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Composite and Prime Polygons

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Abstract. This essay describes an algorithm to determine whether an odd integer is a prime or a composite. It finds the divisors of an odd composite. It presents graphs of primes and composites for the bands in the main partition of a fair odd m-polygon.

Introduction

This essay displays graphs of prime numbers. It usually takes a lot of arithmetic to decide whether a number is a prime. I show that each prime p has a regular polygon with p vertexes and a collection of (p-1)/2 non overlapping round trips made of 1-way arrows. Each round trip visits all the vertexes using arrows of equal length. The number of steps and the length of each step is the same in each round trip but different round trips use arrows of suitable length. Given a regular polygon with m vertexes, m an odd number, my algorithm constructs a simple sequence of steps that can determine whether the m-polygon enables the round trips that show whether or not m is an odd prime.

Preface

I begin with a description of the properties of Hamiltonian [simple] circuits, my principal tool. A regular polygon with an odd number, m, of vertexes is fair if it has m(m-1)/2 arrows and each vertex has the same number of arrows entering and leaving the other vertexes, (m-1)/2 entering, (m-1)/2 exiting. A fair odd m-polygon has an mXm adjacency matrix A[m]=[$a_{i,j}$], whose terms are 0 unless an arrow goes from vertex i, its source, to vertex j, its destination. In this case $a_{i,j}$ = 1. These are 1-way arrows so that when an arrow goes from one vertex to another, no arrow goes the other way. The terms of A[m] parallel to its principal diagonal are called bands. The two examples A[3] and A[5] show the bands

$$A[3] = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad A[5] = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

A simple [Hamiltonian] circuit is a sequence of 1-way arrows such that the source vertex of one arrow is the destination vertex of the preceding arrow. A simple example is a circuit with 3 arrows: $\{1 \leftrightarrow 2,\ 2 \leftrightarrow 3,\ 3 \leftrightarrow 1\}$. It enables a round trip starting from any vertex.

A simple circuit has two properties; i) no arrow appears more than once; ii) each vertex in a circuit appears once as an arrow source and as a destination for another arrow. A simple circuit allows a round trip of its vertexes starting from any vertex. A super circuit is a merger of non overlapping simple circuits that share some vertexes, junctions. Super circuits only allow round trips starting from a junction. A partition of the 1- way arrows in a fair m-polygon is a super circuit that includes all the arrows.

Background and Procedure

1. The number of relatively prime bands is EulerPhi(m)/2.
2. The total number of bands = (m-1)/2. Each band has m arrows.
3. The pairs (1,j), j = 2, 4, 6, ..., m-1 are in the first row of A[m].
4. The length of an arrow (i ⟷ j) ≡ (j-i) (mod m).
5. Band index k = 2 i -1, i = 1, 2, ... , (m-1)/2.
   i. k runs over the odd integers, k=1,3,5,...,m-2.
   ii. Every arrow in Band[k,m] has the same length.
   iii. The common length ≡ k(mod m).
   iv. Arrow lengths sum to zero in each band.
6. GCD (k,m) = number of circuits in band k.

Lemma 1. GCD (k,m) = 1 if k = 1. m-4, m-2 or m-2^a, a runs over the suitable positive integers.
Hence an m-polygon with only 3 bands in its main partition must be a prime.
Two m-polygons meet this condition; the pentagon and the septagon so this proves 5 and 7 are primes.
Lemma 2. The number of circuits in a band must be odd.
Proof. Because m is odd and k is odd, GCD(m, k) must be odd □

Hence a circuit in Band[k,m] has m/GCD[k,m] arrows.
Thus Band[9,15]=3 so it has three 5-circuits.

Lemma 3. If GCD[k,m]=1, then Band k has only one circuit of length k(mod m).
Corollary. Band[k,m] has 3 or more circuits if and only if GCD[k,m]>1.
Theorem. The odd integer m is prime if every Band[k,m] makes only one round trip of the m-polygon.

The procedure requires only addition and sorting.
i. Find the bands in the main partition of a fair odd m-polygon.
ii. Arrange the arrows of each band into simple circuits.
iii. If any band has more than one simple circuit, then the integer m is composite.
iv. If no band has more than one simple m-circuit, then the integer m is an odd prime.

Pentagon

A fair pentagon has two bands:
Band[1,5] = {{1,2},{2,3},{3,4},{4,5},{5,1}}
Band[3,5] = {{1,4},{2,5},{3,1},{4,2},{5,3}}
Its terms are Modulo 5. Note the pattern of the terms in a band, {i,j} ⟷ Mod[{i+1, j+1},5]. The next matrix shows the coordinates for its arrows.

$$A[5] = \begin{pmatrix} 0 & \{1,2\} & 0 & \{1,4\} & 0 \\ 0 & 0 & \{2,3\} & 0 & \{2,5\} \\ \{3,1\} & 0 & 0 & \{3,4\} & 0 \\ 0 & \{4,2\} & 0 & 0 & \{4,5\} \\ \{5,1\} & 0 & \{5,3\} & 0 & 0 \end{pmatrix}$$

15-polygon

The following tables have the composite bands for the 15-polygon; Band[3,15], Band[5,15] and Band[9,15]. The first table introduces the 7 bands in the 15-polygon. For each arrow it shows the band index (mod[15]) and the GCD of the band index with m.

1 / st Band Arrow	1, 2	1, 4	1, 6	1, 8	1, 10	1, 12	1, 14
Band index k	1	3	5	7	9	11	13
GCD[k, m]	1	3	5	1	3	1	1

The next 7 tables give details for each composite band in the 15-polygon and one relatively prime band, B[7,15].

arrow	1, 4	2, 5	3, 6	4, 7	5, 8	6, 9	7, 10	8, 11
arrow length	3	3	3	3	3	3	3	3
arrows	9, 12	10, 13	11, 14	12, 15	13, 1	14, 2	15, 3	–
arrow length	3	3	3	3	−12	−12	−12	sum = 0

k = 3 for all arrows in Band[3, 15]

Three 5 – Circuits in Band[3, 15]

arrow	1, 4	4, 7	7, 10	10, 13	13, 1
arrow length	3	3	3	3	$-12 \equiv 3 \pmod{15}$
arrow	2, 5	5, 8	8, 11	11, 14	14, 2
arrow length	3	3	3	3	$-12 \equiv 3 \pmod{15}$
arrow	3, 6	6, 9	9, 12	12, 15	15, 3
arrow length	3	3	3	3 – 12	$-12 \equiv 3 \pmod{15}$

arrow	1, 6	2, 7	3, 8	4, 9	5, 10	6, 11	7, 12	8, 13
arrow length	5	5	5	5	5	5	5	5
arrow	9, 14	10, 15	11, 1	12, 2	13, 3	14, 4	15, 5	–
arrow length	5	5	−10	−10	−10	−10	$-10 \equiv 5 \pmod{15}$	–

$-10 \pmod{15} \equiv 5$

Five 3 – Circuits in Band[5, 15]

arrow	1, 6	6, 11	11, 1
arrow length	5	5	-10
arrow	2, 7	7, 12	12, 2
arrow length	5	5	-10
arrow	3, 8	8, 13	13, 3
arrow length	5	5	-10
arrow	4, 9	9, 14	14, 4
arrow length	5	5	-10
arrow	5, 10	10, 15	15, 5
arrow length	5	5	-10

arrow	1, 10	2, 11	3, 12	4, 13	5, 14	6, 15	7, 1	8, 2	9, 3
arrow length	9	9	9	9	9	9	-6	-6	-6
arrow	10, 4	11, 5	12, 6,	13, 7,	14, 8	15, 9	-	-	-
arrow length	-6	-6	-6	-6	-6	-6	6 * 9	-6 * 9	sum = 0

