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Winter February 3, 2022

Making Nöther Matrics

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Available at: https://works.bepress.com/lester_telser/145/

2/3/22

Circuits Using Nöther Matrics [NM]

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An arrow, a[i,j], in an m-polygon goes from vertex i, its source, to vertex j, its destination, i ⟷ j. It is not two dimensional in a Nöther Algebra. It is a non-zero term in an sXs matrix whose terms are zero with this one exception, the term in row i, column j equals 1. In my work arrows always belong to circuits. A circuit with s arrows has a given relative order such that the destination of arrow a[i,j], is the source of the next arrow, a[j,k]. The source of the arrow a[j,k] is the destination of its predecessor, a[i,j]. Arrows in a circuit describe a round trip. Every arrow in an m-polygon belongs to at least one circuit. They never are alone. To show this the symbol for an arrow uses a special notation. Arrow a[i,j] in an s-circuit has the special symbol G[i,j,s].

A matric in N-algebra has s rows and columns to represent a one-way arrow in an s-circuit. Matric distinguishes it from matrix. An s-circuit is the sequence

{a[i_1,j_1], a[i_1,j_2], a[i_2,j_3] ,..., a[i_s,i_1]}.

As a matric it becomes the product

G[i_1,j_1,s]. G[i_1,j_2,s]. G[i_2,j_3,s] ,..., G[i_s,j_s,s] = G[i_1, i_1,s], an sXs matric with one term =1 at row i_1, column i_1, a diagonal. All its other terms are 0. Because the same s appears in each G[.], in most cases s is omitted since it is clear from the context. This algebra uses an sXs matrix to identify an arrow in the s-circuit.

1. Simple Circuits

```
In[•]:= Array[0 &, {3, 3}]

Out[•]= {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

In[•]:=
```
MatrixForm[Array[0 &, {3, 3}]]
```

Out[•]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

In[•]:=
```
G[1, 2] := {{0, 1, 0}, {0, 0, 0}, {0, 0, 0}}
```

In[•]:=
```
MatrixForm[G[1, 2]]
```

Out[•]//MatrixForm=

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

In[•]:=
```
G[2, 3] := {{0, 0, 0}, {0, 0, 1}, {0, 0, 0}}
```

In[•]:=
```
MatrixForm[G[2, 3]]
```

Out[•]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

In[•]:=
```
G[3, 1] := {{0, 0, 0}, {0, 0, 0}, {1, 0, 0}}
```

In[•]:=
```
MatrixForm[G[3, 1]]
```

Out[•]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

The symbol for the simple circuit (1, 2, 3) is the product of 3 matrices, one matric for each one-way arrows in the simple 3-circuit. The term indicates a round trip starting from vertex 1. It appears as the first term on the diagonal of the N-matric that shows the product of the 3 one-way arrows in their matrics.

In[∘]:= `MatrixForm[G[1, 2].G[2, 3].G[3, 1]]`

Out[∘]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

In[∘]:= `G[1, 1] := {{1, 0, 0}, {0, 0, 0}, {0, 0, 0}}`

In[∘]:= `MatrixForm[G[1, 1]]`

Out[∘]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

In[∘]:= `G[1, 2].G[2, 3].G[3, 1]`

Out[∘]= `{{1, 0, 0}, {0, 0, 0}, {0, 0, 0}}`

As claimed G[1,1] = **G[1, 2].G[2, 3].G[3, 1]**

For the 3-circuit starting with vertex 3, (3, 1, 2), the matric has its one non zero term in row 3, column 3. The reader can show matric for this product is G[3,3].

In[∘]:= `MatrixForm[G[3, 1].G[1, 2].G[2, 3]]`

Out[∘]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

For the 3-circuit starting with vertex 2, (2, 3, 1,), the matric has one non zero term in row 2, column 2.

In[•]:= `MatrixForm[G[2, 3].G[3, 1].G[1, 2]]`

Out[•]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

These three NA matrices for the same simple 3 - circuit differ. Each has its own matric. An incorrect order of these matrics gives 0 as the next example shows.

In[•]:= `MatrixForm[G[2, 3].G[1, 2].G[3, 1]]`

Out[•]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Let v[i,j] denote the value of the one-way arrow a[i,j]. The term v[1,2] denotes the value of the arrow a[1,2]. The matric showing the value of the round trip starting from vertex 1 and returning to vertex 1 is the following matric V[1]. It has only one non term (1,1) on the diagonal.

$$V[1] := \begin{pmatrix} v[1,2]*v[2,3]*v[3,1] & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The second matric is V[2].

$$V[2] := \begin{pmatrix} 0 & 0 & 0 \\ 0 & v[2,3]*v[3,1]*v[1,2] & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The third matric is V[3]

$$V[3] := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & v[3,1]*v[1,2]*v[2,3] \end{pmatrix}$$

Each V[i] is a different matric for the value of the different round trips of the simple 3-circuit. Although the round trips start from different vertexes, they have equal value shown by their matrics.

2. Super Circuits

Warning: Mathematica arranges the terms in the product in lexicographical order. This is incorrect. The sequence of arrows in the circuit is the correct order.

A super circuit is a set of simple circuits whose *arrows* do not intersect and share at least one *vertex*. The next two simple circuits

{(5,1),(1,4),(4,2),(2,5)} and {(5,3),(3,4),(4,5)}

in the pentagon share no arrows and share vertex 5. They make the following super circuit. Its 5X5 matric is G[5,5]

{(5,1),(1,4),(4,2),(2,5),(5,3),(3,4),(4,5)}.

The procedure step by step follows. To save space I replace G by g in the matric.

In[∘]:=
```
g14 := {{0, 0, 0, a[1, 4], 0}, {0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}}
```

In[∘]:=
```
g53 := {{0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0}, {0, 0, a[5, 3], 0, 0}}
```

In[∘]:=
```
g25 := {{0, 0, 0, 0, 0}, {0, 0, 0, 0, a[2, 5]},
  {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}}
```

In[∘]:=
```
g31 := {{0, 0, 0, 0, 0}, {0, 0, 0, 0, 0},
  {a[3, 1], 0, 0, 0, 0}, {0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0}}
```

In[∘]:=
```
g12 := {{a[1, 2], 0, 0, 0, 0}, {0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}}
```

