August 26, 2014

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MIKHAIL V. BELOLIPETSKY, PAUL E. GUNNELLS, AND RICHARD SCOTT

Abstract. Let C be a one- or two-sided Kazhdan–Lusztig cell in a Coxeter group (W, S), and let $\mathtt{Red}(C)$ be the set of reduced expressions of all $w \in C$, regarded as a language over the alphabet S. Casselman has conjectured that $\mathtt{Red}(C)$ is regular. In this paper we give a conjectural description of the cells when W is the group corresponding to a hyperbolic polygon, and show that our conjectures imply Casselman's.

1. Introduction

Let W be a Coxeter group with generating set S. In their study of representations of Coxeter groups and Hecke algebras, Kazhdan and Lusztig introduced the decomposition of W into *cells* [17]. The cells are equivalence classes in W determined by the left and right descent sets of elements of W and the degrees of the Kazhdan–Lusztig polynomials $P_{x,y}$ (§2). Today cells are known to have many applications in representation theory; for some references, see the bibliography of [14].

This paper addresses the computability of the cells, in the following sense. Given a cell C, one can ask for an efficient way to encode its elements. Since elements of W are easily represented by reduced expressions in the generators S, it is natural to ask for a solution in terms of such expressions. However, since the definition of the cells involves a complicated equivalence relation, it is certainly not clear that this is possible.

Despite this, W. Casselman has conjectured that cells can be efficiently encoded. To state his conjecture, we need some terminology from the theory of formal languages; for more information see [1].

Let A be a finite alphabet of characters. By a *language* $\mathtt{L}$ over A we mean a collection of finite-length ordered words built from elements of A. A *finite state automaton* $\mathscr{A}$ with alphabet A is a finite directed graph on a vertex set $\mathscr{S}$, called states, with edges labeled by elements of $A \cup \{\varepsilon\}$. Different edges leaving a given

Date: June 23, 2014.

2000 *Mathematics Subject Classification.* Primary 20F10, 20F55.

Key words and phrases. Kazhdan–Lusztig cells, hyperbolic groups, finite state automata.

MB was partially supported by CNPq. PG was partially supported by NSF. We thank an anonymous referee for helpful comments.

vertex are assumed to have different labels. One vertex is defined to be the *initial state*; a subset of $\mathscr{S}$ is chosen and defined to be the *accepting states*. A finite state automaton encodes certain words built from A through path traversal: one starts at the initial state and follows a directed path of any length that terminates at an accepting state. As the path is traversed the vertex labels are concatenated into a word (the symbol ε represents a "null-transition;" the word is unaltered if ε is read). The collection of words that can be so constructed forms the language recognized by $\mathscr{A}$. A language is called *regular* if it can be recognized by a finite state automaton.

Regular languages are the simplest infinite languages one encounters in the hierarchy of formal languages. Many languages in algebra are regular. For instance, via an earlier paper of Davis–Shapiro [10], work of Brink–Howlett implies that the language $\texttt{Red}(W)$ of all reduced expressions in the generators S is regular [8]. Any cell C induces a sub language $\texttt{Red}(C) \subset \texttt{Red}(W)$, namely all the reduced expressions of elements in C. We can now state Casselman's conjecture:

1.1. **Conjecture.** *For any Coxeter group W and any (two- or one-sided) cell $C \subset W$, the language $\texttt{Red}(C)$ is regular.*

Casselman's conjecture is known to be true for affine Weyl groups from earlier work of one of us (PG) [15]. In this paper we investigate the case that (W, S) is a Coxeter group corresponding to a hyperbolic polygon. In other words, W can be realized as the discrete subgroup of isometries of the hyperbolic plane $\mathfrak{H}$ generated by the reflections through the side of a geodesic polygon. The cells of such groups have been considered earlier by Bédard [2, 3] and one of us (MB) [4]. We state conjectures due to two of us (MB and PG) that describes the Kazhdan–Lusztig cells of W in terms of reduced expressions. Then we prove (assuming the conjectures) that for any left, right, or 2-sided Kazhdan–Lusztig cell C, the language $\texttt{Red}(C)$ is regular. Moreover, when combined with previous work of two of us (MB and PG), the results in this paper prove the regularity of cells for certain Coxeter groups (cf. Remark 5.5). We note that the proofs in this paper use word-hyperbolicity of W in an essential way, and in particular do not apply to affine Weyl groups.

We now give an overview of the paper. In §2 we give background on Coxeter groups and recall the definition of Kazhdan–Lusztig cells. Section 3 states conjectures for cells in Coxeter groups attached to tessellations of the hyperbolic plane by polygons. In §4 we give background on word hyperbolic groups and state the results we need from geometric group theory. Finally §5 gives our main results.

2. Definitions and basic examples

In this section we recall the basics of Coxeter groups and define Kazhdan–Lusztig cells. For more details we refer to [6, 16, 17].

A *Coxeter group* W is a group generated by a finite subset $S \subset W$ where the defining relations have the form $(st)^{m(s,t)} = 1$ for pairs of generators $s, t \in S$. The exponents $m(s,t)$ are taken from $\mathbb{N} \cup \{\infty\}$, and we require $m(s,s) = 1$, so that each generator s is an involution. Let $I \subset S$ be a subset of the generators. The subgroup of W generated by I is called a parabolic subgroup and is denoted W_I.

