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From the Seven Bridges of Königsberg to the Circuits in Hyde Park

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Fair even m-polygons have no bands, partitions nor permutation matrixes in contrast to fair odd-m polygons that have bands, partitions and permutation matrixes for their bands. They also have perimeters that are simple Hamiltonian m-circuits. This is not the case for fair *even* m-polygons because their vertexes have an *odd* number of arrows unlike fair odd m-polygons whose vertexes have an *even* number of arrows. Their characteristic functions show other differences. Every simple circuit has a positive characteristic function on the semi-open interval (0,1]. The characteristic function for fair *even* m-polygon is negative for all x in (0,1].

Quasi - band is the term I use that corresponds to true bands. Quasi-bands are useful for calculating Eulerian cycles in even m-polygons. The number of quasi-bands = m/2 - 1 = (m-2)/2. The number of arrows in a m-polygon does not depend on the parity of m, m(m-1)/2 for even and odd m.

Sexagon [Latin]

quasi - bands arrow prospects

b1	b3	b5	u2	u4
1, 2	1, 4	1, 6	3, 1	5, 1
2, 3	2, 5	–	4, 2	6, 2
3, 4	3, 6	–	5, 3	–
4, 5	–	–	6, 4	–
5, 6	–	–	–	–

In[•]:=
```
makeA[6]
```

In[•]:=
```
LG1 := {{0, 0, 0, 0, 0, 1}, {0, 0, 1, 0, 0, 0},
  {1, 0, 0, 0, 0, 1}, {0, 0, 0, 0, 1, 0},
  {0, 0, 1, 0, 0, 0}, {0, 1, 0, 1, 0, 0}}
```

