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A method for generalizing bond-percolation problems to include the possibility of infinite-range (equivalent-neighbor) bonds is presented. On Bravais lattices the crossover from nonclassical to classical (mean-field) percolation criticality in the presence of such bonds is described. The Cayley tree with nearest-neighbor and equivalent-neighbor bonds is solved exactly, and a nonuniversal line of percolation transitions with exponents dependent on nearest-neighbor bond occupation probability is observed. Points of logarithmic and exponential singularity are also encountered, and the behavior is interpreted as dimensional reduction due to the breaking of translational invariance by bonds of Cayley-tree connectivity.

I. INTRODUCTION

The percolation problem is related to a number of physical phenomena such as vulcanization, gelation of polymers,[1] and localization,[2] and has been extensively studied.[3,4] There is a close connection between percolation and critical phenomena. The method of Hamiltonian minimization,[5] originally introduced in the context of statistical mechanics, is extended to percolation in this paper. We describe a method for studying bond-percolation problems with both short-range and infinite-range bonds (connecting any pair of sites).[6] On Bravais lattices, any nonzero probability of infinite-range bonds changes the nature of the percolation transition to mean field,[6,7] and the crossover between short-range and infinite-range criticality is described. On the Cayley tree, on the other hand, when the entire lattice (and not just the interior as in the Bethe-Peierls approximation) is considered, there is a nonuniversal line of transitions, with the exponents describing critical quantities such as percolation probability varying with the short-range bond occupation probability. Range-dependent percolation has been studied by other authors.[8]

The connection between percolation and the $q\rightarrow1$ limit of Potts models[9] provides a means of applying methods of statistical mechanics to the study of percolation. The Hamiltonian minimization method[5] has been used to generalize a number of Ising models by introducing infinite-range interactions.[5,10,11] In Sec. II this method is extended to Potts models. In particular, we describe how the bond-percolation problem, with both nearest-neighbor and equivalent-neighbor bonds possible, can be studied. In Sec. III percolation on d-dimensional Bravais lattices in the presence of infinite-range bonds is considered. On these lattices any possibility of infinite-range bonds changes the critical behavior from short range (nonclassical) to classical mean field (infinite range). The crossover phenomena between critical behaviors governed by short-range and infinite-range bonds is studied. Moreover, the one-dimensional problem is solved exactly.

By contrast, on the Cayley tree (Sec. IV) introduction of the infinite-range bonds does not lead to mean-field percolation. In this case there is a novel line of percolation transitions, where the critical exponents are nonuniversal and depend on the probability of bond occupation. In this model, in the absence of any nearest-neighbor bonds all sites are equivalent, while the presence of such bonds breaks this "translational symmetry" and places the sites in the hierarchical structure of the Cayley tree. As the probability of nearest-neighbor bond occupation is increased, the mean-field critical behavior is modified by logarithmic corrections at some point. Then the line of continuously varying criticality, mentioned earlier, is encountered, and terminates at a point of exponential critical singularity. This unusual behavior is interpreted as a form of dimensional reduction due to the breaking of translational symmetry.

II. INFINITE-RANGE BOND PERCOLATION

The connection between percolation and the $q\rightarrow1$ limit of Potts models is well known[9] and provides a means of extending methods from statistical mechanics to the study of percolation. In the Potts model, at each site of the lattice there is a spin $s_i=1,2,\ldots,q$, and the Hamiltonian is

$$-\mathscr{H}/kT=K\sum_{\langle ij\rangle}\delta_{s_i,s_j}+h\sum_i\delta_{s_i,1}\;. \tag{1}$$