Any representation of $w \in W$ as a product of generators is called an *expression*. An expression is called *reduced* if it cannot be made shorter by applying the defining relations of W. The length of a shortest expression for w is denoted $l(w)$. For any $w \in W$, we define the *left descent set* $\mathscr{L}(w) \subset S$ to consist of those $s \in S$ such that $l(sw) < l(w)$. We similarly define the *right descent set* $\mathscr{R}(w)$ to be those s such that $l(ws) < l(w)$. For $w, u, v \in W$, we write $w = u.v$ if $w = uv$ and $l(w) = l(u) + l(v)$.

Given an expression $s_1 \cdots s_N$, a *subexpression* is a (possibly empty) expression of the form $s_{i_1} \cdots s_{i_M}$, where $1 \leq i_1 < \cdots < i_M \leq N$. The *Chevalley–Bruhat order* is the partial order on W defined by putting $v \leq w$ if an expression for v appears as a subexpression of a reduced expression for w. Given any $v, w \in W$, let $[v, w]$ be the interval between v and w, that is $[v, w] = \{x \in W \mid v \leq x \leq w\}$.

The Kazhdan–Lusztig polynomials are most easily defined in terms of an auxiliary family of polynomials, the *R-polynomials*. This family $\{R_{v,w}(q) \in \mathbb{Z}[q] \mid v, w \in W\}$ is defined to be the unique collection of polynomials satisfying the following properties (cf. [6, Theorem 5.1.1]): (i) $R_{v,w}(q) = 0$ if $v \not\leq w$; (ii) $R_{v,w}(q) = 1$ if $v = w$; and (iii) if $s \in \mathscr{R}(w)$, then $R_{v,w}(q) = R_{vs,ws}(q)$ if $s \in \mathscr{R}(v)$, and is $qR_{vs,ws}(q) + (q-1)R_{v,ws}(q)$ otherwise. Given the R-polynomials, the *Kazhdan–Lusztig polynomials* $P_{v,w}(q)$ can be described as the unique family of polynomials satisfying (cf. [6, Theorem 5.1.4]) (i) $P_{v,w}(q) = 0$ if $v \not\leq w$; (ii) $P_{v,w}(q) = 1$ if $v = w$; (iii) $\deg P_{v,w}(q) \leq (l(w) - l(v) - 1)/2$ if $v < w$; and (iv) $q^{l(w)-l(v)} P_{v,w}(q^{-1}) = \sum_{x \in [v,w]} R_{v,x}(q) P_{x,w}(q)$ if $v \leq w$. If $v < w$, we write $\mu(v, w)$ for the coefficient of $q^{(l(w)-l(v)-1)/2}$ in $P_{v,w}(q)$. We write v—w and w—v if $\mu(v, w) \neq 0$.

We are finally ready to define cells. The *left W-graph* $\Gamma_{\mathscr{L}}$ of W is the directed graph with vertex set W, and with an arrow from v to w if and only if v—w and $\mathscr{L}(v) \not\subset \mathscr{L}(w)$. The left cells are extracted from $\Gamma_{\mathscr{L}}$ as follows. Given any directed graph, we say two vertices are in the same *strong connected component* if there exist directed paths from each vertex to the other. Then the *left cells* of W are exactly the strong connected components of the graph $\Gamma_{\mathscr{L}}$. The *right cells* are defined using the analogously constructed *right W-graph* $\Gamma_{\mathscr{R}}$. We say v, w are in the same *two-sided cell* if we can find a sequence $v = w_1, w_2, \ldots, w_k = w$ such that w_i, w_{i+1} lie in either the same left or right cell.

We need one final ingredient to state our conjecture in the next section: the a-function.

Let $\mathscr{H}$ denote the Hecke algebra of W over the ring $\mathscr{A} = \mathbb{Z}[q^{1/2}, q^{-1/2}]$ of Laurent polynomials in $q^{1/2}$. This algebra is a free $\mathscr{A}$-module with a basis $\mathscr{T} = \{T_w \mid w \in W\}$ and with multiplication determined by $T_w T_{w'} = T_{ww'}$ if $l(ww') = l(w) + l(w')$, and $T_s^2 = q + (q-1)T_s$ for $s \in S$. Together with the basis $\mathscr{T}$, we can define in $\mathscr{H}$ another basis $\mathscr{C} = \{C_w \mid w \in W\}$. The element $C_w \in \mathscr{C}$ can be expressed in terms of $\mathscr{T}$ and the Kazhdan–Lusztig polynomials by

$$C_w = \sum_{y \le w} (-1)^{l(w)-l(y)} q^{l(w)/2-l(y)} P_{y,w}(q^{-1}) T_y.$$

Now consider the multiplication of the $\mathscr{C}$-basis elements in $\mathscr{H}$. We can write

$$C_x C_y = \sum_z h_{x,y,z} C_z, \ \ h_{x,y,z} \in \mathscr{A}.$$

Let $a(z)$ be the smallest integer such that $q^{a(z)/2} h_{x,y,z} \in \mathscr{A}^+$ for all $x, y \in W$, where $\mathscr{A}^+ = \mathbb{Z}[q^{1/2}]$. It is a standard conjecture that $\{a(w) \mid w \in W\} \subset \mathbb{Z}$ is bounded for any Coxeter group. The a-function was introduced by Lusztig in [19], where he proved this conjecture for affine Weyl groups. In [4] it was shown that the a-function is bounded for right-angled Coxeter groups. N. Xi recently showed that the a-function is bounded for Coxeter groups with complete Coxeter graphs (i.e. no two generators commute) [23]; this paper has further ramifications for our current article, see Theorem 3.4. P. Zhou has recently proved that the a-function is bounded if W has rank 3 [24].