-6 (Mod 15) ≡ 9

Three 5 - Circuits in Band[9, 15]

arrow	1, 10	10, 4	4, 13	13, 7	7, 1
arrow length	9	-6	9	-6	-6
arrow	2, 11	11, 5	5, 14	14, 8	8, 2
arrow length	9	-6	9	-6	-6
arrow	3, 12	12, 6	6, 15	15, 9	9, 3
arrow length	9	-6	9	-6	-6

One 15 - circuit in Band[7, 15]

arrow	1, 8	2, 9	3, 10	4, 11	5, 12	6, 13	7, 14	8, 15
arrow length	7	7	7	7	7	7	7	7
arrow	9, 1	10, 2	11, 3	12, 4	13, 5	14, 6	15, 7	–
arrow length	−8	−8	−8	−8	−8	−8	−8	sum = 0

arrow	1, 8	8, 15	15, 7	7, 14	14, 6	6, 13	13, 5	5, 12
arrow length	7	7	−8	7	−8	7	−8	7
arrow	12, 4	4, 11	11, 3	3, 10	10, 2	2, 9	9, 1	–
arrow length	−8	7	−8	7	−8	7	−8	sum = 0

15 - polygon

Band[3,15] GCD[3,15]=3 m/GCD[3,15]=5 $\Longrightarrow$ 3 5-circuits

Band[5,15] GCD[3,15]=5 m/GCD[5,15]=3 $\Longrightarrow$ 5 3-circuits

Band[9,15] GCD[3,15]=3 m/GCD[9,15]=5 $\Longrightarrow$ 3 5-circuits

21 - polygon

Band[3,21] GCD[3,21]=3 m/GCD[3,21]=7 $\Longrightarrow$ 3 7-circuits

Band[7,21] GCD[7,21]=7 m/GCD[7,21]=3 $\Longrightarrow$ 7 3-circuits

Band[9,21] GCD[9,21]=3 m/GCD[9,21]=7 $\Longrightarrow$ 3 7-circuits

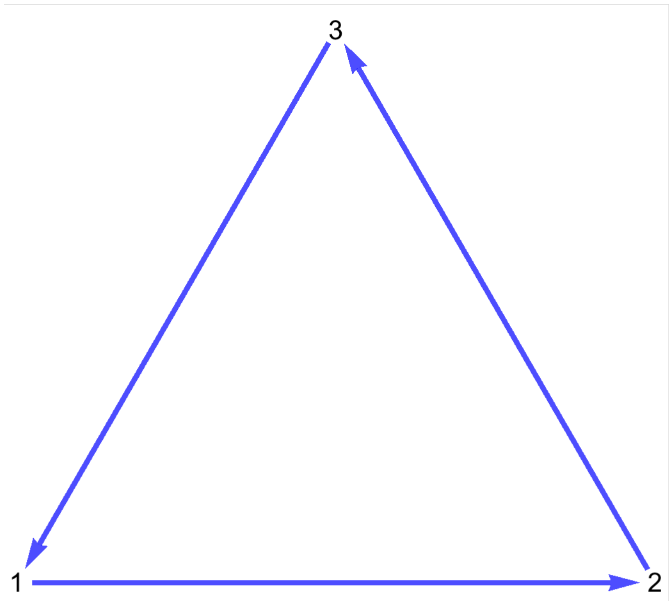
Band[15,21] GCD[15,21]=3 m/GCD[15,21]=7$\Longrightarrow$ 3 7-circuits

Graphs

■ triangle

```
bnd13 := {{0, 1, 0}, {0, 0, 1}, {1, 0, 0}}
```

```
makeGrph[bnd13, 2]
```

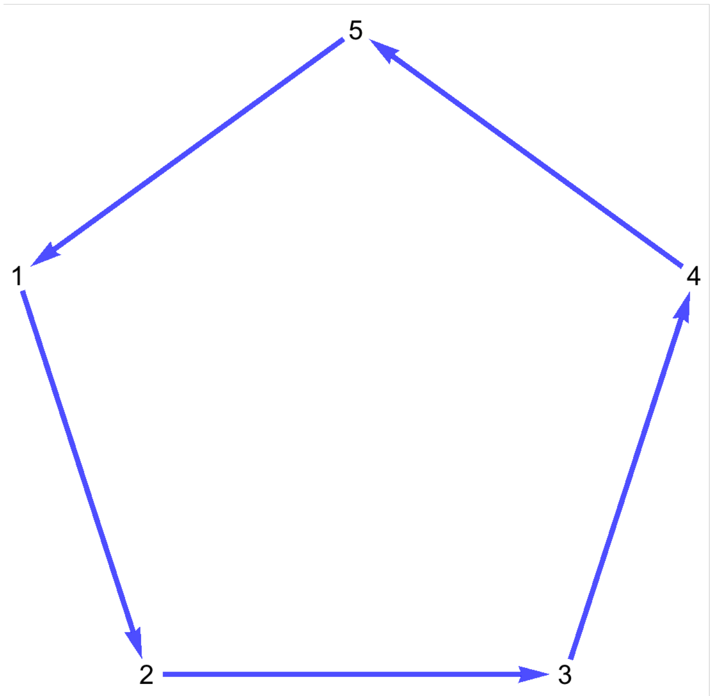



```
Bnd[1, 3] := {{1, 2}, {2, 3}, {3, 1}}

Apply[DirectedEdge, {{1, 2}, {2, 3}, {3, 1}}, 1]

bnd13 := {1 ↔ 2, 2 ↔ 3, 3 ↔ 1}
```

- pentagon

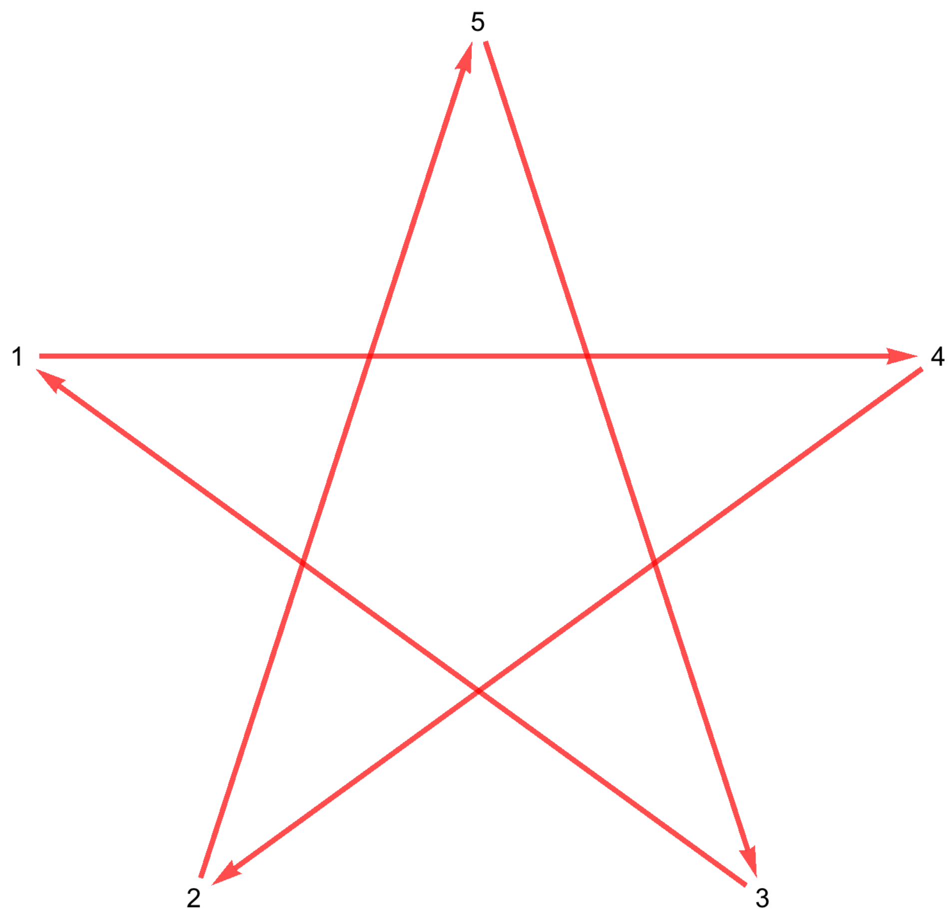
```
b15 := {{0, 1, 0, 0, 0}, {0, 0, 1, 0, 0}, {0, 0, 0, 1, 0},
   {0, 0, 0, 0, 1}, {1, 0, 0, 0, 0}}
```

```
makeGrph[b15, 2]
```




```
b35 := {{0, 0, 0, 1, 0}, {0, 0, 0, 0, 1}, {1, 0, 0, 0, 0},
   {0, 1, 0, 0, 0}, {0, 0, 1, 0, 0}}
```

```
makeGrph[b35, 3]
```