The cluster-size generating function for percolation is related to the $q\rightarrow1$ limit of the free energy $f(K,h)$ of the Potts model by[9]

$$G(p,h)=\sum_{s=1}^{\infty}n_s(p)e^{-sh}=-\left.\frac{\partial f}{\partial q}\right|_{q=1}, \tag{2}$$

where the bond-percolation probability is $p=1-e^{-K}$

$(K>0)$, $1-e^{-h}$ $(h>0)$ is the ghost bond probability,[12] and $n_s(p)$ is the mean number of clusters of size s divided by the total number of sites. The percolation probability $P(p)$ and the mean finite-cluster size $S(p)$ are related to derivatives of G by

$$P(p)=1+\frac{\partial}{\partial h}G(p,0^+)\ ,\quad S(p)=\frac{\partial^2 G}{\partial h^2}(p,0^+)\ . \tag{3}$$

We also note, for future purposes, that the definition of G [Eq. (2)] implies that

$$(-1)^n\frac{\partial^n G}{\partial h^n}\bigg|_{h\to 0^+}$$

is a positive quantity. With the use of the method of Hamiltonian minimization,[5] the Potts Hamiltonian [Eq. (1)] can be generalized to include infinite-range interactions. This procedure has been used in the context of Ising models,[10,11] and is easily generalized to Potts models. We consider a system of N Potts spins $\{s_i\}$ and evaluate

$$Z=\int_{-\infty}^{\infty}\prod_{\alpha=1}^{q}(dx_\alpha e^{-(NJ/2)x_\alpha^2})\sum_{\{s_i\}}\exp\left[K\sum_{\langle ij\rangle}\delta_{s_i,s_j}+\sum_i[(h+Jx_1)\delta_{s_i,1}+\cdots+Jx_q\delta_{s_i,q}]\right]\ . \tag{4}$$

Evaluating the Gaussian integral first and ignoring terms of order $(\ln N)/N$ in the exponent, we obtain

$$\begin{aligned}Z&=\sum_{\{s_i\}}\exp\left[K\sum_{\langle ij\rangle}\delta_{s_i,s_j}+h\sum_i\delta_{s_i,1}+\frac{J}{2N}\sum_{i,j}(\delta_{s_i,1}\delta_{s_j,1}+\cdots+\delta_{s_i,q}\delta_{s_j,q})\right]\\&=\sum_{\{s_i\}}\exp\left[K\sum_{\langle ij\rangle}\delta_{s_i,s_j}+h\sum_i\delta_{s_i,1}+\frac{J}{2N}\sum_{i,j}\delta_{s_i,s_j}\right]=e^{-Nf(K,J,h)}\ ,\end{aligned} \tag{5}$$

where f is the free energy of the original Potts model, with an additional equivalent-neighbor interaction J/N. If the summation over spins in Eq. (4) is carried out first, the result is

$$\begin{aligned}Z=\int dx_1\cdots dx_q\exp[&-(NJ/2)(x_1^2+\cdots+x_q^2)\\&-Nf_0(K,h+Jx_1,\ldots,Jx_q)]\ ,\end{aligned} \tag{6}$$

where $f_0(K,h+Jx_1,\ldots,Jx_q)$ is the free energy of the nearest-neighbor Hamiltonian in magnetic fields $h_1=h+Jx_1,\ldots,h_q=Jx_q$. In the thermodynamic limit $N\to\infty$, the saddle-point method relates the free energies in Eqs. (5) and (6) by

$$f(K,J,h)=\min\left[\frac{J}{2}\sum_{n=1}^{q}x_n^2+f_0(K,h+Jx_1,\ldots,Jx_q)\right]_{\{x_n\}}\ . \tag{7}$$

Therefore, in principle, the free energy is obtained by a q-parameter minimization. This is, however, considerably simplified by noting that x_α are the magnetizations of the Potts model[5] $[x_\alpha=\langle(1/N)\sum_i\delta_{s_i,\alpha}\rangle]$, and hence we can choose a parametrization

$$x_1=1/q+(q-1)m\ ,\quad x_2=\cdots=x_q=1/q-m\ . \tag{8}$$

Equation (7) can now be rewritten as

$$f(K,J,h)=-\frac{J}{2q}+\min\left[Jm+\frac{Jq(q-1)}{2}m^2+f_0(K,h+qJm)\right]_m\ . \tag{9}$$