```
In[•]:= g23 := {{0, 0, 0, 0, 0}, {0, 0, a[2, 3], 0, 0},
          {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}}
```

```
In[•]:= g34 := {{0, 0, 0, 0, 0}, {0, 0, 0, 0, 0},
          {0, 0, 0, a[3, 4], 0}, {0, 0, 0, 0, 0},
          {0, 0, 0, 0, 0}}
```

```
In[•]:= g45 := {{0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0},
          {0, 0, 0, 0, a[4, 5]}, {0, 0, 0, 0, 0}}
```

```
In[•]:= g51 := {{a[5, 1], 0, 0, 0, 0}, {0, 0, 0, 0, 0},
          {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}}
```

```
In[•]:= g42 := {{0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0},
          {0, a[4, 2], 0, 0, 0}, {0, 0, 0, 0, 0}}
```

The next 3 steps make the following super circuit.
(g51.(g14.g42)).g25) (g53.g34).g45) =
((g51.(g14.g42)).g25).((g53.g34).g45)
Note the positions of the parentheses.

```
In[•]:= ((g51.(g14.g42)).g25)

{{0, 0, 0, 0, a[1, 4] * a[2, 5] * a[4, 2] * a[5, 1]},
 {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0}}
```

```
In[•]:= (g53.g34).g45

{{0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0}, {0, 0, 0, 0, a[3, 4] * a[4, 5] * a[5, 3]}}
```

In[•]:= ((g51.(g14.g42)).g25).((g53.g34).g45)

```
{{0, 0, 0, 0, a[1, 4] * a[2, 5] * a[3, 4] * a[4, 2] *
   a[4, 5] * a[5, 1] * a[5, 3]}, {0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}}
```

This gives the super circuit in the first term on the diagonal of the matric product **((g51.(g14.g42)).g25).((g53.g34).g45)**. Mathematica shows the factors in lexicographical order, not the correct super circuit order. Next is the procedure for g23.g31 = g21

In[•]:= g23.g31

