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# Movable Band Matrixes in a Circuit Core Model

L. G. Telser

## General

A band matrix is a permutation matrix for a band in an odd adjacency matrix A[m]. It has (m-1)/2 bands, B[k,m], where k runs over the odd integers from 1 to m-2. Each band B[k,m] has a permutation matrix P[k,m]. The latter has a polynomial of degree m in the variable x as follows:
(1) f[x, k, m] = Det[B[k,m] - x I[m], I[m] is the mXm identity matrix.
This polynomial is positive for all x ∊ (0,1]. It shows the net gain of (m-1)/2 business enterprises, U'B[k,m]V-x U'V, where U and V are positive m-vectors that represent inputs and outputs. Revenue is U'B[k,m]V and Cost is x U'V. Net gains move togther with x. While net gains differ among them so some are losses, some break even and some are profits, all move up or down together depending on whether x moves up or down. These net gains depend on x as if x were a number drawn at random from (0,1] .

The function defined in (1) is often called the characteristic function of the matrix. I call it the Band Characteristic Function, BCF. The adjacency matrix A[m] also has a characteristic function for A[m], ACF, the Adjacency Characteristic Function.

A collection of m points yields m(m-1)/2 pairs. A pair is a one-way arrow from its first to its second coordinate. A set of points that has a permutation matrix includes at least one simple (Hamiltonian) circuit. Let P[*v*,m] denote an mXm permutation matrix that represents *v* simple circuits each of size s so that s *v* = m. An s-circuit has an sXs permutation matrix. Each band has a unique permutation matrix. A collection of (m-1/2 mXm permutation matrixes that do not intersect forms a partition of the arrows in an m-polygon. Arrows that traverse the circuit perimeter and return to their starting vertex must have the same orientation.

Next are some pertinent properties of the bands in A[m]. All bands in the m-polygon have the same number of arrows, m. All simple circuits in band k have the same number of arrows, s, for size. The number of simple circuits in band k is GCD[k,m]=$v$, so m = $v$ s. Now an interesting fact enters. If m is prime, then all the bands are relatively prime so that GCD[k,m]=1 for all k. In this case every band in a prime m-polygon has a single circuit of size m. If m is composite, then the number of relatively prime bands is $\varphi$[m]/2, where $\varphi$[m] denotes the Euler totient function for m. Thus $\varphi$(m) is the number of band indexes relatively prime to m. The number of composite bands is equals m - $\varphi$[m]/2. Hence formula (2) applies to a BCF.

(2) f[x, k, m] = $(1-x^s)^v$ where m = $v$ s.

## A Useful Simple Result

A fair m-polygon has (m-1)/2 bands, k = 1,3,5, ...,m-2 in its adjacency matrix A[m]. Together with the bands for A'[m], k' = -1,-3,-5, ... , -(m-2), they form a commutative, multiplicative group.

(1) f[x, k, m] = Det[B[k,m] - x I[m]].

The transpose B'[k,m] = B[-k,m]. Each band has a unique mXm permutation matrix. The inverse of the permutation matrix for B[k,m] is its transpose.

(2) B[k, m] B'[k,m] = B[k,m] B[-k,m] = I[m].

Write the mXm matrix

(3) F[x, k, m] = B[k,m] - x I[m].

so that

(4) Det[F[x,k,m]] = f[x,k,m].

Every B[k,m] is a product of two bands described in the Multiplicative Groups of Circuits notebook.

Theorem. Every band and its inverse in the group for the fair odd m-polygon has the same BCF.

Proof. From (2) and (3) it follows that

F[x, k, m] = x B[k,m] B[-k,m] - B[k,m] = B[k, n] [x B[-k,m] - I[m]]

(5) = B[k,m] [x I[m] - B[-k, m]] B[k,m]

Since B[k,m] and [B[-k,m] are permutation matrixes, their determinants equal 1. Consequently, taking determinants

(6) Det [F[x, k, m]] = Det [F[x, -k, m].

and

(7) f[x, k, m] = f[x, -k, m]

## Pentagon Example

In[∘]:=
```
makeA[5]
```

In[∘]:=
```
B[3, 5] := {{0, 0, 0, 1, 0}, {0, 0, 0, 0, 1},
   {1, 0, 0, 0, 0}, {0, 1, 0, 0, 0}, {0, 0, 1, 0, 0}}
```