It is therefore sufficient to do just a one-parameter minimization. For $K=0$, the pure equivalent-neighbor Potts model is obtained. In this case $f_0=-\ln[e^{qJm+h}+(q-1)]$, and Eq. (9) results in the familiar mean-field expression for Potts models.[13] To make the connection with percolation the $q\to 1$ limit of Eq. (9) has to be considered. The nearest-neighbor bond occupation probability is $p_s=1-e^{-K}$, while the equivalent-neighbor bond occupation probability is $2p_l/N=1-e^{-J/N}=(J/N)$ $(N\to\infty)$, implying that on average p_lN equivalent-neighbor bonds are occupied. Also,

$$\lim_{q\to 1}f_0(K,h+qJm)=-Jm-h-\frac{N_b}{N}K$$

(N_b is the number of bonds), and Eq. (2) implies

$$G(p_s,p_l,h)=-\min[p_l(m-1)^2-G_0(p_s,h+2p_lm)]_m\ . \tag{10}$$

Equation (10) is the main result that will be used to generalize percolation problems to include infinite-range bonds. The case $K=0$ of infinite-range percolation has been studied by Wu.[6] In this case

$$G_0=e^{-(h+2p_lm)}\ , \tag{11}$$

and the cluster-size generating function is

$$G=-\min[p_l(m-1)^2-e^{-(2p_lm+h)}]_m\ , \tag{12}$$

which is the result previously obtained.[6] The value of m that minimizes the above expression (denoted $\overline{m}$) is indeed the percolation probability P. We now apply Eq. (10) to study cases with both nearest-neighbor (short-range) and equivalent-neighbor (infinite-range) bond occupation possible.

III. BRAVAIS LATTICES

The percolation problem has only been solved exactly for a number of one-dimensional systems.[12,14] In particular, the one-dimensional bond-percolation problem can be solved easily using the connection to Potts models. The free energy of the q-state Potts model [Eq. (1)], calculated in one dimension by the transfer-matrix method, is

$$f_0(K,h)=\ln 2-\ln(q-2+e^K(1+e^h)+\{[(q-2)+e^K(1-e^h)]^2+4(q-1)e^h\}^{1/2}) . \tag{13}$$

Differentiating f_0 with respect to q at $q=1$, we obtain the generating function [Eq. (2)]

$$G_0(p_s,h)=\frac{(1-p_s)^2}{e^h-p_s} . \tag{14}$$

The generating function for the one-dimensional problem with nearest-neighbor and equivalent-neighbor bond percolation is now obtained using Eq. (10) as

$$G(p_s,p_l,h)=-\min\left[p_l(m-1)^2-\frac{(1-p_s)^2}{e^{h+2p_lm}-p_s}\right]_m . \tag{15}$$

The parameter m corresponds to the percolation probability, and is thus confined to the range zero to one. We consider below the case of zero ghost bond probability ($h=0$). Close to the percolation threshold $\overline{m}=P(p_s,p_l)$ is a small quantity, and it is sufficient to make an expansion in m as follows:

$$G(p_s,p_l,0)=1-p_s-p_l-\min\left[p_lm^2\left[1-2p_l\frac{1+p_s}{1-p_s}\right]+\frac{4p_l^3m^3}{3}\frac{1+4p_s+p_s^2}{(1-p_s)^2}+O(m^4)\right]_m . \tag{16}$$