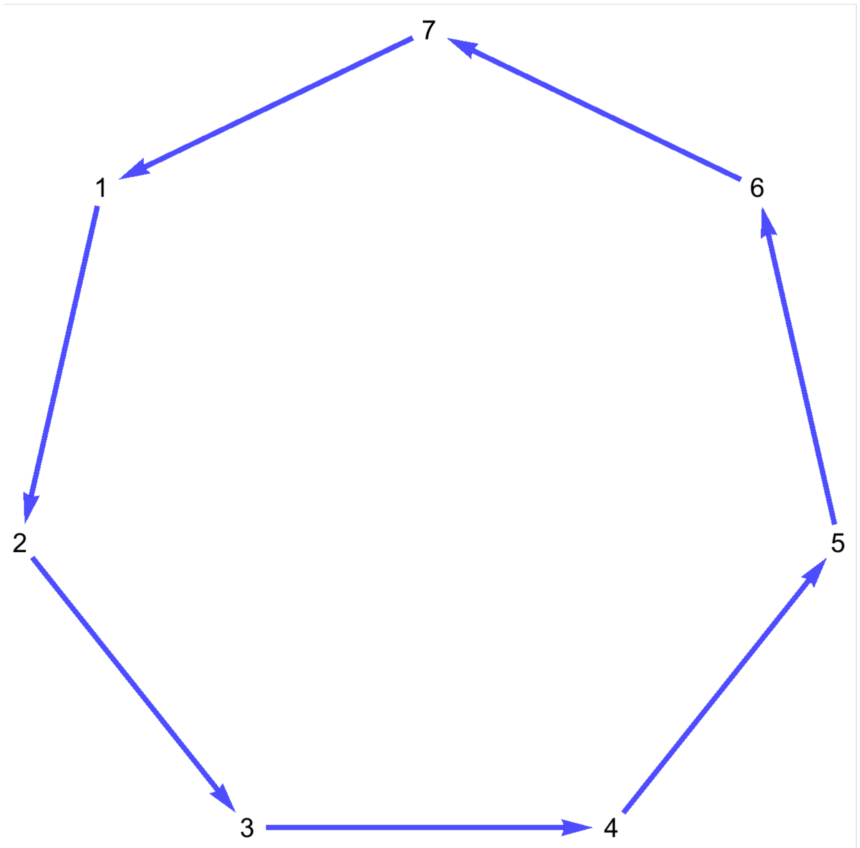



```
bnd15 := {1 ⟷ 2, 2 ⟷ 3, 3 ⟷ 4, 4 ⟷ 5, 5 ⟷ 1}

bnd35 := {1 ⟷ 4, 2 ⟷ 5, 3 ⟷ 1, 4 ⟷ 2, 5 ⟷ 3}
```

- septagon

```
b17 := {{0, 1, 0, 0, 0, 0, 0}, {0, 0, 1, 0, 0, 0, 0},
   {0, 0, 0, 1, 0, 0, 0}, {0, 0, 0, 0, 1, 0, 0},
   {0, 0, 0, 0, 0, 1, 0}, {0, 0, 0, 0, 0, 0, 1},
   {1, 0, 0, 0, 0, 0, 0}}
```

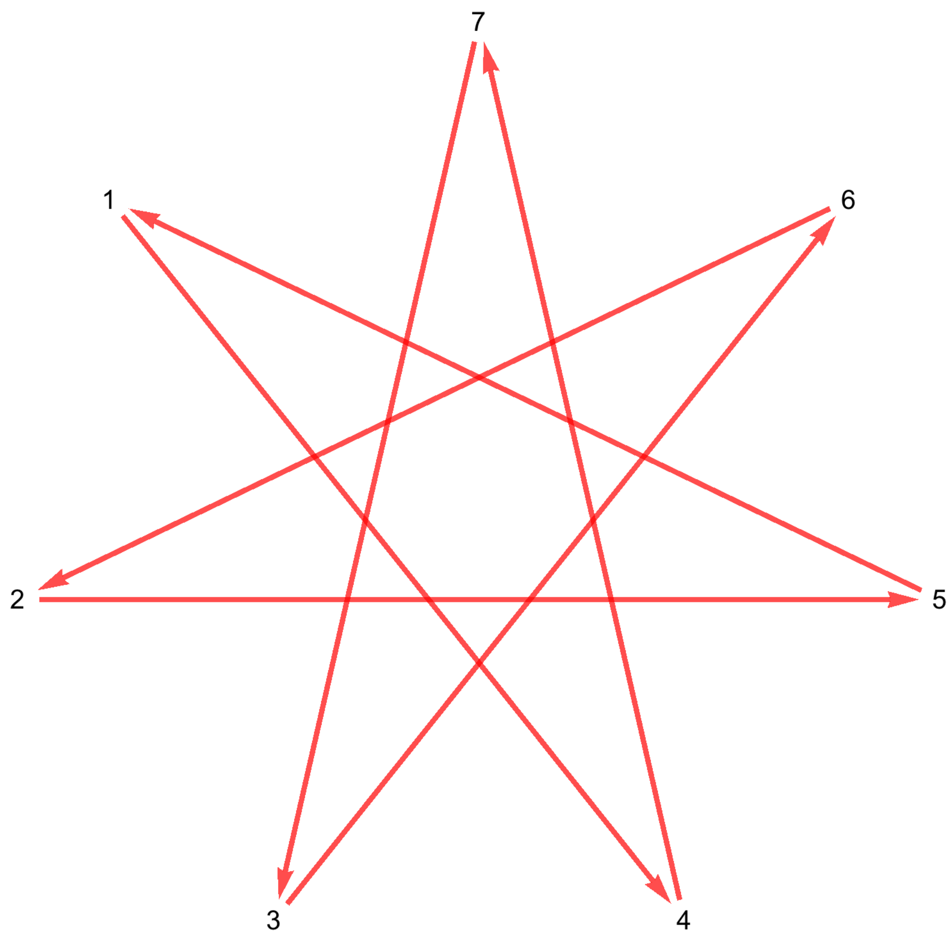
```
makeGrph[b17, 2]
```



Band[1, 7]

```
b37 := {{0, 0, 0, 1, 0, 0, 0}, {0, 0, 0, 0, 1, 0, 0},
  {0, 0, 0, 0, 0, 1, 0}, {0, 0, 0, 0, 0, 0, 1},
  {1, 0, 0, 0, 0, 0, 0}, {0, 1, 0, 0, 0, 0, 0},
  {0, 0, 1, 0, 0, 0, 0}}
```