3. Conjectures about cells of hyperbolic polygon groups

In this paper we take W to be a hyperbolic polygon group. This means the following. Let $\mathfrak{H}$ be the hyperbolic plane, and let $\Delta \subset \mathfrak{H}$ be an n-sided geodesic polygon with angles $\alpha_i = \pi/a_i$, $i = 1, \dots, n$. (We omit the conditions the denominators a_i satisfy to make Δ hyperbolic; we also allow the angles to vanish, in which case the polygon has ideal vertices.) Label the sides of Δ by $\sigma_1, \dots, \sigma_n$, such that the angle α_i sits between the sides σ_i, σ_{i+1}, and where the subscripts are taken mod n as necessary. Then the generating set S of W has n elements $s_1, \dots s_n$, corresponding to the sides σ_i. We put $m(s_i, s_j) = \infty$ unless σ_i and σ_j meet at the angle $\alpha_k \neq 0$. In the latter case we put $m(s_i, s_j) = a_k$.

It is not hard to see that W is isomorphic to the discrete subgroup of isometries of $\mathfrak{H}$ generated by reflections in the lines through the σ_i. Thus there is an action of W on $\mathfrak{H}$ by reflections, the polygon Δ is a fundamental domain, and the translates $\{w \cdot \Delta \mid w \in W\}$ form a tessellation of $\mathfrak{H}$ (note our convention that the reflection action of W on $\mathfrak{H}$ is a left action). The correspondence $w \mapsto w \cdot \Delta$ is a bijection between W and the tiles in the tessellation. Using this we identify W with the set of all tiles.

We can also use this identification to define certain subsets of W. Recall that $\mathscr{L}(w)$ denotes the set of left descents of an element w. Given any subset $T \subset S$, we let W^T be the (possibly empty) set of all $w \in W$ such that $\mathscr{L}(w) = T$. The tessellation allows us to identify the sets W^T as follows. First, $W^{\varnothing}$ consists of Δ itself. Next, any edge of Δ corresponds to a generator $s \in S$. Extending this edge to a line divides the plane $\mathfrak{H}$ into two half-spaces, one containing Δ and one not. The half-space H_s not containing Δ contains all elements w such that $s \in \mathscr{L}(w)$. Any (non-ideal) vertex of Δ corresponds to an order 2 subset T with $W^T \neq \varnothing$. Namely we have $W^T = H_s \cap H_{s'}$, where $T = \{s, s'\}$ and s, s' label the edges of Δ meeting this vertex. Finally, if T, T' have order 2 and $T \cap T' = \{s\}$ has order 1, then $W^{\{s\}}$ consists of $H_s \smallsetminus W^T \cap W^{T'}$. These give all subsets T such that $W^T \neq \varnothing$.

We call a subgroup $D \subset W$ *finite dihedral* if D is the parabolic subgroup for an order 2 subset T with W^T nonempty. Let $\mathscr{F}$ be the set of finite dihedral subgroups and let $\mathscr{T}$ be the set of order 2 subsets indexing them. Assume that the distinct nonzero exponents are $e_1 < e_2 < \cdots < e_m$, where $m \leq n$. This means there are m isomorphism classes of dihedral subgroups of W. We write $\mathscr{F} = \mathscr{F}_1 \cup \cdots \cup \mathscr{F}_m$, where $\mathscr{F}_i$ is the set of finite dihedral subgroups of exponent e_i. We also let $\mathscr{T} = \mathscr{T}_1 \cup \cdots \cup \mathscr{T}_m$ be the corresponding partition of $\mathscr{T}$. For any $D \in \mathscr{F}$, let $w_D \in D$ be the longest element. For $i = 1, \ldots, m$ let $W_i = \{w_D \mid D \in \mathscr{F}_i\}$ be the sets of longest elements. We also write w_T for w_D if $D = \langle T \rangle$.

We are now ready to give a conjectural description of the two-sided cells of W. We define a sequence of subsets $C_m, \ldots, C_1$ of W as follows. First, C_m is defined by

$$w \in C_m \text{ if and only if } w = u.w_D.v \text{ for some } w_D \in W_m.$$

In other words, w is in C_m if and only if there is some reduced expression of w that contains a reduced expression for w_D as a subword, where D is finite dihedral subgroup of maximal exponent e_m. Next, for $i < m$ we define C_i by

$$\begin{aligned} w \in C_i \text{ if and only if } w = u.w_D.v \text{ for some } w_D \in W_i, \\ \text{and } w \neq x.w_{D'}.y \text{ for any } w_{D'} \in W_k \text{ with } k > i. \end{aligned}$$

Thus $w \in C_i$ if it has a reduced expression containing a subword of w_D with D finite dihedral of exponent e_i, and has *no* reduced expression containing any $w_{D'}$ as a subword with D' finite dihedral of exponent $> e_i$. We also define subsets $C_{\mathrm{id}} = \{\mathrm{id}\}$ and

$$C_0 = \{w \mid w \text{ has a unique reduced expression}\}.$$

Let $\mathscr{C}$ be the collection $\{C_{\mathrm{id}}, C_0, \ldots, C_m\}$.