There is a percolation transition when the coefficient of the m^2 term changes sign at

$$2p_l=\frac{1-p_s}{1+p_s}=S_0^{-1}(p_s) . \tag{17}$$

The "phase diagram" is indicated in Fig. 1(a). The pure one-dimensional problem ($p_l=0$) percolates only for $p_s=1$, while pure equivalent-neighbor percolation ($p_s=0$) takes place for $p_l>\frac{1}{2}$.[6] The expression in Eq. (16) is similar to a Landau expansion of the free energy in spin models. However, the presence of the cubic term *does not* imply that the transition is first order. This is because the variable m is confined to the range zero to one, and negative values of m are meaningless. The percolation transition is continuous, and the critical behavior is determined by balancing the m^2 and m^3 terms as follows:

$$G^{\text{sing}}\sim\begin{cases}0\,, & t<0\\ t^3\,, & t>0\,,\end{cases}$$

$$P=\overline{m}\sim\begin{cases}0\,, & t<0\\ t\,, & t>0\,,\end{cases} \tag{18}$$

$$S\sim t^{-1}\,,$$

where $t=2p_lS_0(p_s)-1$. The critical exponents $\alpha=-1$, $\beta=1$, and $\gamma=1$ are characteristic of mean-field percolation.[6,7]

For higher-dimensional lattices, the exact solution to $G_0(p_s,h)$ is not known. However, certain properties of $G_0(p_s,h)$ are known, and it is possible to make general statements about the behavior in the presence of long-range percolation for Bravais lattices of arbitrary dimension. For $p_s<p^*$ (percolation threshold of the short-range model) it is possible to make an expansion as in Eq. (16), and

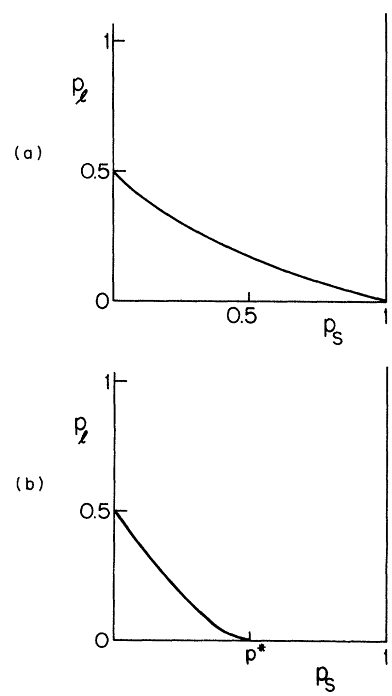

FIG. 1. Phase diagrams for percolation with nearest-neighbor and equivalent-neighbor occupation probabilities p_s and $2p_l/N$ for (a) the one-dimensional lattice (exact result) and (b) a higher-dimensional lattice (schematic).

$$G(p_s,p_l,0)=-p_l+G_0(p_s,0)-\min\{p_l m^2[1-2p_l S_0(p_s)]+\tfrac{4}{3}p_l^3 U_0(p_s)m^3+O(m^4)\}_m\ , \tag{19}$$

where

$$S_0(p_s)=\left.\frac{\partial^2 G_0}{\partial h^2}\right|_{h=0^+}$$

is the mean size of finite clusters in the nearest-neighbor model, and

$$U_0=-\left.\frac{\partial^3 G_0}{\partial h^3}\right|_{h=0}>0\ .$$

The percolation transition occurs for $2p_l=S_0(p_s)^{-1}$, and the critical behavior is governed by mean-field exponents [Eq. (18)]. A representative phase diagram is given in Fig. 1(b). For $p_s=0$, the cluster size $S_0(0)=1$ and $p_l=\frac{1}{2}$ of the equivalent-range model is recovered. The line terminates at the percolation point of the short-range model since $S_0(p^*)^{-1}=0$. The crossover phenomena in the vicinity of the short-range percolation transition can also be studied. In this neighborhood there are two scaling exponents, y_t and y_h, corresponding to the thermal and magnetic directions in the Potts model.[15] For $p_l\bar{m}\gg(p^*-p_s)^{\phi}$, where $\phi=y_h/y_t=\beta+\gamma$ is the crossover exponent, the expansion of Eq. (19) is no longer valid. The correct expansion gives