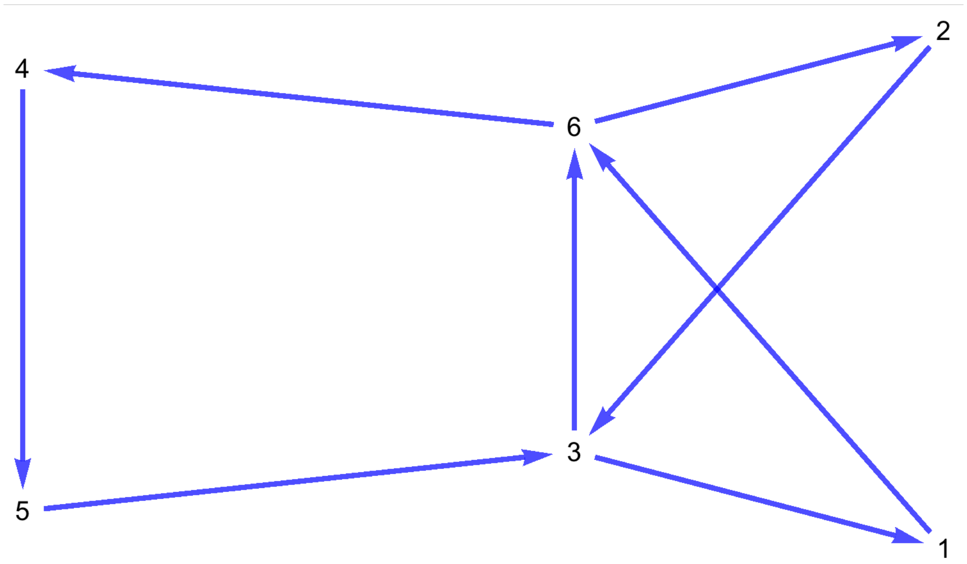
In[∘]:=
```
MatrixForm[LG1]
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

In[∘]:=
```
makeGrph[LG1, 2]
```

Out[∘]=



In[∘]:=
```
Apply[DirectedEdge, arrw, 1]
```

In[∘]:=
```
unused := {1 ↔ 2, 3 ↔ 4, 5 ↔ 6}
```

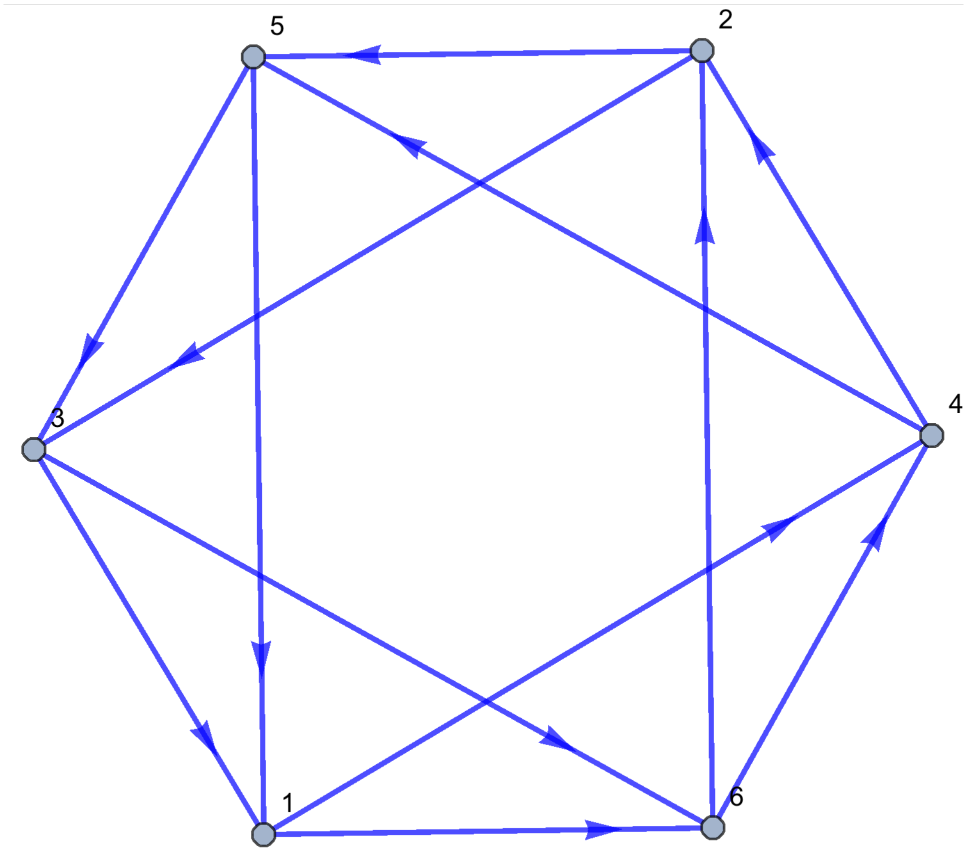
lg1 is not a simple circuit.

In[∘]:=
```
lg1 := {1 ↔ 6, 6 ↔ 2, 2 ↔ 3, 3 ↔ 1, 1 ↔ 4, 4 ↔ 5,
   5 ↔ 3, 3 ↔ 6, 6 ↔ 4, 4 ↔ 2, 2 ↔ 5, 5 ↔ 1}
```

In[∘]:=
```
l1 := fig[lg1, 2]
```

In[∘]:= l1

Out[∘]=



Graph of Maximal Eulerian cycle. It is not a partition of the sexagon

In[∘]:= FindEulerianCycle[l1]

Out[∘]= {{1 ⧴ 6, 6 ⧴ 2, 2 ⧴ 3, 3 ⧴ 1, 1 ⧴ 4, 4 ⧴ 2, 2 ⧴ 5, 5 ⧴ 3, 3 ⧴ 6, 6 ⧴ 4, 4 ⧴ 5, 5 ⧴ 1}}