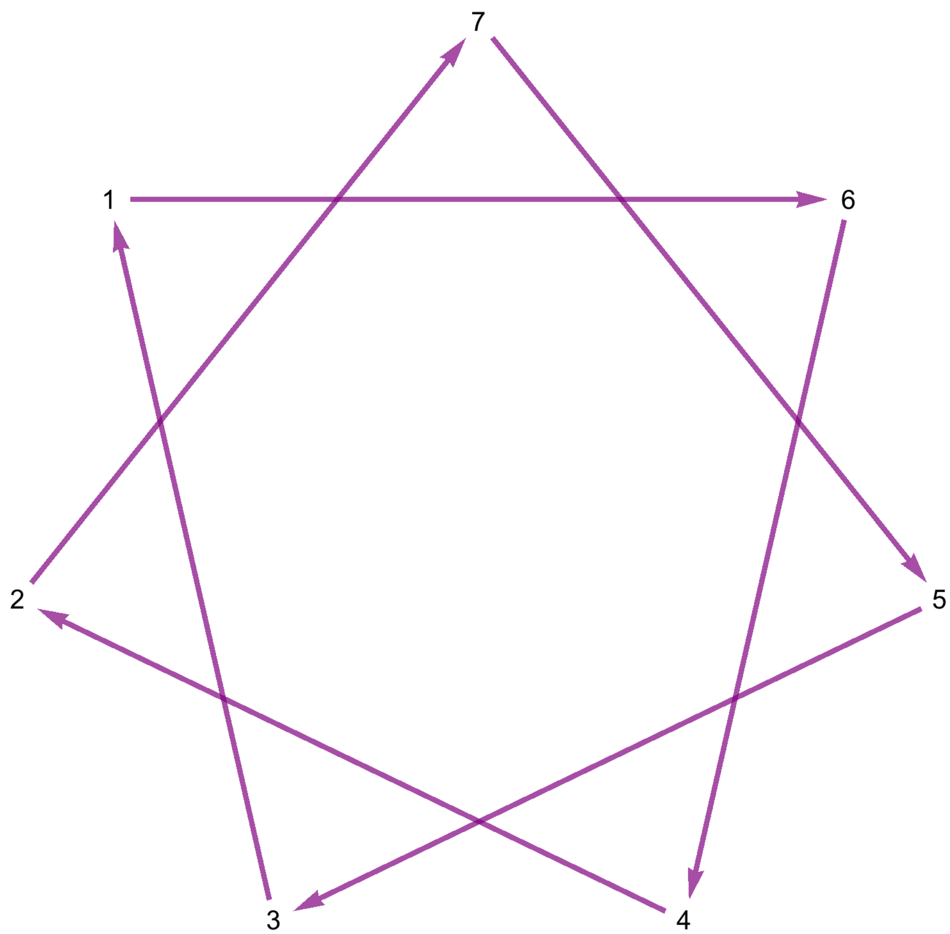
```
makeGrph[b37, 3]
```



Band[3, 7]

```
b57 := {{0, 0, 0, 0, 0, 1, 0}, {0, 0, 0, 0, 0, 0, 1},
   {1, 0, 0, 0, 0, 0, 0}, {0, 1, 0, 0, 0, 0, 0},
   {0, 0, 1, 0, 0, 0, 0}, {0, 0, 0, 1, 0, 0, 0},
   {0, 0, 0, 0, 1, 0, 0}}
```

```
makeGrph[b57, 7]
```



Band[3, 7]

```
bnd17 := {1 ↔ 2, 2 ↔ 3, 3 ↔ 4, 4 ↔ 5, 5 ↔ 6, 6 ↔ 7, 7 ↔ 1}

bnd37 := {1 ↔ 4, 2 ↔ 5, 3 ↔ 6, 4 ↔ 7, 5 ↔ 1, 6 ↔ 2, 7 ↔ 3}

bnd57 := {1 ↔ 6, 6 ↔ 4, 4 ↔ 2, 2 ↔ 7, 7 ↔ 5, 5 ↔ 3, 3 ↔ 1}
```

11 - Polygon

```
ur[11]
```

```
{{{1, 2}, {2, 3}, {3, 4}, {4, 5}, {5, 6}, {6, 7},
  {7, 8}, {8, 9}, {9, 10}, {10, 11}, {11, 12}},
 {{1, 4}, {2, 5}, {3, 6}, {4, 7}, {5, 8}, {6, 9},
  {7, 10}, {8, 11}, {9, 12}, {10, 13}, {11, 14}},
 {{1, 6}, {2, 7}, {3, 8}, {4, 9}, {5, 10}, {6, 11},
  {7, 12}, {8, 13}, {9, 14}, {10, 15}, {11, 16}},
 {{1, 8}, {2, 9}, {3, 10}, {4, 11}, {5, 12}, {6, 13},
  {7, 14}, {8, 15}, {9, 16}, {10, 17}, {11, 18}},
 {{1, 10}, {2, 11}, {3, 12}, {4, 13}, {5, 14}, {6, 15},
  {7, 16}, {8, 17}, {9, 18}, {10, 19}, {11, 20}}}
```

```
circt[11]
```

```
bnd[1, 11] := {{1, 2}, {2, 3}, {3, 4}, {4, 5}, {5, 6},
  {6, 7}, {7, 8}, {8, 9}, {9, 10}, {10, 11}, {11, 1}}
```

```
bnd[3, 11] := {{1, 4}, {2, 5}, {3, 6}, {4, 7}, {5, 8},
  {6, 9}, {7, 10}, {8, 11}, {9, 1}, {10, 2}, {11, 3}}
```

```
bnd[5, 11] := {{1, 6}, {2, 7}, {3, 8}, {4, 9},
  {5, 10}, {6, 11}, {7, 1}, {8, 2}, {9, 3}, {10, 4},
  {11, 5}}
```

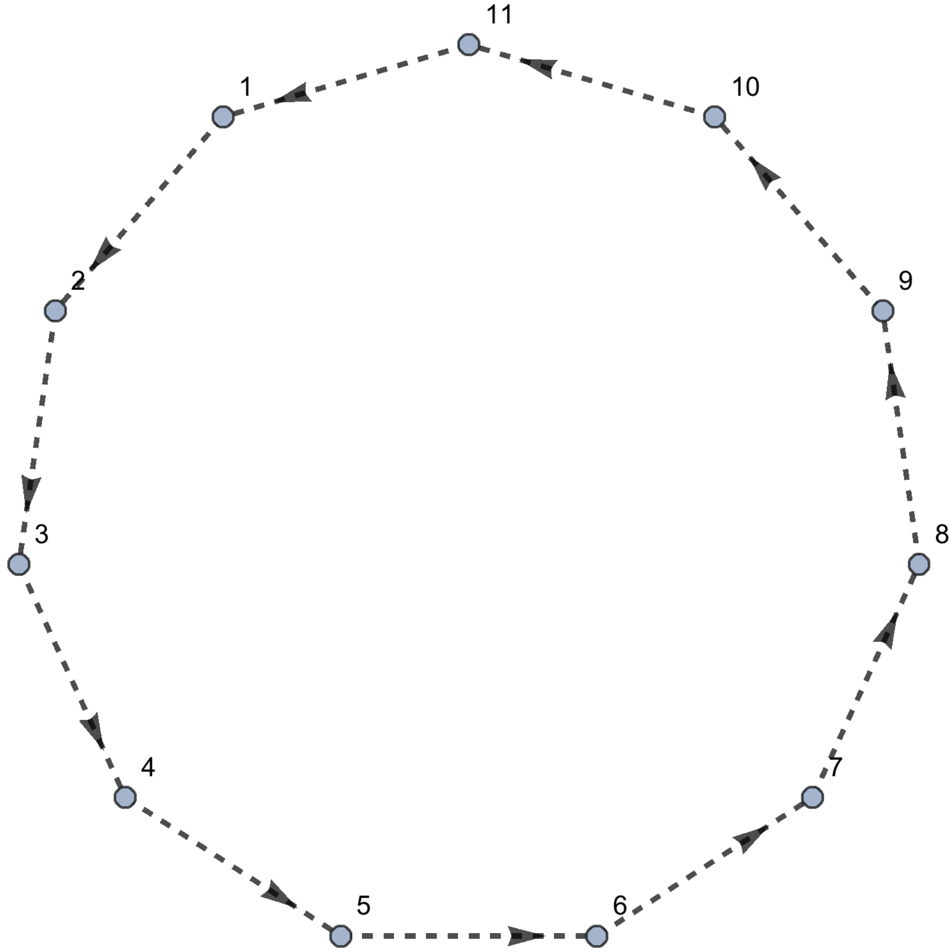
```
bnd[7, 11] := {{1, 8}, {2, 9}, {3, 10}, {4, 11},
  {5, 1}, {6, 2}, {7, 3}, {8, 4}, {9, 5}, {10, 6},
  {11, 7}}
```

```
bnd[9, 11] := {{1, 10}, {2, 11}, {3, 1}, {4, 2},
  {5, 3}, {6, 4}, {7, 5}, {8, 6}, {9, 7}, {10, 8},
  {11, 9}}
```