$$G(p_s,p_l,0)\simeq-p_l+G_0(p^*,0)-\min[p_l m^2-c(2p_l m)^{d/y_h}]_m\ , \tag{20}$$

where c is a positive constant. The percolation probability in this region is $P=\bar{m}\sim p_l^{\beta/\gamma}$, independent of p_s. For $p_s-p^*\sim p_l^{1/\gamma}$ there is another crossover to a region where the percolation probability of the pure short-range model is nonzero, and for $p_s-p^*\gg p_l^{1/\gamma}$ the proper expansion is

$$G(p_s,p_l,0)\simeq-p_l+G_0(p_s,0)-\min\{-2p_l P_0(p_s)m+p_l[1-2p_l S_0(p_s)]m^2+O(m^3)\}_m\ , \tag{21}$$

where $P_0(p_s)$ is the percolation probability of the short-range model. In this regime $\bar{m}\simeq P_0(p_s)/[1-2p_l S_0(p_s)]$. Figure 2(a) shows the behavior of $\bar{m}$ as a function of p_s for p_l close to zero. Similarly, the crossover phenomena encountered for the finite-cluster size S is depicted in Fig. 2(b). The crossover behavior is very similar to that encountered in the Ising model with equivalent-neighbor interactions.[10,16]

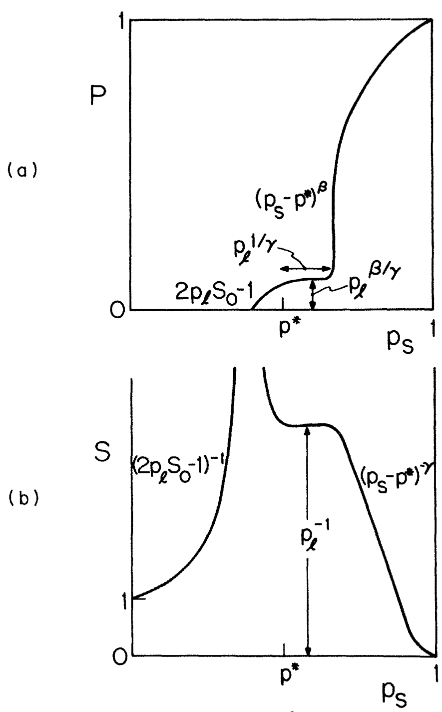

FIG. 2. Crossover phenomena for a small equivalent-neighbor occupation probability p_l in (a) the percolation probability P and (b) the mean finite-cluster size S.

IV. CAYLEY TREE

Various forms of percolation have been studied on Cayley trees[7,17,18] [Fig. 3(a)]. Cayley trees are special cases of

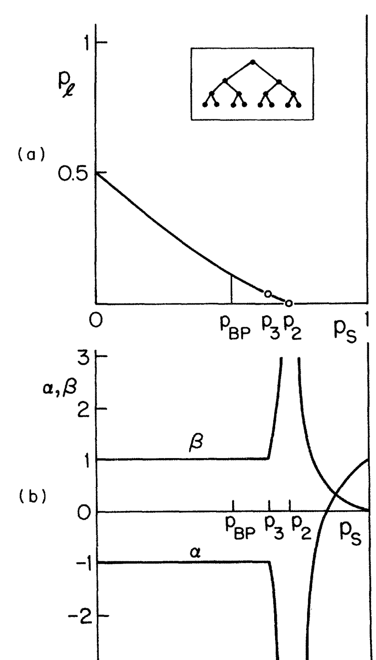

FIG. 3. (a) Phase diagrams for the Cayley tree with $z=3$ (inset). (b) The dependence of the exponents α and β on the nearest-neighbor bond occupation probability.

hierarchical lattices[19,20] with no loops present. An important feature of the Cayley tree and of all hierarchical lattices is its lack of translational symmetry.[20] As a result such properties as local magnetization and percolation probability will also be site dependent. In the Bethe-Peierls approximation[21,7] this site dependence is ignored and a uniform magnetization (or percolation probability) is assumed. The results are valid only for sites at the center of the Cayley tree. However, a finite fraction of the total number of sites lies at the surface of the tree and drastically affects the thermodynamic behavior. Such quantities as the net magnetization (or the percolation probability averaged over all lattice sites) are very different from the local properties obtained for the central sites by the Bethe-Peierls approximation.[21] The correct thermodynamic behavior was obtained by Müller-Hartmann and Zittartz[22] for the Ising model on the Cayley tree, and was extended later on to general Potts models[23] (including percolation).