```
{{0, 0, 0, 0, 0}, {a[2, 3] * a[3, 1], 0, 0, 0, 0},
 {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}}
```

g23.g31 = g21

In[•]:= (g23.g31).g12

```
{{0, 0, 0, 0, 0}, {a[1, 2] * a[2, 3] * a[3, 1], 0, 0, 0, 0},
 {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}}
```

(g23.g31).g12 = g21.g12 = g22

In[•]:= (g51.g14).g45

```
{{0, 0, 0, 0, a[1, 4] * a[4, 5] * a[5, 1]}, {0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}}
```

The final step is (g51.g14).g45 = g54.g45. It is the fifth term of the diagonal matric g[5,5].

In[•]:= ((g51.g14).g45)

```
{{0, 0, 0, 0, a[1, 4] * a[4, 5] * a[5, 1]}, {0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}}
```

((g51.g14).g45).((g23.g31).g12)

```
{{0, 0, 0, 0, 0},
 {0, 0, 0, 0, a[1, 2] * a[1, 4] * a[2, 3] * a[3, 1] *
   a[4, 5] * a[5, 1]}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0}}
```

3. Valuation of a Circuit

The results in the preceding two sections show that the value of a circuit is given by its matric whose terms are arrow values.

Let $1+\rho_{\mathrm{i,j}}$ denote the value of arrow a[i, j]. The value of the 3 – circuit, V[θ], is

$$V[\theta] = (1+\rho_{12}) * (1+\rho_{23}) * (1+\rho_{31}) =$$
$$= 1+\rho_{12}+\rho_{23}+\rho_{31}+\rho_{12}\,\rho_{23}+\rho_{12}\,\rho_{31}+\rho_{23}\,\rho_{31}+\rho_{12}\,\rho_{23}\,\rho_{31}.$$

The arithmetic mean of the value of the arrows in a circuit exceeds the geometric mean of their values unless $\rho_{\mathrm{ij}}=\rho$. Given the mean, the geometric mean varies inversely with the dispersion of its terms. The closer together are $\rho_{i,j}$, the closer the geometric mean to the arithmetic mean. Given the arithmetic mean, the more nearly equal are $(1+\rho_{i,j})$ in the circuit, the bigger the circuit value.

For $\rho_l \geq 0$,

(1) $\Sigma_1^n(1+\rho_i) < \Pi_1^n(1+\rho_i)$.

The subscript identifies the owner of the arrow . Let w_i denote the return to owner i, i = 1, 2, ... , n, n = m(m-1)/2.

Let V(θ) in (2) denote the value of circuit θ .

(2) $V(\theta) = \Pi_1^n(1+\rho_i)$.

The core constraint is inequality (3)

(3) $\Sigma_{\mathrm{i}\epsilon\theta}\, w_i \geq V(\theta)$

If $w_i=1+\rho_i$, then the smallest acceptable payoff to owner i cannot satisfy (3) owing to inequality (1). The core constraints are not feasible. Changing to logs gives feasibility. Let the owner of vertex i get log w_i instead of w_i. The log of the value replaces value. Instead of (3), there is

(4) $\Sigma_{\mathrm{i}\epsilon\theta}\log w_i \geq \log V(\gamma) = \log \Pi_{\mathrm{i}\epsilon\theta}(1+\rho_i)=\Sigma_{\mathrm{i}\epsilon\theta}\log(1+\rho_i)$.

Log w_i can satisfy (4) with equality even if some ρ_i=0. Such arrows are unproductive. Their value =1 and log 1 = 0.

4 Undirected Edges

The undirected edge from vertex i to vertex j is denoted by i•—•j. It differs from the pair of oppositely oriented arrows i•→j and j•→ i. The Nöther matric for the undirected edge connecting vertex i and vertex j is

```
In[•]:= a := {{0, 1}, {1, 0}}

In[•]:= MatrixForm[a]

Out[•]//MatrixForm=
```

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The matric for the arrow 1•→2 is a. The matric for the arrow 2•→1 is b

```
In[•]:= b := {{1, 0}, {0, 1}}

In[•]:= MatrixForm[b]

Out[•]//MatrixForm=
```

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

```
In[•]:= a.b

Out[•]= {{0, 1}, {1, 0}}
```

Thus a.b = a and {1•→2, 2•→1} = 1•→2 as required.