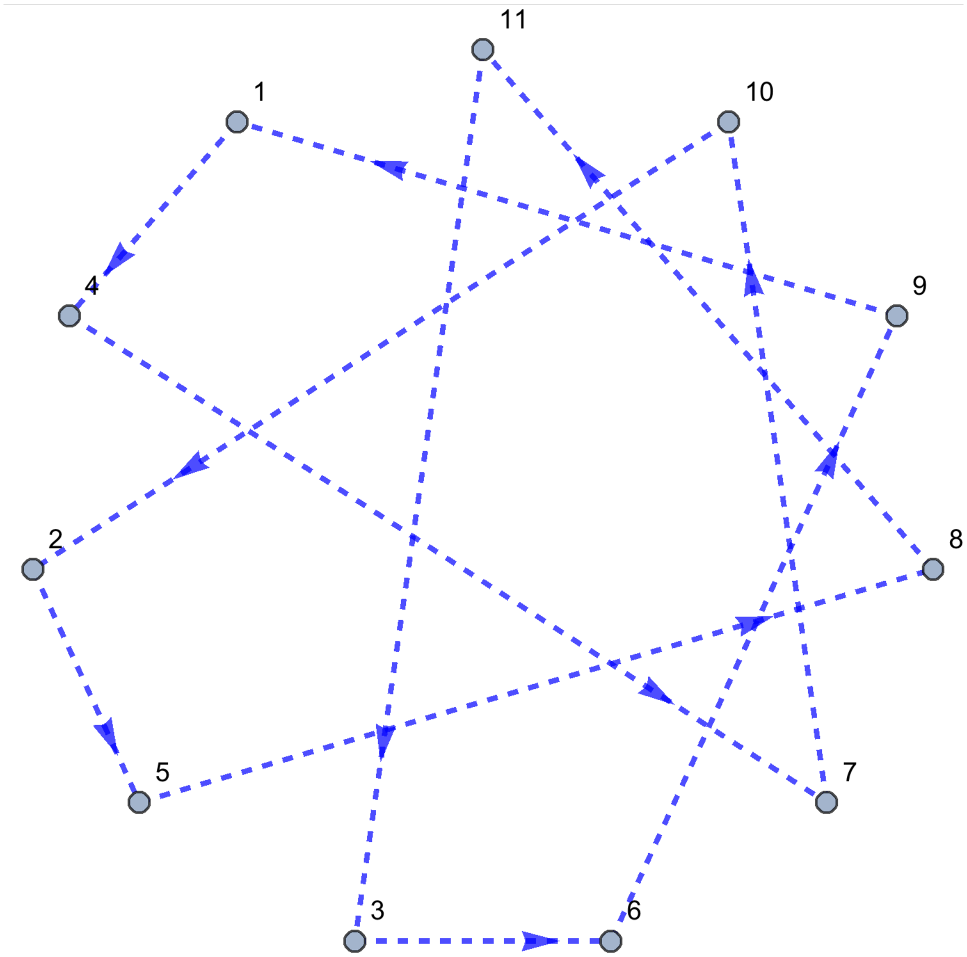
The next 5 graphs show the 5 bands in the Main Partition of the 11-

Polygon

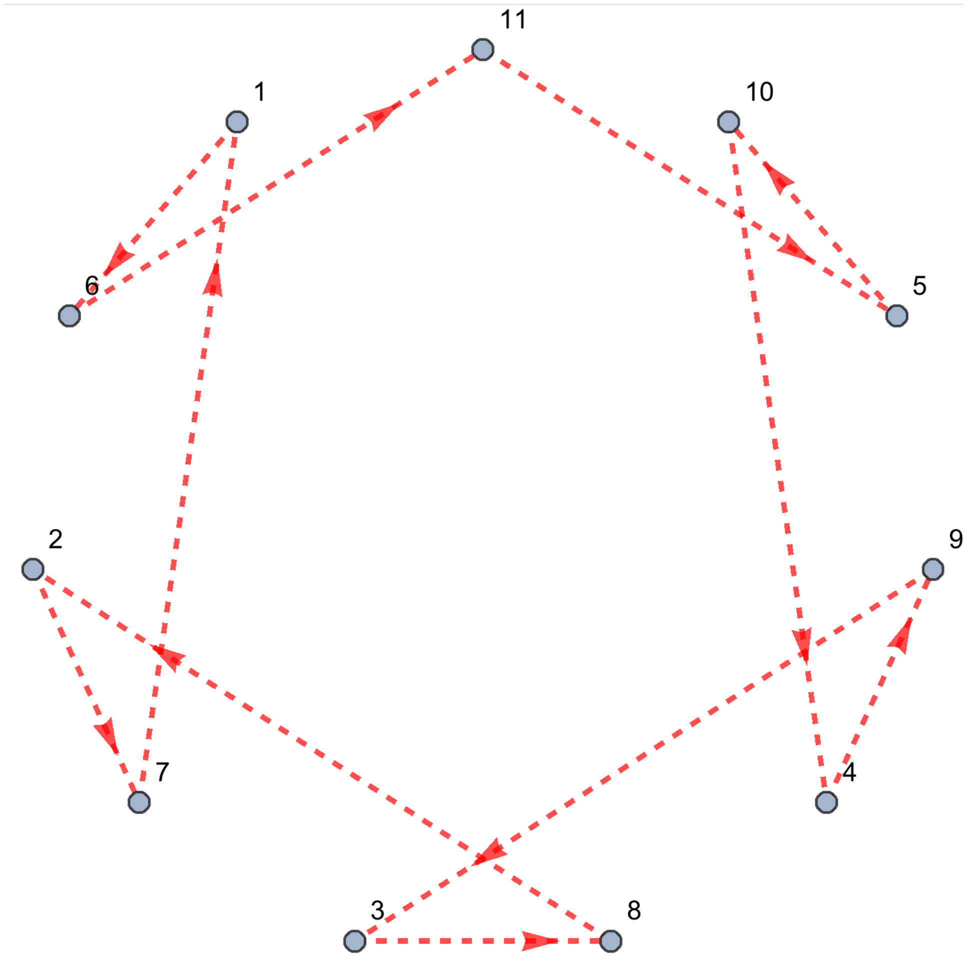
```
fig[Apply[DirectedEdge, bnd[1, 11], 2], 1]
```