We now briefly review the properties of the cluster-generating function $G_0(p_s,h)$ on a Cayley tree of coordination number z. For $p_s<p_{\rm BP}=1/(z-1)$, the Bethe-Peierls percolation threshold $G_0(p_s,h)$ is analytic in h and can be expended as

$$G_0(p_s,h)=G_0(p_s,0)+[P_0(p_s)-1]h+\frac{S_0(p_s)}{2}h^2-\frac{U_0(p_s)}{6}h^3+O(h^4)\,, \tag{22}$$

where

$$\begin{aligned}
&G_0(p_s,0)=1-p_s\,,\\
&P_0(p_s)=0\,,\\
&S_0(p_s)=\left.\frac{\partial^2 G_0}{\partial h^2}\right|_{h=0}=\frac{(1+p_s)^2}{1-p_s^2(z-1)}\quad [p_s<p_2=(z-1)^{-1/2}]\,,\\
&U_0(p_s)=-\left.\frac{\partial^3 G_0}{\partial h^3}\right|_{h=0}=3\frac{p_s(1+p_s)^2}{[1-p_s^2(z-1)]^2}+\frac{(1-p_s)[1+2p_s+2p_s^2(z-1)+p_s^3(z-1)]}{[1-p_s^2(z-1)][1-p_s^3(z-1)^2]}\\
&\qquad\qquad\times\left\{p_s+\frac{(1+p_s)[1+2p_s^2(z-1)]}{1-p_s^2(z-1)}\right\}\quad [p_s<p_3=(z-1)^{-2/3}]\,.
\end{aligned} \tag{23}$$

For $p_s>1/(z-1)$, there is an additional singular term,[23]

$$G_0^{\rm sing}(p_s,h)\underset{h\to 0^+}{\simeq} Ah^{\Delta}\,, \tag{24}$$

where the exponent Δ varies continuously with p_s as

$$\Delta(p_s)=\frac{\ln(z-1)}{\ln[(z-1)p_s]}\,. \tag{25}$$

Furthermore, when $\Delta(p_s)$ equals an integer n, the leading singularity is modified by a logarithm as follows:

$$G_0^{\rm sing}(p_s,h)\underset{h\to 0^+}{\simeq} a_n h^n \ln(1/h)\,. \tag{26}$$

To examine the behavior of the percolation problem on the Cayley tree with equivalent-neighbor bonds, the form of $G_0(p_s,h)$ from Eqs. (22)–(26) can be substituted into Eq. (10). As for Bravais lattices, the percolation threshold occurs for $2p_l=S_0(p_s)^{-1}$, and is given by

$$2p_l=\begin{cases}\dfrac{1-p_s^2(z-1)}{(1+p_s)^2}\,, & p_s<p_2\\[2mm] 0\,, & p_s>p_2\end{cases} \tag{27}$$

where $p_2=(z-1)^{-1/2}$. The critical boundary for $z=3$ is given in Fig. 3(a). The critical line can be divided into the following segments.

(i) For $p_s<p_{\rm BP}=1/(z-1)$, the expansion of $G_0(p_s,h)$ is analytic in h, and the critical properties will be mean-field-like ($\alpha=-1,\beta=1,\gamma=1$).