```
In[•]:= Det[a.a]

Out[•]= 1

In[•]:= Transpose[a]

Out[•]= {{0, 1}, {1, 0}}
```

The transpose of a is b.

```
In[•]:= tl := {1 ⟷ 2, 2 ⟷ 3, 3 ⟷ 4, 4 ⟷ 2, 2 ⟷ 1}

In[•]:= Array[0 &, {4, 4}]

Out[•]= {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}

In[•]:= g12 := {{0, 1, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0},
          {0, 0, 0, 0}}

In[•]:= g23 := {{0, 0, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 0},
          {0, 0, 0, 0}}

In[•]:= g34 := {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 1},
          {0, 0, 0, 0}}

In[•]:= g42 := {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0},
          {0, 1, 0, 0}}

In[•]:= g21 := {{0, 0, 0, 0}, {1, 0, 0, 0}, {0, 0, 0, 0},
          {0, 0, 0, 0}}

In[•]:= g12.g23

In[•]:= g13 := {{0, 0, 1, 0}, {0, 0, 0, 0}, {0, 0, 0, 0},
          {0, 0, 0, 0}}

In[•]:= g13.g34

In[•]:= g14 := {{0, 0, 0, 1}, {0, 0, 0, 0}, {0, 0, 0, 0},
          {0, 0, 0, 0}}

In[•]:= g14.g42

In[•]:= g12 := {{0, 1, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0},
          {0, 0, 0, 0}}
```

In[•]:= `g12.g21`

In[•]:= `g11 := {{1, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}`

The Nöther matric of tl is the product of the 5 matrics; g12 . g23 . g34 . g42 .g21 = g11, the circuit tl

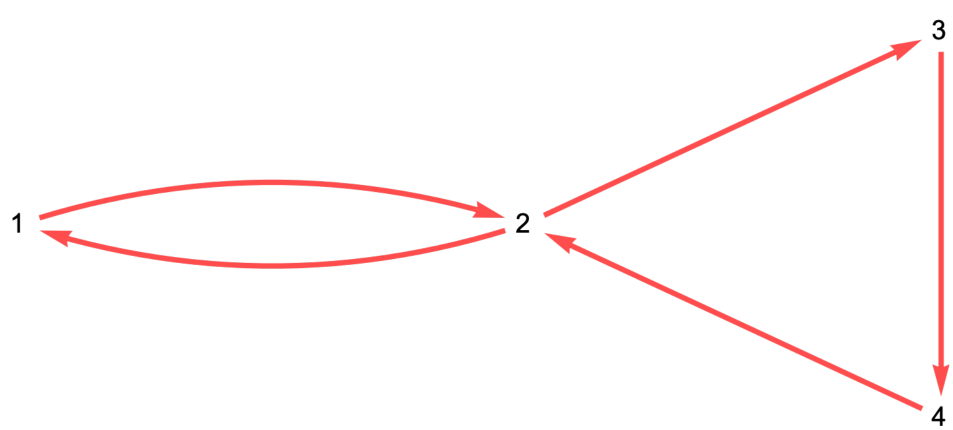
`tl := {1 ⟷ 2, 2 ⟷ 3, 3 ⟷ 4, 4 ⟷ 2, 2 ⟷ 1}`

The adjacency matrix for tl is atl

In[•]:= `atl := {{0, 1, 0, 0}, {1, 0, 1, 0}, {0, 0, 0, 1}, {0, 1, 0, 0}}`

In[•]:= `gatl := makeGrph[atl, 3]`

In[•]:= `gatl`

Out[•]=


The Undirected Edge 1⟷ 2 becomes the pair of oppositely Directed Edges1⟶ 2 & 2⟶ 1 in the graph.

In[•]:= `EulerianGraphQ[gatl]`

Out[•]= `True`

In[•]:=
```
HamiltonianGraphQ[gatl]
```

Out[•]=
```
False
```

Therefore, an Eulerian graph can have two oppositely oriented arrows (Undirected Edge) made by two Directed Edges. This is impossible for a Hamiltonian graph.

Program