```
In[∘]:= u6 := {{0, 0, 0, 1, 0, 1}, {0, 0, 1, 0, 1, 0},
   {1, 0, 0, 0, 0, 1}, {0, 1, 0, 0, 1, 0},
   {1, 0, 1, 0, 0, 0}, {0, 1, 0, 1, 0, 0}}
```

In[∘]:= Det[u6 - x IdentityMatrix[6]]

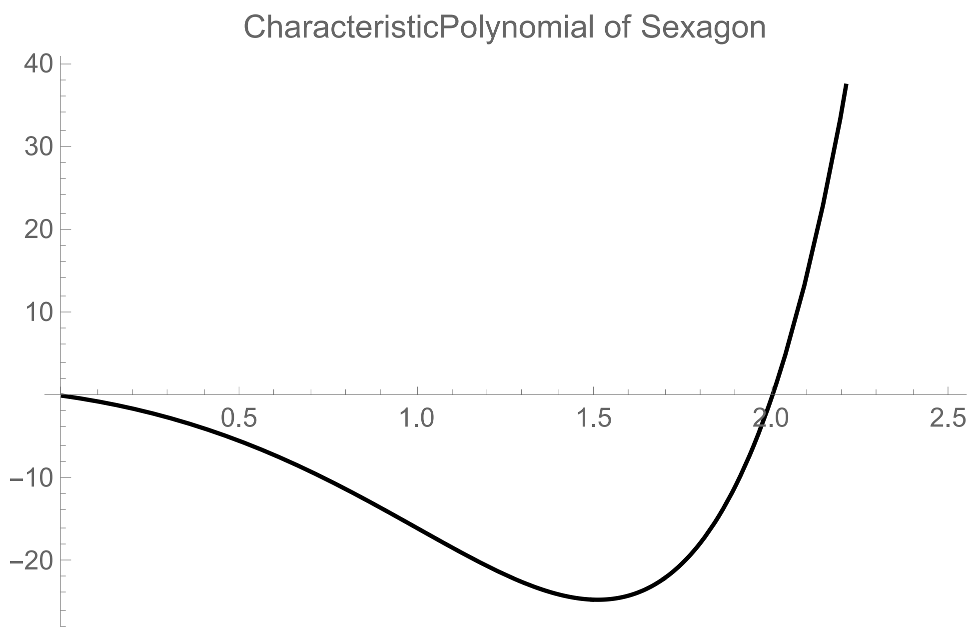
Out[∘]= $-6x - 9x^2 - 2x^3 + x^6$

In[•]:=
```
Factor[-6 x - 9 x^2 - 2 x^3 + x^6]
```

Out[•]= $(-2+x)\, x\, (1+x)^2 \left(3+x^2\right)$

In[15]:=
```
Plot[-6 x - 9 x^2 - 2 x^3 + x^6, {x, 0, 2.5},
 PlotStyle → {Black, Medium},
 PlotLabel → "CharacteristicPolynomial of Sexagon"]
```

Out[15]=



The list ur[top] shows the arrows in the perimeter of a fair even m-polygon whose adjacency matrix is A[m]. Make its arrows. The terms in ur[m] are its arrows from {1⟷2} to {m-1, m}. It is easier to describe the next step using the sexagon.

In[1]:=
```
ur[top_Integer] := Table[{i ⧴ i + 1}, {i, 1, top - 1}]
```

In[•]:=
```
ur[6]
```

Out[•]= {{1 ⧴ 2}, {2 ⧴ 3}, {3 ⧴ 4}, {4 ⧴ 5}, {5 ⧴ 6}}

In[•]:=
```
makeA[6]
```

In[•]:=
```
Apply[DirectedEdge, arrw, 1]
```

Out[•]=
```
{1 ⟷ 2, 1 ⟷ 4, 1 ⟷ 6, 2 ⟷ 3, 2 ⟷ 5, 3 ⟷ 1, 3 ⟷ 4,