3.1. **Conjecture.**
(1) *The decomposition $\mathscr{C}$ gives the partition of W into two-sided cells.*
(2) *On the two-sided cell C_i, the a-function equals the length of any element of W_i.*

3.2. **Example.** To illustrate Conjecture 3.1, we consider the hyperbolic triangle group

$$W = W_{237} = \langle r, s, t \mid r^2 = s^2 = t^2 = (rs)^3 = (rt)^2 = (st)^7 = 1 \rangle.$$

There are three finite dihedral subgroups, of orders $4, 6, 14$, corresponding to the exponents $2, 3, 7$. The longest words in these subgroups have reduced expressions rt, rsr, and $ststst$. One knows from the theory of Coxeter groups that one can pass between any two reduced expressions of a given element by applying the substitutions $rt = tr$, $rsr = srs$, and $ststst = tststs$.

Figure 1 shows the partition $\mathscr{C}$ for W. The simplest subsets are the white and yellow triangles. The white triangle corresponds to the identity in W and is the set C_{id}. The yellow triangles correspond to those elements of W with unique reduced expressions, such as r, s, t, rs, and srt. There are 27 such elements in W, and together they form the set C_0.

The blue triangles are the set C_1. They consist of elements of W with a reduced expression containing rt as a subword, but with no reduced expressions containing either rsr or $ststst$ as subwords. One way to think about what this means is the following. Elements of C_1 do not have unique reduced expressions, but they almost do: when moving between reduced expressions for these elements, one only applies relations of the form $rt = tr$, and never uses the other two relations $rsr = srs$ and $ststst = tststs$. Thus the blue triangles correspond to those elements of W with the simplest possible nonunique reduced expressions. Elements in a Coxeter group such that any two reduced expressions are connected by those moves that exchange two adjacent commuting generators are called *fully commutative* in the literature [22]; their special relationship with Kazhdan–Lusztig cells was investigated by Green–Losonczy [12] and Shi [21].

Next we have the green triangles, which form the set C_2. These are the elements with the next most complicated reduced expressions: when rewriting a reduced expression for any of these elements, one uses the relations $rt = tr$ and $rsr = srs$, but never the relation $ststst = tststs$. Equivalently, no element in the green set has a reduced expression containing the subword $ststst$, and every element has at least one reduced expression containing rsr as a subword.

Finally we come to the set C_3, which is made of the red triangles. Any element in the red set has at least one reduced expression with $ststst$ as a subword.

The computation of the two-sided cells of W has unfortunately not been carried out. Nevertheless, the partition $\mathscr{C}$ experimentally agrees with the two-sided cells in the following sense. One can naively compute an approximation to the two-sided cells by computing many Kazhdan–Lusztig polynomials and descent sets, and thus computing approximations to the W-graphs $\Gamma_{\mathscr{L}}$ and $\Gamma_{\mathscr{R}}$; the only limitations to improving these approximations are computer time and memory. Doing this one finds that Figure 1 agrees with the partition into two-sided cells in a bounded region around C_{id}. Actually one can do significantly better than the naive computation: techniques of [5] lead to predictions of nonzero μ-values for pairs v, w with $|l(w)-l(v)|$ arbitrarily large. That these are in fact nonzero can be checked in any given example by computer, again with the only limitation being computing resources. This then leads to much better approximations to the two-sided cells that continue to agree with the partition $\mathscr{C}$.[1]

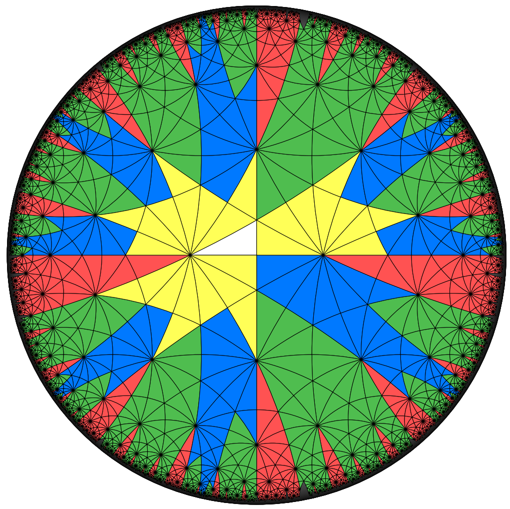

FIGURE 1. The alcoves for the triangle group $W = W_{237}$ and the partition $\mathscr{C}$. The sets $C_{\mathrm{id}}, C_0, C_1, C_2, C_3$ are (respectively) the white triangle, the yellow triangles, the blue triangles, the green triangles, and the red triangles.

3.3. **Example.** For an examples where some exponents are equal, consider the Coxeter group $W' = W_{2224}$ generated by reflections in the sides of the hyperbolic quadrilateral with angles $\pi/2, \pi/2, \pi/2, \pi/4$. This group has the presentation

$$W' = W_{2224} = \langle a, b, c, d \mid a^2 = b^2 = c^2 = d^2 = (ab)^2 = (bc)^2 = (cd)^2 = (ad)^4 = 1\rangle.$$

There are four finite dihedral subgroups of orders 4, 4, 4, and 8. Since there are two distinct exponents, Conjecture 3.1 predicts that there are four two-sided cells. These are the four sets

$$C_{\mathrm{id}}, C_0, C_1, C_2$$

shown in Figure 2; the colors are (respectively) white, yellow, blue, and green.

Returning now to the general discussion, we remark that both C_{id} and C_0 are known to be two-sided cells, the former for trivial reasons and the latter from work of Lusztig [18, §§3.7–3.8]. It also follows from computations of Lusztig [19] that the two-sided cells of the planar affine Weyl groups $\widetilde{A}_2$, $\widetilde{B}_2$, $\widetilde{G}_2$ can be described by

[1]We remark that if one could prove that all our claimed μ-coefficients from [5] were nonzero, one could then prove that $\mathscr{C}$ matches the decomposition into two-sided cells.