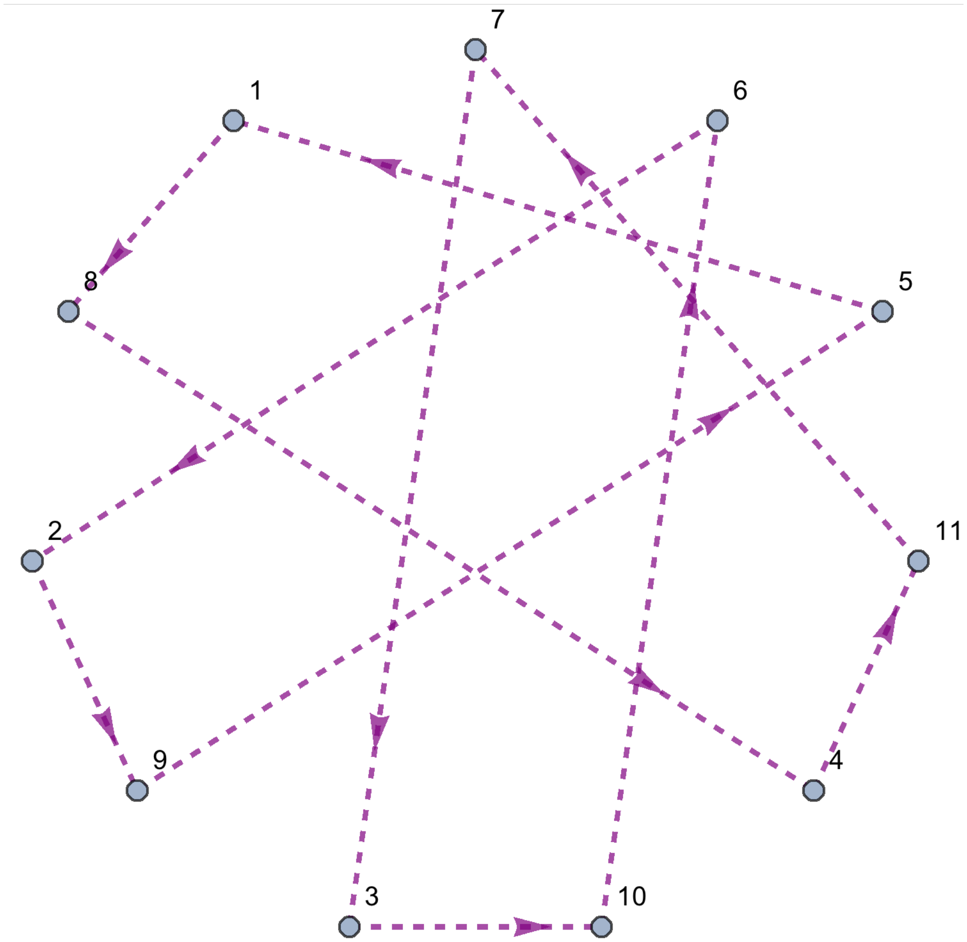
```
fig[Apply[DirectedEdge, bnd[3, 11], 2], 2]
```



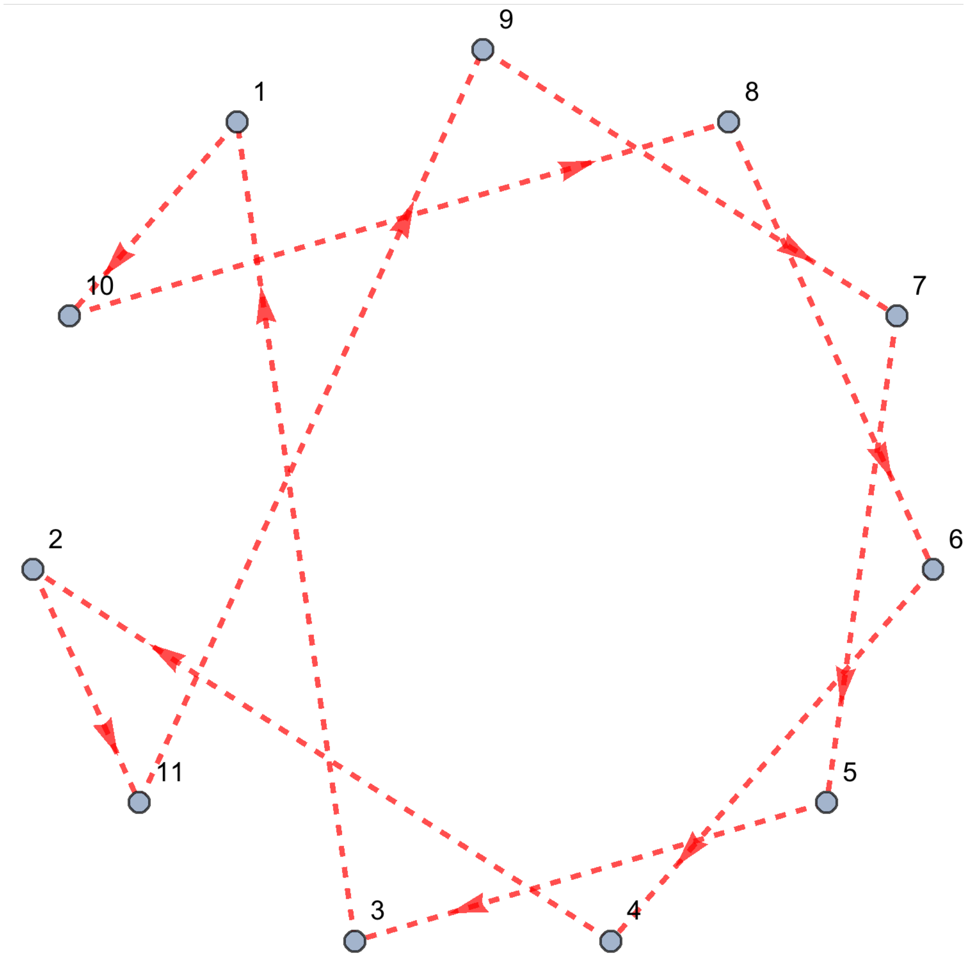
```
fig[Apply[DirectedEdge, bnd[5, 11], 2], 3]
```



```
fig[Apply[DirectedEdge, bnd[7, 11], 2], 7]
```



```
fig[Apply[DirectedEdge, bnd[9, 11], 2], 10]
```



Can non band 7-circuits in the septagon intersect?

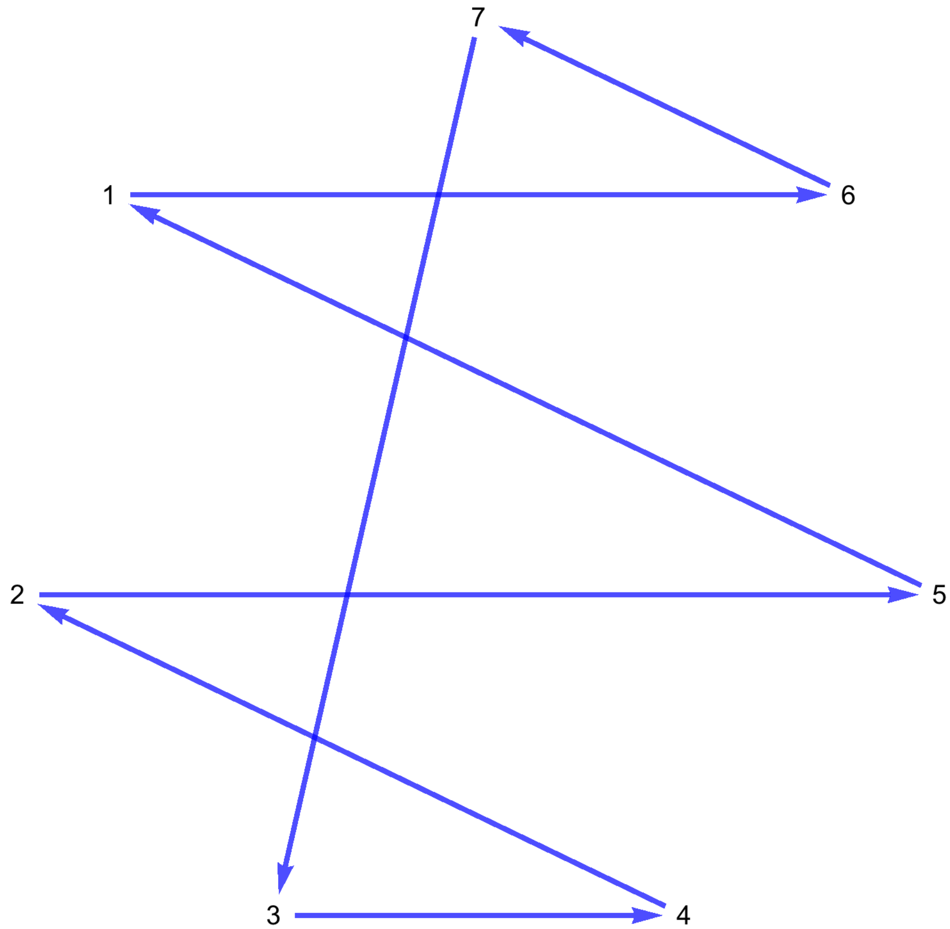
Every band in a prime p-polygon has one p-circuit. In a composite m-polygon only the relatively prime bands have m-circuits. In the latter case the number of these circuits is Φ[m]/2. This raises the question of whether all relatively prime bands have a regular shape. The septagon has 17 7-circuits. The results for the next 3 cases support the proposition that non band m-circuits have intersecting arrows whereas circuits in relatively prime bands do not intersect.

```
cyc7 := MatrixForm[findSomeCycles[A, 7]]
```

```
A

c1 := {{0, 0, 0, 0, 0, 1, 0}, {0, 0, 0, 0, 1, 0, 0},
  {0, 0, 0, 1, 0, 0, 0}, {0, 1, 0, 0, 0, 0, 0},
  {1, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 1},
  {0, 0, 1, 0, 0, 0, 0}}

MatrixForm[c1]
```