In[∘]:=
```
MatrixForm[B[3, 5]]
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

In[∘]:=
```
B[1, 5] := {{0, 1, 0, 0, 0}, {0, 0, 1, 0, 0},
   {0, 0, 0, 1, 0}, {0, 0, 0, 0, 1}, {1, 0, 0, 0, 0}}
```

In[∘]:=
```
MatrixForm[B[1, 5]]
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

In[∘]:=
```
U := Array[u, 5]
```

In[∘]:=
```
V := Array[v, 5]
```

In[∘]:=
```
Cost := U.V
```

In[∘]:=
```
Cost
```

In[∘]:=
```
cost := (u[1] × v[1]) + (u[2] × v[2]) + (u[3] × v[3]) +
   (u[4] × v[4]) + (u[5] × v[5])
```

In[•]:=
```
U.B[1, 5]
```

Out[•]=
```
{u[5], u[1], u[2], u[3], u[4]}
```

In[•]:=
```
{u[5], u[1], u[2], u[3], u[4]}.V
```

Out[•]=
```
u[5] × v[1] + u[1] × v[2] +
 u[2] × v[3] + u[3] × v[4] + u[4] × v[5]
```

In[•]:=
```
revenue1 :=
  {(u[5] × v[1]) + (u[1] × v[2]) + (u[2] × v[3]) +
    (u[3] × v[4]) + (u[4] × v[5])}
```

B1Gain := revenue1 - x Cost

In[•]:=
```
U.B[3, 5]
```

Out[•]=
```
{u[3], u[4], u[5], u[1], u[2]}
```

In[•]:=
```
{{u[3], u[4], u[5], u[1], u[2]}}.V
```

In[•]:=
```
revenue3 :=
  {(u[3] × v[1]) + (u[4] × v[2]) + (u[5] × v[3]) +
    (u[1] × v[4]) + (u[2] × v[5])}
```

## Net Gain

Next are the gains from the two bands in the pentagon. The second term in each pair is the same. The first terms differ. The costs are the same. To see which offers the bigger gain depends only on the inputs. There is dominance if and only if it is possible to order the u’s by size. Cost depends on x, a number between 0 and 1. There may be some x that yields a positive gain in this interval. If so, and if x is a random variable, then sometimes only one band is active, sometime two and sometimes none.

B1 Gain = {(u[5] × v[1]) + (u[1] × v[2]) + (u[2] × v[3]) +- x Cost
 (u[3] × v[4]) + (u[4] × v[5])}

B3 Gain ={ (u[3] × v[1]) + (u[4] × v[2]) + (u[5] × v[3]) +- x Cost
(u[1] × v[4]) + (u[2] × v[5]) }