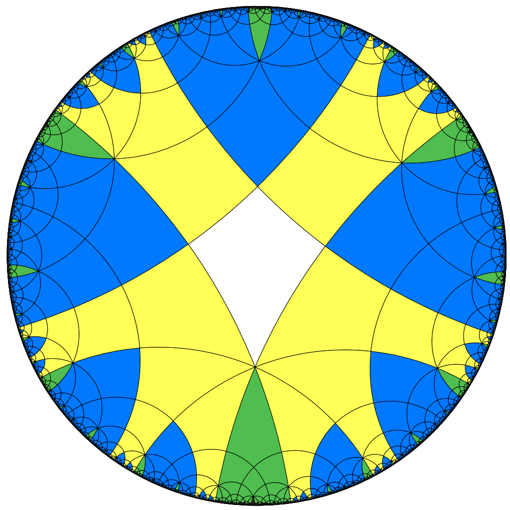

FIGURE 2. The partition $\mathscr{C}$ determined by Conjecture 3.1 for the Coxeter group W_{2224}.

Conjecture 3.1, even though they are of course not hyperbolic. We also have the following theorem of Xi, which gives confirmation of Conjecture 3.1 for certain (not necessarily hyperbolic) W:

3.4. **Theorem.** [23] *Suppose W is crystallographic and that no exponent of W is 2. Then $\mathscr{C}$ gives the partition of W into two-sided cells.*

Next we turn to the one-sided cells. Given any $T \in \mathscr{T}_i$, define

$$U^T = W^T \smallsetminus \bigcup_{j>i} C_j \cap W^T. \tag{1}$$

In particular if $i = m$, we have $U^T = W^T$. Let $\Omega_T = \{w^{-1}w_T \mid w \in U^T\}$ and let $\Omega_i = \bigcup_{T\in\mathscr{T}_i} \Omega_T$. The one-sided cells will be built from the sets $w \cdot U^T$, where $w \in \Omega_T$ and T ranges over $\mathscr{T}$. We put a partial order on Ω_i by $w \preceq w'$ if $l(w) \leq l(w')$ and there exists $T, T' \in \mathscr{T}_i$ such that $(w \cdot U^T) \subseteq (w' \cdot U^{T'})$. We define Ω_i° to be the minimal elements in Ω_i with respect to this partial order, and we write Ω_T° for the minimal elements of Ω_i appearing in Ω_T.

3.5. **Conjecture.** *The subsets $\{w \cdot U^T \mid w \in \Omega_T^\circ, T \in \mathscr{T}_i\}$ are the right cells in C_i.*

3.6. **Example.** Figures 3–4 illustrate Conjecture 3.5 for $W = W_{237}$. First we consider the two-sided cell C_3, which consists of all the red regions in Figure 1. It is clear from Figure 1 that C_3 is a disjoint union of geodesically convex regions in the hyperbolic plane. Further, a little experimentation suggests that each connected component in

C_3 can be taken to any other by a sequence of reflections. This is the main motivation behind Conjecture 3.5, which uses finitely many one-sided cells to generate all others.

Let $T \subset S$ be $\{s, t\}$. The region U^T from (1) is the connected component of C_3 in Figure 1 that shares a vertex with the identity triangle C_{id}. This is our initial one-sided cell; we build the others by taking the reflected images $w^{-1}w_T \cdot U^T$, where w ranges over U^T; this set of prefixes $\{w^{-1}w_T \mid w \in U^T\}$ is the set we denote Ω_T. Since there is only one finite dihedral subgroup of exponent 7, we have $\Omega_3 = \Omega_T$.

What happens is in depicted in Figures 3–4. The red region in Figure 3, together with the purple triangles inside it, is the subset U^T. Let Δ be the purple triangle that shares a vertex with C_{id}, i.e. the purple triangle at the tip of the red region U^T. Then all the purple triangles correspond to $w \cdot \Delta$ as w ranges over Ω_3.

Figure 4 shows the regions $\{w \cdot U^T \mid w \in \Omega_3\}$ (colored randomly). What has happened is that the original red region has been reflected so that its tip has been taken to one of the purple triangles in Figure 3. Thus some images meet others; indeed, if this happens then one translate of U^T is entirely contained in another. The partial order that determines Ω_3° selects the reflections corresponding to the purple triangles in Figure 3 lying at the tips of the orange regions. Flipping the original red region by the reflections in Ω_3° recovers all the red regions in Figure 1.

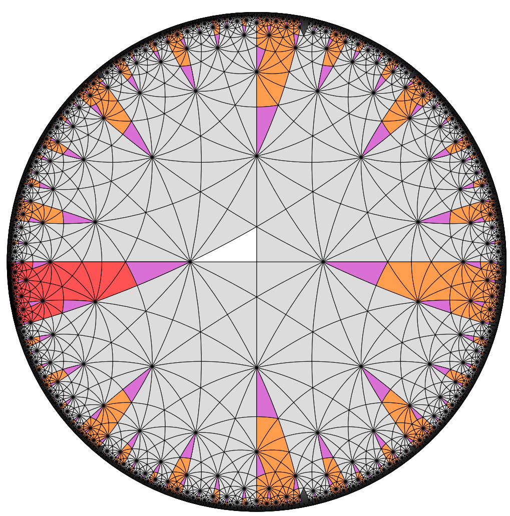

FIGURE 3. The decomposition of the two-sided cell C_3 in W_{237} into right cells. We have $T = \{s, t\}$. The red region, including the purple trianges inside it, is U^T. The purple triangles are those of the form $w \cdot \Delta$, where $w \in \Omega_3$ and Δ is the purple triangle at the tip of the red region.