 3 ⟷ 6, 4 ⟷ 2, 4 ⟷ 5, 5 ⟷ 1, 5 ⟷ 3, 5 ⟷ 6, 6 ⟷ 2,
 6 ⟷ 4}
```

First step in general procedure

Replace {1⟷ 2} by {1⟷6},{6⟷2},
{3⟷4{ by {3⟷1}, {1⟷4}
{5⟷6} by {5⟷3}, {3⟷6}

Following {3⟷ 6}, insert {6⟷4},{4⟷2} & {2⟷5}, {5⟷1},
This makes an Eulerian cycle for the sexagon.
Starting from the arrows in the m-polygon, replace alternate pairs beginning with the first {1⟷2} and continuing to the last, {m-1⟷m}, we finish by alternating the remaining terms in the sequence of arrows starting with {m ⟷ m+1} to the last term in the arrow list, {m, m-2}. The result is

m	6	8	10	12
left out	3	4	5	6

The total number of arrows is m(m-1)/2. As we shall see, m/2 arrows are left out so the ratio is 1/(m-1).

Octagon

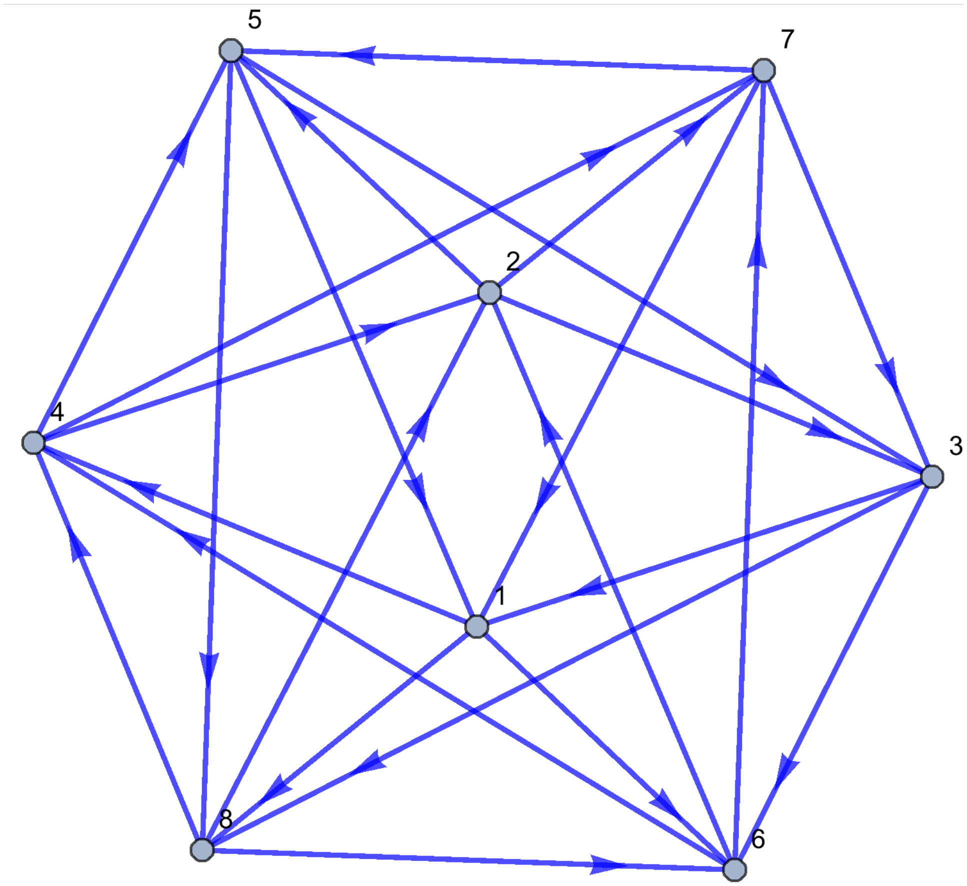
Although Mathematica could not find an Eulerian Cycle, I did. The maximal Eulerian cycle excludes 4 arrows.

quasi - bands				prospects		
b1	b3	b5	b7	u2	u4	u6
1, 2	1, 4	1, 6	1, 8	3, 1	5, 1	7, 1
2, 3	2, 5	2, 7	–	4, 2	6, 2	8, 2
3, 4	3, 6	3, 8	–	5, 3	7, 3	–
4, 5	4, 7	–	–	6, 4	8, 4	–
5, 6	5, 8	–	–	7, 5	–	–
6, 7	–	–	–	8, 6	–	–
7, 8	–	–	–	–	–	–