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

```
makeGrph[c1, 2]
```



A

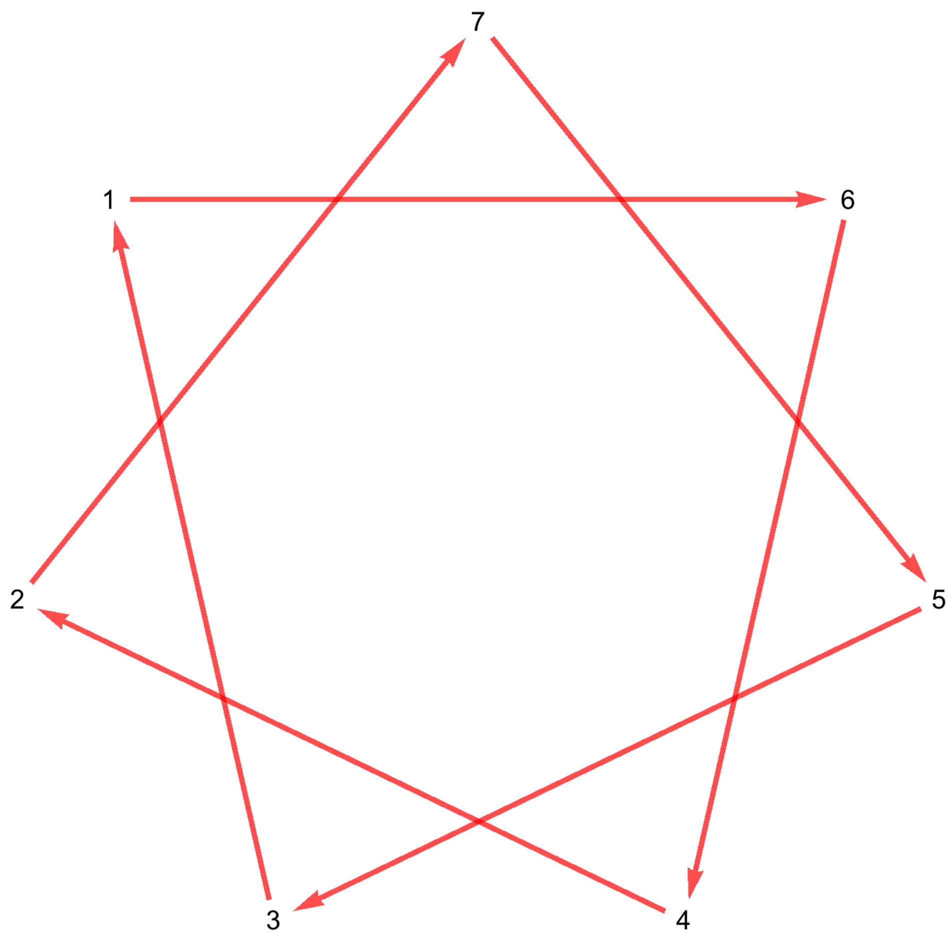
```
c2 := {{0, 0, 0, 0, 0, 1, 0}, {0, 0, 0, 0, 0, 0, 1},
   {1, 0, 0, 0, 0, 0, 0}, {0, 1, 0, 0, 0, 0, 0},
   {0, 0, 1, 0, 0, 0, 0}, {0, 0, 0, 1, 0, 0, 0},
   {0, 0, 0, 0, 1, 0, 0}}
```

Next is Band[5,7].

```
cyc7[[1, 2]]
```

{1 ⧟ 6, 6 ⧟ 4, 4 ⧟ 2, 2 ⧟ 7, 7 ⧟ 5, 5 ⧟ 3, 3 ⧟ 1}

```
makeGrph[c2, 3]
```




```
cyc7[[1, 1]]
```