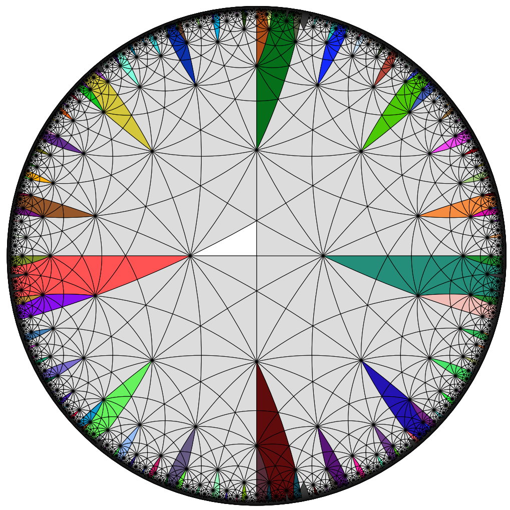

FIGURE 4. The images $w \cdot U^T$, as w ranges over Ω_3.

3.7. **Example.** Now we consider the one-sided cells in $W' = W_{2224}$ that form the blue two-sided cell C_1 in Figure 2. This time there are three relevant dihedral subgroups, since there are three of exponent 2: the subsets of the generators are $T = \{a, b\}$, $T' = \{b, c\}$, and $T'' = \{c, d\}$. We focus on T and T'', since T' can be treated by symmetry. The region U^T (respectively, $U^{T''}$) is the blue region in Figure 2 immediately to the right (respectively, above) the identity cell C_{id}. Figures 5–6 are the analogues of Figure 4. We see the original regions (colored the same blue as in Figure 2) and all their reflected images (colored randomly). Comparing with Figure 2, one observes that if one image meets another, then one is entirely contained in the other, just as in Figure 4.

Conjectures 3.5 and 3.1 should be considered as a special case of conjectures from [5] applied to hyperbolic polygon groups.

4. WORD HYPERBOLIC GROUPS AND AUTOMATA

In this section we prove that certain languages in word hyperbolic groups are regular. We will then apply these results to the languages $\mathtt{Red}(C)$ where C is a Kazhdan–Lusztig cell in a hyperbolic polygon group. First we define word hyperbolic groups and recall some of the standard facts we shall need. Details and additional properties can be found, for example, in [7, 13].

Let (X, d) be a geodesic metric space. A *geodesic triangle* consists of 3 points in X together with geodesics joining each pair of points. A geodesic triangle is called

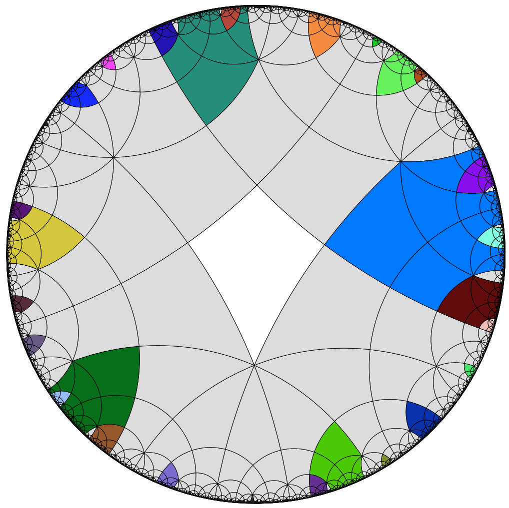

FIGURE 5. The images $w \cdot U^T$ where $T = \{a, b\}$ and $w \in \Omega_T$.

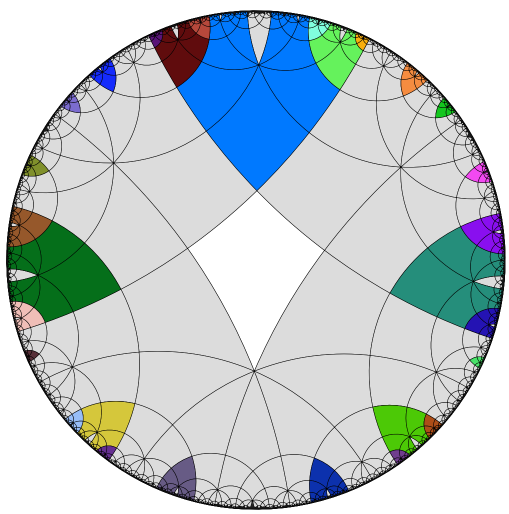

FIGURE 6. The images $w \cdot U^{T''}$ where $T'' = \{c, d\}$ and $w \in \Omega_{T''}$.

δ-thin ($\delta \in \mathbb{R}_{>0}$) if every side is in a δ-neighborhood of the other two sides. The metric space X is called *δ-hyperbolic* if every geodesic triangle is δ-thin.

Given a group W and a generating set S, we let $\mathrm{Cay}(W, S)$ denote the corresponding Cayley graph, which we regard as a geodesic metric space by identifying each edge with a unit length interval. Note that the metric restricts to the *word metric*

$d_S : W \times W \to \mathbb{Z}_{\geq 0}$ on the vertices of the Cayley graph; that is, for any $u, v \in W$ the distance $d_S(u, v)$ is the minimal length of a geodesic from u to v in $\mathrm{Cay}(W, S)$. We define the *length* of an element $w \in W$ by $l(w) = d_S(1, w)$.

