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The Grammar of Mathematics

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Whole numbers have names using one or more words. These names usually do not describe relations among numbers. Sometimes the definitions are accurate descriptions of their relations to other names of numbers. For example, three is the same number as two plus one. How do we know this definition is accurate. How do we know that four does not equal eight minus three. Given the names of numbers, their relations come from statements such as 'This number is the same as that number plus one.'
Is this a description of a relation between words or is it a verifiable assertion about the relation between certain pairs of numbers and the operation 'plus'? Verification is by experiment or by the rules of the grammar for these special words, names of numbers. The more common name for these grammatical rules is mathematics. Verification can be difficult.

A finite number of either digits or words often do not suffice. We can express numbers with power series that have powers of 10^{-n} and the 10 digits for the power coefficients. An exact numerical result is possible only for rationals because rationals require only a finite number of terms in their decimal power series.

Consider pi. The word pi is the name of a set of infinite series that approximate the circumference of a circle with a positive error that decreases as the number of terms in the series increases. The error remains positive as long as the number of terms in the series remains finite. The name of this set is not a number.

By definition a rational number is a ratio of two integers $\frac{a}{b}$, b ≠ 0. A rational number produces a series with a finite number of terms using the division algorithm. A decimal power series with a finite number of terms also represents a rational number. To use a decimal power series to represent an irrational number admits infinity as a number. Once you accept the axiom that one plus a number is a number, numbers have no finite upper bound so acceptance of infinity as a number follows. The Axiom of Choice allows independent random random draws from the ten digits {0, 1, 2, ... , 9} for the coefficients of a decimal power series to represent numbers in the open interval (0, 1).

No coefficient in the decimal power series for an irrational number is predictable. Each is an independent, random draw with probability 1/10 from the 10 digits {0, 1, ..., 9}. Therefore, a random walk does not generate the coefficients in the decimal power series. In a random walk each term takes a random step from its current position contradicting sequential independence of the random draws. A moving average of the terms in a perfect random sequence remains constant. Successive averages also remain constant. A more demanding test employs auto regressions. An autoregression with n terms, n as big as you like, could not predict the (n+1)/st term in the series. The autoregression would have the same standard deviation as the set of 10 digits, $\sqrt{\frac{55}{6}}$. Non parametric tests for independent, identically distributed random draws would give equivalent results (See Goodman & Kruskal, 1979).

The power series with n terms whose coefficients are independent random draws from the ten digits is a rational number because it can be converted into a fraction a/b. It could differ from an irrational number by a positive amount ϵ_n as small as you please by choosing n big enough. The infimum for ϵ_n is zero. This procedure does not contradict the view that predictable terms can also yield a rational number. A finite subsequence of the power series whose coefficients are random, independent draws is a rational number close to an irrational number whose decimal power series has an infinite number of terms each an independent random draw from the 10 integers. Every irrational number has a sup given by an irrational number. To show this, increase a coefficient by one in the power series. This is possible in a series with at least one integer coefficient not equal to 9.

Let a given sequence X = $\{x_n$: n = 1, 2, ..., } be a candidate for irrational status. Presented with the sequence of coefficients but not knowing its source, even if it could pass the preceding tests for random sequences, this would not reveal the status of X. Statisticians are familiar with this situation. Given a sequence X that may include numbers from a deterministic non linear process, it would be difficult to infer its structure by trial and error alone. Ignorance is not the same as randomness. No one could predict the coefficients of the random sequence defined in advance even in principle. A non linear sequence may be hard, but is not impossible, to detect. It does not resemble a random sequence.

As n increases the decimal power series for a rational approaches an irrational. A predictable sequence is a rational number. It is nearly impossible to obtain it by chance. A finite sub sample of a power series for $\sqrt{2}$ would not differ from a rational number, Pythagoras notwithstanding.

The irrational number pi is the same for Julius Caesar and Napoleon Bonaparte two thousand years apart. Given X, compute the moving average $\overline{X_n}$. Make the sequence $\{x_n - \overline{X_n}\}$. If it passes tests for independent random draws from {0, 1, ..., 9}, then it is an irrational number.

Because the width of a cut is positive for all finite n, exact numerical results that involve irrationals are impossible. Applications of mathematics to problems in the real world typically use irrationals so cannot obtain exact numerical results. Just as real world experience led to the discovery and invention of irrationals, so too we must accept imperfect numerical results.

Summary
1. Each decimal power series for an irrational has a finite sub series that represents a rational. The number of these subseries is countably infinite. The number of permutations of these series is countably infinite. Hence the total number is Aleph_o^2.
2. Each irrational has a decimal power series whose coefficients are generated by independent random draws from the 10 integers 0 to 9.

The number of these series is Aleph_1.

References

Goodman, Leo A . & William H. Kruskal, 1979, Measures of Association for Cross Classifications. New York: Springer-Verlag.

Kolmogorov, A. N. 2018 [1933]. Foundations of the Theory of Probability. 2nd English ed. Trans. Nathan Morrison. New York: Dover.
Kolmogorov A . N. & S. V. Fomin. 1975. Introductory Real Analysis. Trans. Richard A. Silverman. New York: Dover.

Appendix

$S[n] = 1 + r + r^2 + \ldots + r^n$
$r\,S[n] = r + r^2 + \ldots + r^n + r^{n+1}$
$rS[n] - S[n] = r^{n+1} - 1$
Multiply by -1 and obtain
$S[n]\,(1 - r) = 1 - r^{n+1} > 0$ if $0 < r < 1$.
$S[n] = 1 - r^{n+1} \big/ (1 - r)$
In my case $r = 1/10$.

In[•]:= **Range[0, 9]**

Out[•]= {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}

In[•]:= **StandardDeviation[{0, 1, 2, 3, 4, 5, 6, 7, 8, 9}]**

Out[•]= $\sqrt{\frac{55}{6}}$

In[•]:= $\mathbf{N}\left[\sqrt{\frac{55}{6}}, 6\right]$

Out[•]= 3.02765