```
In[•]:= makeA[8]

In[•]:= ur[8]

Out[•]= {{1 ⟼ 2}, {2 ⟼ 3}, {3 ⟼ 4},
         {4 ⟼ 5}, {5 ⟼ 6}, {6 ⟼ 7}, {7 ⟼ 8}}

In[•]:= e1 := {1 ⟼ 8, 8 ⟼ 2, 2 ⟼ 7, 7 ⟼ 3, 3 ⟼ 8, 8 ⟼ 4,
          4 ⟼ 7, 7 ⟼ 5, 5 ⟼ 8, 8 ⟼ 6, 6 ⟼ 7, 7 ⟼ 1}

In[•]:= e2 := {1 ⟼ 6, 6 ⟼ 2, 2 ⟼ 3, 3 ⟼ 1, 1 ⟼ 4, 4 ⟼ 2,
          2 ⟼ 5, 5 ⟼ 3, 3 ⟼ 6, 6 ⟼ 4, 4 ⟼ 5, 5 ⟼ 1}

        excluded arrows := {1 ⟼ 2, 3 ⟼ 4, 5 ⟼ 6, 7 ⟼ 8}

In[•]:= g12 := fig[e1 ⋃ e2, 2]
```

In[∘]:= g12

Out[∘]=



In[∘]:= FindEulerianCycle[g12]

Out[∘]= {{1 ⟷ 4, 4 ⟷ 2, 2 ⟷ 3, 3 ⟷ 1, 1 ⟷ 6, 6 ⟷ 4, 4 ⟷ 5, 5 ⟷ 1,
1 ⟷ 8, 8 ⟷ 4, 4 ⟷ 7, 7 ⟷ 3, 3 ⟷ 6, 6 ⟷ 2, 2 ⟷ 5, 5 ⟷ 8,
8 ⟷ 6, 6 ⟷ 7, 7 ⟷ 5, 5 ⟷ 3, 3 ⟷ 8, 8 ⟷ 2, 2 ⟷ 7, 7 ⟷ 1}}

excluded arrows := {1 ⟷ 2, 3 ⟷ 4, 5 ⟷ 6, 7 ⟷ 8}

In[∘]:= exa := {{0, 0, 0, 1, 0, 1, 0, 1}, {0, 0, 1, 0, 1, 0, 1, 0},
{1, 0, 0, 0, 0, 1, 0, 1}, {0, 1, 0, 0, 1, 0, 1, 0},
{1, 0, 1, 0, 0, 0, 0, 1}, {0, 1, 0, 1, 0, 0, 1, 0},
{1, 0, 1, 0, 1, 0, 0, 0}, {0, 1, 0, 1, 0, 1, 0, 0}}

In[∘]:=
```
Det[exa - x IdentityMatrix[8]]
```

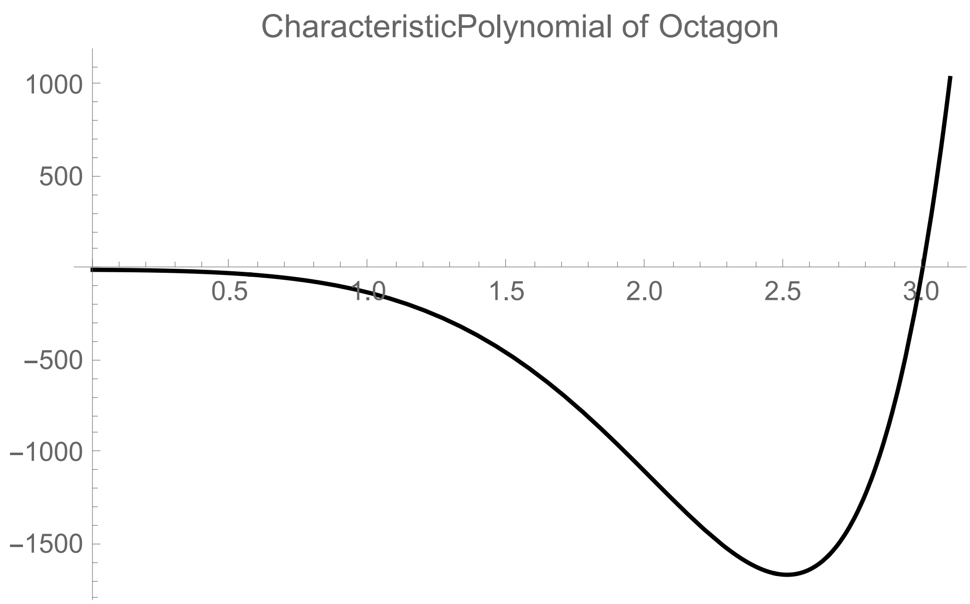
Out[∘]= $-3-8x-24x^2-48x^3-38x^4-8x^5+x^8$

In[∘]:=
```
Factor[-3 - 8 x - 24 x^2 - 48 x^3 - 38 x^4 - 8 x^5 + x^8]
```

Out[∘]= $(-3+x)(1+x)^3(1+6x^2+x^4)$

In[16]:=
```
Plot[-3 - 8 x - 24 x^2 - 48 x^3 - 38 x^4 - 8 x^5 + x^8, {x, 0, 3.1},
 PlotStyle → {Black, Medium}, AxesOrigin → {0, 0},