4.1. **Definition.** A finitely generated group W is *word hyperbolic* if for some (equivalently, any) finite generating set S, there exists a δ such that $\mathrm{Cay}(W, S)$ is δ-hyperbolic.

It is known that a word hyperbolic group cannot contain a subgroup isomorphic to $\mathbb{Z} \times \mathbb{Z}$. For Coxeter groups, this condition is also sufficient.

4.2. **Proposition.** [9, Corollary 12.6.3] *A Coxeter group W is word hyperbolic if and only if it contains no subgroup isomorphic to $\mathbb{Z} \times \mathbb{Z}$.*

In particular, if a Coxeter group W is a lattice in the isometry group of $\mathfrak{H}$ (for example, a hyperbolic polygon group), then W is word hyperbolic.

The key property of hyperbolic groups that we shall need is the *fellow-traveler property.* For a group W with generating set S, we let S^* denote the language of all words over the alphabet S. Any word $\alpha \in S^*$ determines a path in $\mathrm{Cay}(W, S)$ that starts at the identity vertex $1 \in W$. We let $|\alpha|$ denote the length of this path and $\overline{\alpha} \in W$ denote the terminal vertex. Keeping the terminology for Coxeter groups, we say that a word $\alpha \in S^*$ is an *expression for $w \in W$* if $\overline{\alpha} = w$. (Note that $S = S^{-1}$, so that every element of W is represented by some $\alpha \in S^*$, i.e. so that the map $\alpha \mapsto \overline{\alpha}$ from S^* to W is surjective.) An expression α for w is a *reduced expression for w* if the corresponding path in $\mathrm{Cay}(W, S)$ is a minimal length geodesic between 1 and w. In other words, α satisfies $|\alpha| = d_S(1, w) = l(w)$ and $\overline{\alpha} = w$; in particular, there is no conflict between the use of the notation $l(w)$ to mean distance from 1 to w in $\mathrm{Cay}(W, S)$ and to mean the length of a reduced expression for w.

A subset $\mathtt{L} \subseteq S^*$ is called a *normal form* for W if the map $\alpha \mapsto \overline{\alpha}$ from $\mathtt{L}$ to W is surjective. The normal form we are interested in most is the geodesic normal form, denoted by $\mathtt{Red}(W)$, consisting of all reduced expressions for all elements in W. More generally, for any subset $X \subseteq W$, we define $\mathtt{Red}(X)$ to be the set of all reduced expressions for elements of X.

Two words $\alpha, \beta \in S^*$ with $|\alpha| \leq |\beta|$, can be written uniquely as $\alpha = s_1 s_2 \cdots s_n$ and $\beta = t_1 t_2 \cdots t_{n+p}$ where $s_i, t_j \in S$. We say that α and β are *(synchronous) k-fellow-travelers* if $d_S(\overline{s_1 \cdots s_i}, \overline{t_1 \cdots t_i}) \leq k$ for all $i = 1, \ldots, n$ and $d_S(\overline{s_1 \cdots s_n}, \overline{t_1 \cdots t_{n+i}}) \leq k$ for $i = 1, \ldots, p$. In other words, the corresponding paths for α and β in the Cayley graph are never more than k-apart.

4.3. **Definition.** Given a group W with generating set S, a normal form $\mathtt{L} \subseteq S^*$ is said to have the *fellow-traveler property* (respectively, *two-sided fellow-traveler property*) if there exists a $k > 0$ such that for any $\alpha, \beta \in \mathtt{L}$ with $\overline{\alpha} = \overline{\beta t}$ for some $t \in S \cup \{1\}$ (respectively, $\overline{\alpha} = \overline{s \alpha t}$ for some $s, t \in S \cup \{1\}$), the words α and β are k-fellow-travelers.

4.4. *Remark.* The (two-sided) fellow-traveler property for a normal form $\mathtt{L}$ is known to be equivalent to W having an *automatic structure* (resp., *biautomatic structure*) with respect to $\mathtt{L}$ in the sense of [11]. In particular, such a normal form must be recognized by a finite-state automaton, hence is a regular language. Obviously, biautomatic implies automatic.

The key fact we shall need is that word hyperbolic groups are biautomatic *with respect to the geodesic normal form.*

4.5. **Proposition.** *If W is word hyperbolic, and S is any finite generating set, then* $\mathtt{Red}(W)$ *has the two-sided fellow-traveler property.*

Proof. The two-sided fellow-traveler property is equivalent to both the normal form and its inverse language having the (one-sided) fellow-traveler property [11, Definition 2.5.4 and Lemma 2.5.5]. Since the geodesic language is closed under taking inverses, it is enough to show that $\mathtt{Red}(W)$ satisfies the fellow-traveler property, and this is well-known [11, Theorem 3.4.5]. $\square$

5. Hyperbolic polygon cells and regular languages

In this final section we prove our main results, Theorems 5.1 and 5.3. The first uses only the one-sided fellow traveler property, but the second requires the stronger two-sided property. We then combine them with Conjectures 3.1 and 3.5 to deduce the regularity of certain Kazhdan–Lusztig cells.

Given a group W and finite generating set S, let μ be any reduced word in S^*. As above, we write $w = u.v$ in W if $w = uv$ and $l(w) = l(u) + l(v)$. We then define the subset X_μ of W by

$$X_\mu = \{w \in W \mid w = u.\overline{\mu}.v \text{ for some } u, v \in W\}.$$

In other words, X_μ consists of all elements of W that have *some* reduced expression containing μ as a (consecutive) subword. The language $\mathtt{Red}(X_\mu)$ therefore consists of all reduced expressions that are equivalent to a reduced expression containing μ as a subword.

