial $A_{\Gamma,\mathbf{v}_\mu}$ of the complete bipartite supernova quiver is given by*

$$A_{\Gamma,\mathbf{v}_\mu}(q) = \langle \mathcal{A}_r, h_{\tilde{\boldsymbol{\mu}}}\rangle, \tag{5.0.3}$$

where h_μ denotes the complete symmetric function, $h_{\tilde{\boldsymbol{\mu}}} := h_{\tilde{\mu}^1}\cdots h_{\tilde{\mu}^k}$ with $\tilde{\boldsymbol{\mu}} = (\tilde{\mu}^1, \ldots, \tilde{\mu}^k)$ and $\tilde{\mu}^i$ is the partition of r defined by $(r - |\mu^i|, \mu^i_1, \mu^i_2, \ldots)$.

The right hand side of (5.0.3) gives the coefficient of $m_{\tilde{\boldsymbol{\mu}}}$ when writing $\mathcal{A}_r$ in terms of the monomial symmetric functions. For example, for $k = 1$ we obtain the following

$$\begin{aligned}
\mathcal{A}_1 &= m_1\\
\mathcal{A}_2 &= m_{1^2}\\
\mathcal{A}_3 &= (q+4)m_{1^3} + m_{12}\\
\mathcal{A}_4 &= (q^3+6q^2+20q+33)m_{1^4} + (q^2+5q+11)m_{1^22} + (q+4)m_{2^2} + m_{13}
\end{aligned}$$

In particular we see the polynomial $q^2+5q+11$ corresponding to the example discussed in §4.4. The coefficient of m_{1^4} on the other hand corresponds to a dandelion quiver with four short legs and a long leg with dimension vector $(3, 2, 1)$ along its vertices corresponding to the full flag variety GL_4/B. Here is the list of permutations w of block structure $(1, 1, 1, 1)$ with connected inversion graphs and their corresponding R-polynomials.

w	R_w
4321	q^3+3q^2+6q+6
4312	q^2+3q+4
4231	q^2+3q+4
4213	$q+2$
4132	$q+2$
4123	1
3421	q^2+3q+4
3412	$q+3$
3241	$q+2$
3142	1
2431	$q+2$
2413	1
2341	1

We verify that indeed the sum of these polynomials is $q^3+6q^2+20q+33$.

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