 PlotLabel → "CharacteristicPolynomial of Octagon"]
```

Out[16]=



Decagon

quasi - bands   prospects

b1	b3	b5	b7	b9	u2	u4	u6	u8
1, 2	1, 4	1, 6	1, 8	1, 10	3, 1	5, 1	7, 1	9, 1
2, 3	2, 5	2, 7	2, 9	–	4, 2	6, 2	8, 2	10, 2
3, 4	3, 6	3, 8	3, 10	–	5, 3	7, 3	9, 3	–
4, 5	4, 7	4, 9	–	–	6, 4	8, 4	10, 4	–
5, 6	5, 8	5, 10	–	–	7, 5	9, 8	–	–
6, 7	6, 9	–	–	–	8, 6	10, 6	–	–
7, 8	7, 10	–	–	–	9, 7	–	–	–
8, 9	–	–	–	–	10, 8	–	–	–
9, 10	–	–	–	–	–	–	–	–

In[•]:=
```
makeA[10]
```

In[•]:=
```
Apply[DirectedEdge, arrw, 1]
```

In[•]:=
```
c1 := {1 ⟷ 10, 2 ⟷ 3, 3 ⟷ 10, 4 ⟷ 5, 5 ⟷ 10, 6 ⟷ 7,
  7 ⟷ 10, 8 ⟷ 9, 9 ⟷ 1, 10 ⟷ 2, 10 ⟷ 4, 10 ⟷ 6, 10 ⟷ 8}
```

In[•]:=
```
u10 := {9 ⟷ 10, 7 ⟷ 8, 5 ⟷ 6, 3 ⟷ 4, 1 ⟷ 2}
```

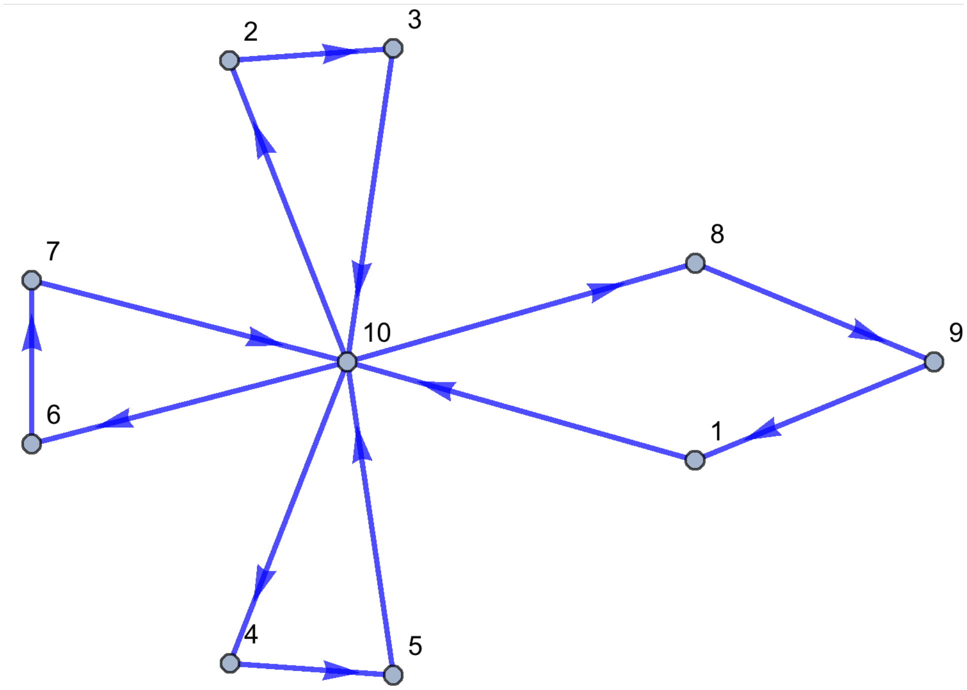
The quasi-bands {c1, c2, c3, c4} are the maximal Eulerian cycle. Neither c1 nor c2 is a simple Hamiltonian circuit

In[•]:=
```
c4 := {1 ⟷ 8, 8 ⟷ 6, 6 ⟷ 9, 9 ⟷ 3, 3 ⟷ 1}
```

In[•]:=
```
c3 := {1 ⟷ 6, 6 ⟷ 2, 2 ⟷ 7, 7 ⟷ 3, 3 ⟷ 8, 8 ⟷ 4,
  4 ⟷ 9, 9 ⟷ 5, 5 ⟷ 1}
```

In[•]:=
```
c2 := {1 ⟷ 4, 4 ⟷ 2, 2 ⟷ 5, 5 ⟷ 3, 3 ⟷ 6, 6 ⟷ 4,
  4 ⟷ 7, 7 ⟷ 5, 5 ⟷ 8, 8 ⟷ 2, 2 ⟷ 9, 9 ⟷ 7, 7 ⟷ 1}
```

In[•]:=
```
fig[c1, 2]
```

Out[•]=


As required every vertex in the Eulerian cycle has an even number of arrows.

In[•]:=
```
c1234 := fig[c1 ⋃ c2 ⋃ c3 ⋃ c4, 3]
```

In[•]:=