5.1. **Theorem.** *Let W be a word hyperbolic group, let S be any finite generating set S satisfying $S = S^{-1}$, and let μ be any word in* $\mathtt{Red}(W)$*. Then* $\mathtt{Red}(X_\mu)$ *is a regular language.*

Proof. Since W is word hyperbolic, $\mathtt{Red}(W)$ is a regular language. The sublanguage $\mathtt{Red}_\mu(W)$ consisting of all reduced words that contain μ as a subword (i.e., that match the regular expression $.{*}\mu.{*}$) is also a regular language.

Now let A be a finite state automaton accepting $\mathtt{Red}(W)$, and let k be a positive integer such that $\mathtt{Red}(W)$ has the k-fellow-traveler property. Let N_k be the set of all reduced expressions in S^* with length $\leq k$. Then the standard automaton M_ϵ based on (A, N_k) (see [11, Definition 2.3.3]) accepts the language

$$\mathtt{L} = \{(\alpha, \beta) \in \mathtt{Red}(W)^2 \mid \overline{\alpha} = \overline{\beta} \text{ and } \alpha \text{ and } \beta \text{ are } k\text{-fellow travelers}\},$$

which is therefore regular. But since $\mathtt{Red}(W)$ satisfies the k-fellow-traveler property, this language consists precisely of pairs of reduced expressions having the same image in W, i.e.,

$$\mathtt{L} = \{(\alpha, \beta) \in \mathtt{Red}_S(W) \times \mathtt{Red}_S(W) \mid \overline{\alpha} = \overline{\beta}\}.$$

The language $\mathtt{Red}(X_\mu)$ is obtained by intersecting $\mathtt{L}$ with the language $\mathtt{Red}(W) \times \mathtt{Red}_\mu(W)$ and then projecting onto the first factor. By standard predicate calculus for regular languages (see, e.g., [11, Theorem 1.4.6]), the language $\mathtt{Red}(X_\mu)$ is therefore regular. □

5.2. **Corollary.** *Let W be a hyperbolic polygon group, and let C be a conjectural two-sided cell in the decomposition $\mathscr{C}$ of Conjecture 3.1. Then the language $\mathtt{Red}(C)$ is regular.*

Proof. For each finite dihedral subgroup D, let μ_D be any reduced expression for the longest element w_D. Then

$$\mathtt{Red}(C_m) = \bigcup_{\overline{\mu}_D \in W_m} \mathtt{Red}(X_{\mu_D}),$$

and for $1 \leq i < m$,

$$\mathtt{Red}(C_i) = \bigcup_{\overline{\mu}_D \in W_i} \mathtt{Red}(X_{\mu_D}) \setminus \bigcup_{i < j \leq m} \mathtt{Red}(C_j).$$

Since these are all obtained using finite unions, complements, and intersections of regular languages, they are regular. For $\mathtt{Red}(C_{\mathrm{id}})$ and $\mathtt{Red}(C_0)$, we note that the former is finite, and the latter is the complement of $\mathtt{Red}(C_{\mathrm{id}}) \cup \mathtt{Red}(C_1) \cup \cdots \cup \mathtt{Red}(C_m)$ in $\mathtt{Red}(W)$. It follows that both are regular as well. □

5.3. **Theorem.** *Let W be a word hyperbolic group, and let S be any generating set. Suppose $X \subseteq W$ is such that $\mathtt{Red}(X)$ is a regular language. Then for any $w \in W$, the language $\mathtt{Red}(w \cdot X)$ is also regular.*

Proof. In fact, the theorem holds for any normal form on a group that satisfies the two-sided fellow-traveler property (i.e., is biautomatic). The proof is fairly immediate from the definitions; a reference is [20, Lemma 1.2]. □

5.4. **Corollary.** *Let W be a hyperbolic polygon group and let $w \cdot U^T$ (for $w \in \Omega_i^\circ$ and $T \in \mathscr{T}_i$) be one of the conjectured one-sided cells in C_i. Then $\mathtt{Red}(w \cdot U^T)$ is regular.*

Proof. First, we claim that the language $\mathtt{Red}(W^T)$ is regular. This is easily seen using the *canonical automaton* $\mathscr{A}_{\mathrm{can}}$ that accepts $\mathtt{Red}(W)$ [6, Theorem 4.8.3]. The states of this automaton, all of which are accepting, are given by the regions that are the connected components of the complement of the hyperplane arrangement determined by the *small roots* [6, §4.7]. Since the simple roots are small, the subset of the tessellation of $\mathfrak{H}$ corresponding to W^T is given by a union of states of $\mathscr{A}_{\mathrm{can}}$. Hence we can make an automaton accepting $\mathtt{Red}(W^T)$ by starting with $\mathscr{A}_{\mathrm{can}}$ and only making certain states accepting. Thus $\mathtt{Red}(W^T)$ is regular. Since all of the $\mathtt{Red}(C_j)$ are regular, it follows that $\mathtt{Red}(U^T)$ is regular. Hence, by Theorem 5.3, $\mathtt{Red}(w \cdot U^T)$ is also regular. □

5.5. *Remark.* Conjectures 3.5 and 3.1 are true for right-angled polygon groups by [4], and more generally for polygons with equal angles that satisfy the crystallographic condition by [5, §4]. Thus we can apply the results of this section to show regularity of the languages attached to the cells for those groups.

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