(ii) For $p_{\rm BP}<p_s<p_3=(z-1)^{-2/3}$, there will be nonanalytic terms in $G_0(p_s,h)$. However, the exponent Δ is larger than 3, and the leading terms in the expansion for $G_0(p_s,p_l,0)$ will be

$$G(p_s,p_l,0)\simeq 1-p_s-p_l-\min\{p_l m^2[1-2p_lS_0(p_s)]+\tfrac{4}{3}p_l^3U_0(p_s)m^3\}_m\,, \tag{28}$$

resulting in classical criticality, although there will be nonclassical corrections in higher-order derivatives.

(iii) As $p_s\to p_3$, the third-order derivative $U_0(p_s)$ diverges, and at $p_s=p_3$,

$$G(p_s,p_l,0)\simeq 1-p_s-p_l-\min\{p_l m^2[1-2p_lS_0(p_s)]-a_3(2p_lm)^3\ln(1/2p_lm)\}_m\,, \tag{29}$$

where

$$a_3=-\frac{1}{2}\frac{1}{\ln(z-1)}\left[\frac{z-2}{z-1}\right]^2\left[\frac{(z-1)^{1/3}+1}{(z-1)^{1/3}-1}\right]^3 \tag{30}$$

is a negative constant. For $t=2p_lS_0(p_s)-1$ negative $\overline{m}$ is zero, while for positive t it behaves as

$$\overline{m}\sim t/\ln t\,, \tag{31}$$

while the generating function behaves as

$$G^{\rm sing}(p_s,p_l,0)\sim t^3/(\ln t)^2\,. \tag{32}$$

Similar modifications of classical criticality by logarithms have been indicated for percolation at the upper critical dimension.[24]

(iv) For $p_3 < p_s < p_2$, the exponent Δ is less than 3, and the leading expansion in m gives

$$G(p_s,p_l,0) \simeq 1-p_s-p_l-\min\{p_l m^2[1-2p_l S_0(p_s)] -A(2p_l m)^{\Delta}\}_m \ , \tag{33}$$

with A negative. This is a consequence of the fact that for $2<\Delta<3$,

$$\left.\frac{\partial^3 G_0}{\partial h^3}\right|_{h\to 0^+} \simeq A\Delta(\Delta-1)(\Delta-2)h^{\Delta-3}<0 \ ,$$

which implies $A<0$. For $t>0$, the expression (33) gives $\overline{m}\sim t^{\beta}$ and $G^{\rm sing}\sim t^{2-\alpha}$, where the exponents

$$\alpha=\frac{4-\Delta}{2-\Delta} \ , \quad \beta=\frac{1}{\Delta-2} \tag{34}$$

vary continuously with the probability p_s. Nonuniversal critical behavior has been encountered in vertex and Ising models,[25] but is a new result in percolation problems (there are one-dimensional models with exponents dependent on the range,[12,14] but not on the magnitude of the percolating bonds).

(v) As $p_s \to p_2$, the cluster size $S_0(p_s)$ diverges, and for $p_s=p_2$ the proper expansion is

$$G(p_s,p_l,0) \simeq 1-p_s-p_l -\min[p_l m^2-a_2(2p_l m)^2\ln(1/2p_l m)]_m \ , \tag{35}$$

where

$$a_2=\frac{(\sqrt{z-1}+1)^2}{(z-1)\ln(z-1)}$$

is positive. The behavior of critical quantities as $p_l \to 0$ is

$$\overline{m}\sim\exp\left[-\frac{1}{4a_2 p_l}\right] \ , \quad G^{\rm sing}\sim\exp\left[-\frac{1}{2a_2 p_l}\right] \ . \tag{36}$$

The exponential singularity of critical quantities is characteristic of spin systems at their lower critical dimensionality.[26,27] The reason it is not encountered in one-dimensional percolation [Eq. (14)] is that the conventional choice for the small deviation from criticality is $1-p_s$ rather than e^{-K}.