```
FindEulerianCycle[c1234]
```

Out[•]=
```
{{1 ⧴ 4, 4 ⧴ 2, 2 ⧴ 3, 3 ⧴ 1, 1 ⧴ 6, 6 ⧴ 4, 4 ⧴ 5,
  5 ⧴ 1, 1 ⧴ 10, 10 ⧴ 4, 4 ⧴ 7, 7 ⧴ 10, 10 ⧴ 6,
  6 ⧴ 2, 2 ⧴ 5, 5 ⧴ 10, 10 ⧴ 2, 2 ⧴ 7, 7 ⧴ 5, 5 ⧴ 3,
  3 ⧴ 10, 10 ⧴ 8, 8 ⧴ 6, 6 ⧴ 7, 7 ⧴ 8, 8 ⧴ 9, 9 ⧴ 1}}
```

In[•]:=
```
makeA[10]
```

In[•]:=
```
chf := {{0, 0, 0, 1, 0, 1, 0, 1, 0, 1},
  {0, 0, 1, 0, 1, 0, 1, 0, 1, 0},
  {1, 0, 0, 0, 0, 1, 0, 1, 0, 1},
  {0, 1, 0, 0, 1, 0, 1, 0, 1, 0},
  {1, 0, 1, 0, 0, 0, 0, 1, 0, 1},
  {0, 1, 0, 1, 0, 0, 1, 0, 1, 0},
  {1, 0, 1, 0, 1, 0, 0, 1, 0, 1},
  {0, 1, 0, 1, 0, 1, 0, 0, 0, 0},
  {1, 0, 1, 0, 1, 0, 1, 0, 0, 0},
  {0, 1, 0, 1, 0, 1, 0, 1, 0, 0}}
```

In[•]:=
```
Det[chf - x IdentityMatrix[10]]
```

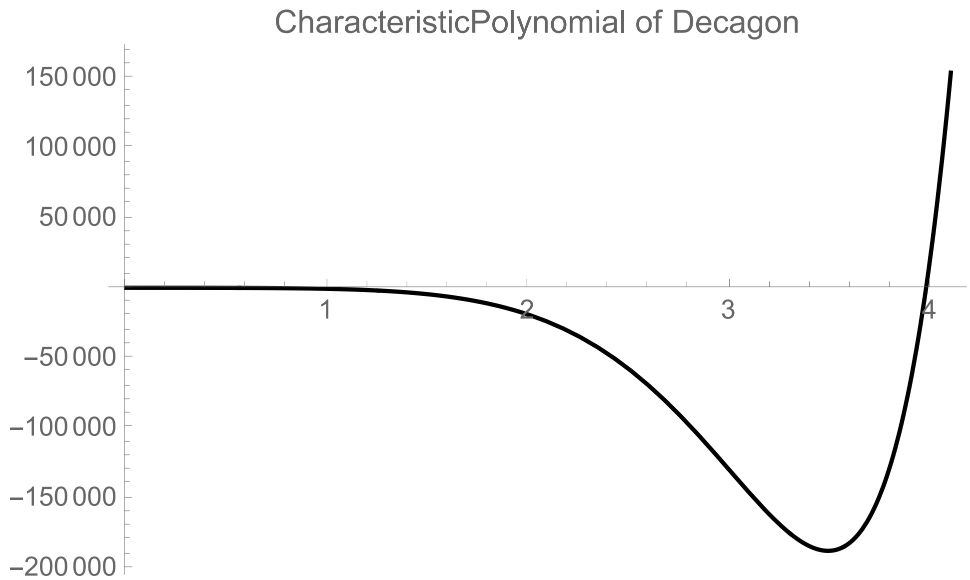
Out[•]= $-2-18x-63x^2-132x^3-198x^4-198x^5-106x^6-20x^7+x^{10}$

In[•]:=
```
Factor[-2 - 18 x - 63 x^2 - 132 x^3 - 198 x^4 - 198 x^5 - 106 x^6 -
  20 x^7 + x^10]
```

Out[•]= $(1+x)^3\left(-2-12x-21x^2-31x^3-30x^4+6x^5-3x^6+x^7\right)$

In[14]:=
```
Plot[-2 - 18 x - 63 x^2 - 132 x^3 - 198 x^4 - 198 x^5 - 106 x^6 -
   20 x^7 + x^10, {x, 0, 4.1}, PlotStyle → {Medium, Black},
  PlotLabel → "CharacteristicPolynomial of Decagon"]
```

Out[14]=



Conclusions