{1 ⧴ 6, 6 ⧴ 7, 7 ⧴ 3, 3 ⧴ 4, 4 ⧴ 2, 2 ⧴ 5, 5 ⧴ 1}

```
cyc7[[1, 2]]
```

{1 ⧴ 6, 6 ⧴ 4, 4 ⧴ 2, 2 ⧴ 7, 7 ⧴ 5, 5 ⧴ 3, 3 ⧴ 1}

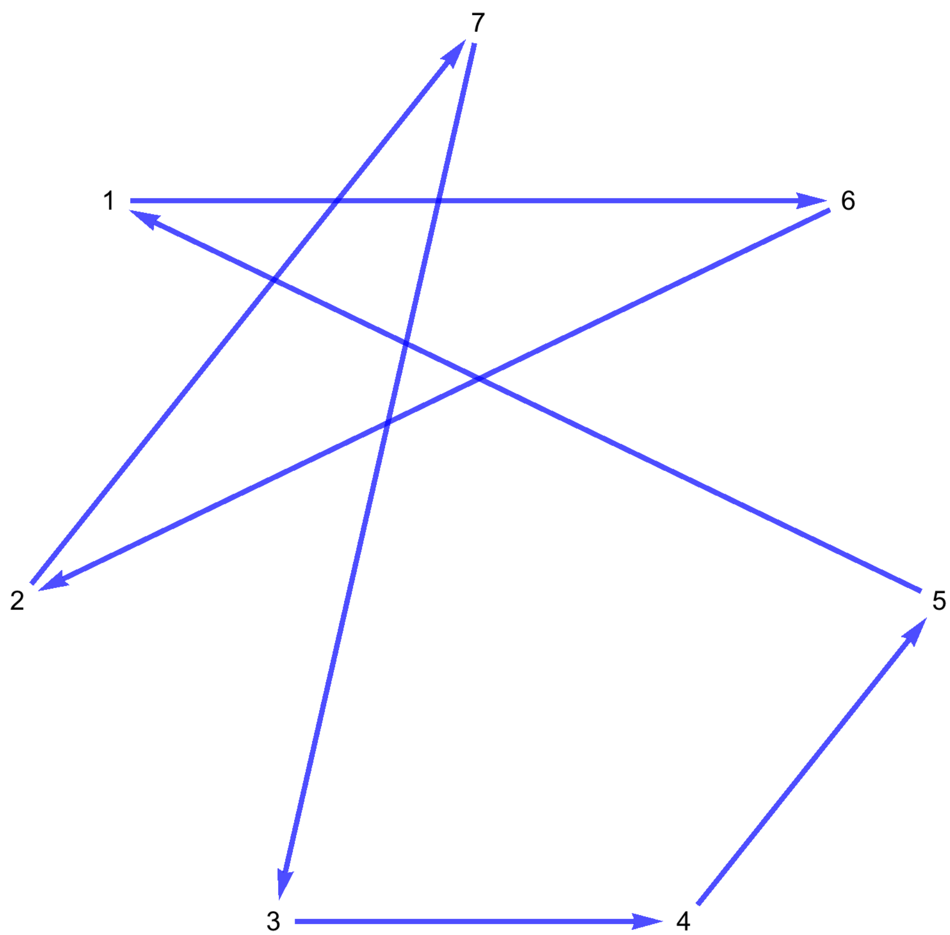
```
cyc7[[1, 3]]
```

{1 ⧴ 6, 6 ⧴ 2, 2 ⧴ 7, 7 ⧴ 3, 3 ⧴ 4, 4 ⧴ 5, 5 ⧴ 1}

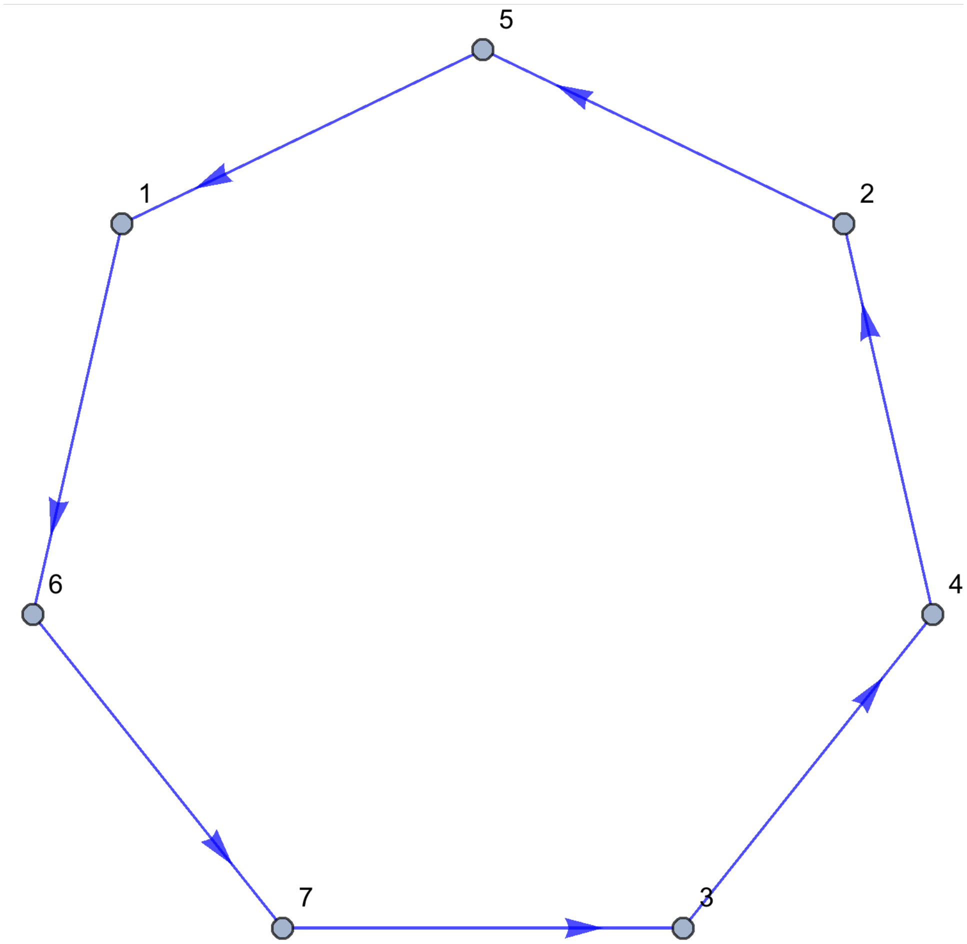
A

```
c3 := {{0, 0, 0, 0, 0, 1, 0}, {0, 0, 0, 0, 0, 0, 1},
  {0, 0, 0, 1, 0, 0, 0}, {0, 0, 0, 0, 1, 0, 0},
  {1, 0, 0, 0, 0, 0, 0}, {0, 1, 0, 0, 0, 0, 0},
  {0, 0, 1, 0, 0, 0, 0}}
```

```
makeGrph[c3, 2]
```



```
fig[cyc7[[1, 1]], 2]
```

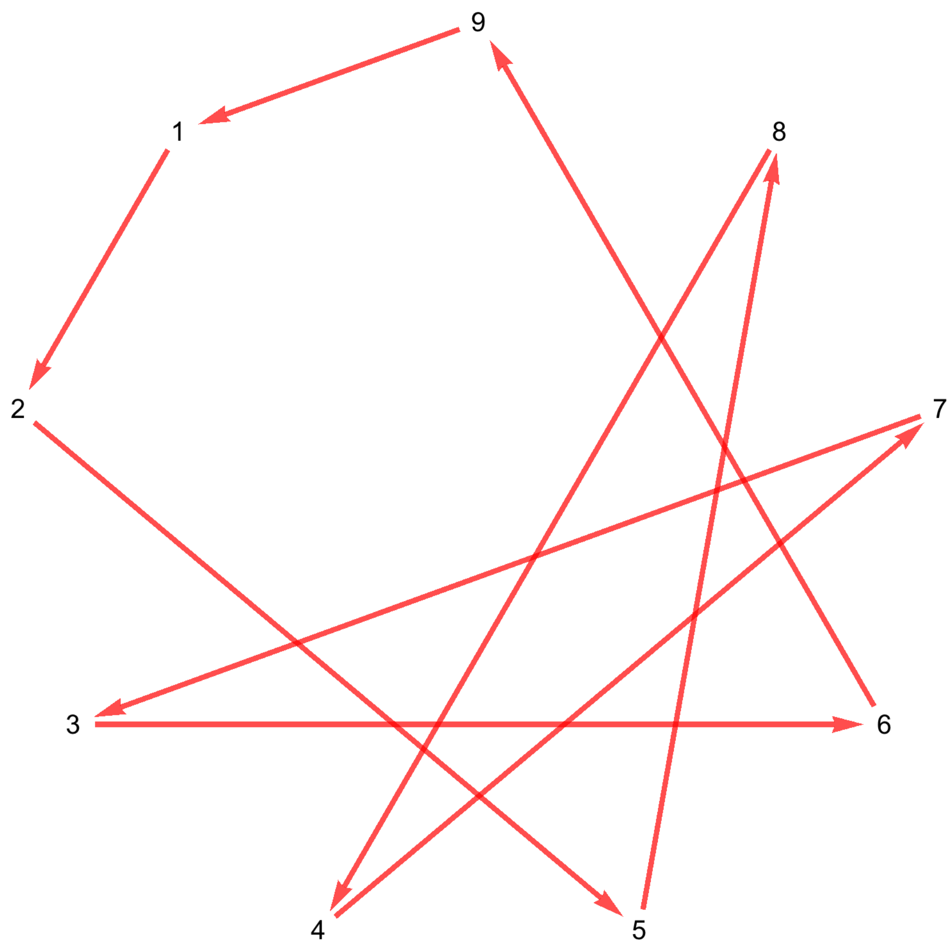



```
makeA[9]

A

c9t := {{0, 1, 0, 0, 0, 0, 0, 0, 0},
   {0, 0, 0, 0, 1, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 1, 0, 0, 0},
   {0, 0, 0, 0, 0, 0, 1, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 1, 0},
   {0, 0, 0, 0, 0, 0, 0, 0, 1}, {0, 0, 1, 0, 0, 0, 0, 0, 0},
   {0, 0, 0, 1, 0, 0, 0, 0, 0}, {1, 0, 0, 0, 0, 0, 0, 0, 0}}
```

```
MatrixForm[c9t]
```

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

```
makeGrph[c9t, 3]
```



Non band 9-circuits intersect.