(vi) For $p_s>p_2$, the exponent Δ is less than 2, and

$$G(p_s,p_l,0) \simeq 1-p_l-p_s-\min[p_l m^2-A(2p_l m)^{\Delta}]_m \ . \tag{37}$$

For $1<\Delta<2$,

$$\left.\frac{\partial^2 G_0}{\partial h^2}\right|_{h=0^+} \simeq \Delta(\Delta-1)Ah^{\Delta-2}>0 \ ,$$

which implies $A>0$. The system is now percolating for $p_l>0$, while as $p_l\to 0$, the critical quantities behave as $\overline{m}\sim p_l^{\beta}$ and $G^{\rm sing}\sim p_l^{2-\alpha}$, where

$$\alpha=\frac{4-3\Delta}{2-\Delta} \ , \quad \beta=\frac{\Delta-1}{2-\Delta} \ . \tag{38}$$

These exponents are also nonuniversal as Δ varies with the probability p_s [Eq. (25)]. In this model the exponent γ always retains its mean-field value of unity. The exponents α and β are plotted in Fig. 3(b). As in the Ising model on a Cayley tree,[28] we expect the amplitude A [Eqs. (24), (33), and (37)] to equal a constant plus a numerically small periodic function of $\ln(2p_l m)$. This will cause small but rather interesting modifications in the critical behavior.[29]

The results obtained above are analogous to those observed for the Ising model on the Cayley tree with equivalent-neighbor interactions.[11] For $p_s=0$ all sites are identical and connected by equivalent-neighbor bonds, and the classical percolation transition at $p_l=\frac{1}{2}$ is that of an infinite-dimensional system. Increasing p_s breaks the translational invariance between the sites and places them in the hierarchical structure of the Cayley tree. As p_s is increased the classical percolation transition is modified by logarithms (resembling percolation at the upper critical dimension of six),[24] then there is a line of continuously varying exponents terminating in a point of exponential singularity (characteristic of spin systems at the lower critical dimension).[26,27] Thus breaking the translational invariance by bonds of Cayley-tree connectivity appears to have effects on the critical percolation behavior which are similar to the lowering of spatial dimensionality on Bravais lattices. This is precisely the conclusion reached in the Ising problem,[11] and will also be extended to other Potts models.[29]

Again, in this case we must emphasize the meaning of the percolation probability $P(p_s,p_l)=\overline{m}$, as an *average* and not a *local* percolation probability. Owing to the lack of translational invariance, the local percolation probabilities $P_i(p_s,p_l)$ are site dependent. For $p_s>p_{\rm BP}=1/(z-1)$ the probability that the sites at the center of the tree belong to an infinite cluster is finite (and given by the Bethe-Peierls approximation),[7] while due to the large number of surface sites, the average percolation probability $P=(1/N)\sum_i P_i$ is zero. Thus for $p_s>p_{\rm BP}$ there is a finite probability of an infinite cluster on the tree.[7]

V. CONCLUSION

Using the relation between percolation and the $q\to 1$ limit of Potts models,[9] we describe a method for generalizing and solving exactly a number of percolation problems. In this paper the bond-percolation problem in the presence of infinite-range bonds was studied on Bravais lattices and on the Cayley tree. The absence of translation symmetry between the sites of the Cayley tree causes percolation criticality dramatically different from Bravais lattices, and leads to a nonuniversal line of percolation transitions. Such nonuniversal transition lines have been encountered in problems in statistical mechanics.[25] The procedure described in this paper can be further generalized by the Hamiltonian minimization method to include the possibility of bonds simultaneously connecting three or more sites.[5] In the language of Potts models these would correspond to interactions connecting three or more spins in an equivalent-neighbor fashion. In such generalized problems percolation transitions with "first-order"[18] or even "tricritical" type of criticality might be encountered. The application of Eq. (9) to general Potts models on Bravais and Cayley-tree lattices also leads to a number of interesting results which will be discussed elsewhere.[29] Percolation and Potts models on another hierarchical lattice, the diamond,[20] will be discussed in Ref. 30.

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