A fair m-polygon has m(m-1)/2 arrows. A fair even m-polygon leaves out m/2 arrows to obtain the maximal Eulerian cycle. The left-out arrows measure the minimal structural unemployment in the m-polygon network. Let σ denote the structural unemployment rate divided by total number of arrows in the polygon. Thus σ is the minimal structural unemployment rate. Equation (1) shows σ decreases as m increases.

(1) $\sigma = (m/2) \div (m(m-1)/2) = 1/(m-1)$.

Because the maximal Eulerian cycle in an odd m-polygon is a partition, it includes all arrows so that odd polygons have no structural unemployment. Since even and odd numbers alternate, simple algebra can show cyclical behavior of σ. An even m-polygon has m(m-1)/2 arrows. The following odd (m+1)-polygon has m(m+1)/2 arrows. The difference between the odd and the preceding m number, [m(m+1)-m(m-1)]/2 is m. The successor odd m-polygon has m more arrows than its predecessor even m-polygon. The Eulerian Cycle, (E-cycle) in the odd m-polygon is a partition so it includes all its arrows. The preceding even m-polygon has a maximal E-cycle that leaves out m/2 arrows. The difference between E-cycle sizes odd minus even is m + m/2, the new arrows in the odd polygon + the arrows left out of the preceding maximal

E-cycle is (3/2). The table shows that structural unemployment decreases as m increases.

m	4	5	6	7	8	9	10	11	12	…
σ	1/3	0	1/5	0	1/7	0	1/9	0	1/11	…