In[•]:=
```
Det[B[1, 5] - x IdentityMatrix[5]]
```

Out[•]= $1 - x^5$

In[•]:=
```
Det[B[3, 5] - x IdentityMatrix[5]]
```

Out[•]= $1 - x^5$

In[•]:=
```
Det[Transpose[B[1, 5]]] - Det[B[1, 5]]
```

Out[•]= 0

In[•]:=
```
Det[Transpose[B[3, 5]]] - Det[B[3, 5]]
```

Out[•]= 0

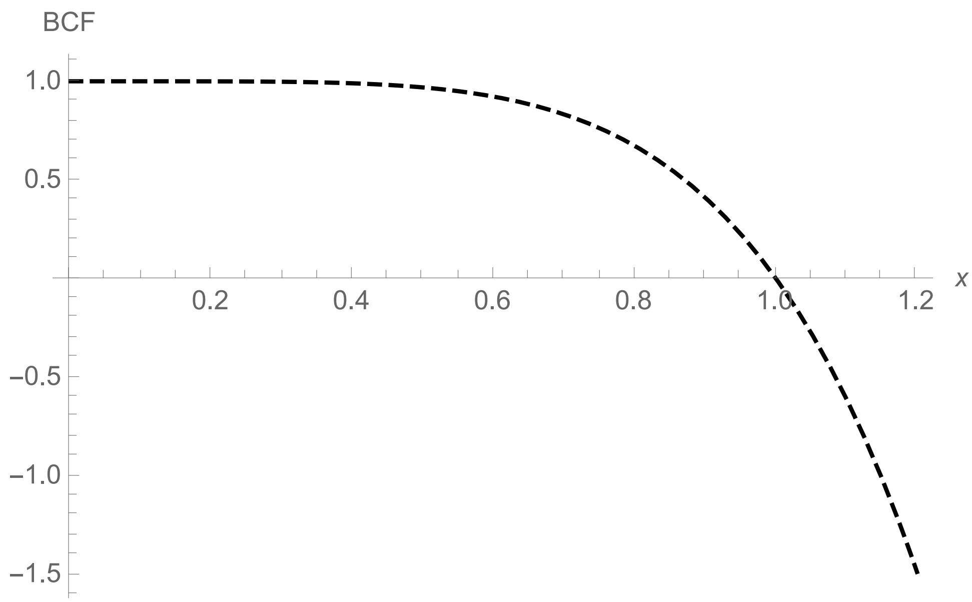
```
TB[3, 5] := {{0, 0, 1, 0, 1}, {0, 0, 0, 1, 0},
  {0, 0, 0, 0, 0}, {1, 0, 0, 0, 0}, {0, 1, 0, 0, 0}}
```

In[•]:=
```
Det[Transpose[B[3, 5]] - x IdentityMatrix[5]]
```

Out[•]= $1 - x^5$

BCF is the same for B[3, 5] and B'[3, 5]

In[∘]:= Plot[1 - x^5, {x, 0, 1.2}, PlotStyle → {Black, Dashed}, AxesLabel → {x, BCF}]

Out[∘]=



## A[7] Septagon

In[∘]:= makeA[7]

In[∘]:= Det[A - x IdentityMatrix[7]]

Out[∘]= $3 + 14 x + 28 x^2 + 28 x^3 + 14 x^4 - x^7$

In[∘]:= Factor[$3 + 14 x + 28 x^2 + 28 x^3 + 14 x^4 - x^7$]

Out[∘]= $-(-3 + x)(1 + 5 x + 11 x^2 + 13 x^3 + 9 x^4 + 3 x^5 + x^6)$

In[∘]:= $\partial_x (1 + 5 x + 11 x^2 + 13 x^3 + 9 x^4 + 3 x^5 + x^6)$

Out[∘]= $5 + 22 x + 39 x^2 + 36 x^3 + 15 x^4 + 6 x^5$

In[∘]:= Factor[$5 + 22 x + 39 x^2 + 36 x^3 + 15 x^4 + 6 x^5$]

Out[∘]= $(1 + 2 x)(5 + 12 x + 15 x^2 + 6 x^3 + 3 x^4)$

In[∘]:= `Solve[5 + 12 x + 15 x^2 + 6 x^3 + 3 x^4 == 0, x]`

Out[∘]=
$$\left\{\left\{x \to \frac{1}{6}\left(-3 - i\sqrt{3\left(21 - 4\sqrt{21}\right)}\right)\right\},\right.$$
$$\left\{x \to \frac{1}{6}\left(-3 + i\sqrt{3\left(21 - 4\sqrt{21}\right)}\right)\right\},$$
$$\left\{x \to \frac{1}{2}\left(-1 - i\sqrt{\frac{1}{3}\left(21 + 4\sqrt{21}\right)}\right)\right\},$$
$$\left.\left\{x \to \frac{1}{2}\left(-1 + i\sqrt{\frac{1}{3}\left(21 + 4\sqrt{21}\right)}\right)\right\}\right\}$$

## Bands for A[7]

```
Band[1, 7] := {{1, 2}, {2, 3}, {3, 4}, {4, 5}, {5, 6},
  {6, 7}, {7, 1}}
Band[3, 7] := {{1, 4}, {4, 7}, {7, 3}, {3, 6}, {6, 2},
  {2, 5}, {5, 1}}
Band[5, 7] := {{1, 6}, {6, 4}, {4, 2}, {2, 7}, {7, 5},
  {5, 3}, {3, 1}}
```

In[∘]:=
```
B[5, 7] := {{0, 0, 0, 0, 0, 1, 0}, {0, 0, 0, 0, 0, 0, 1},
  {1, 0, 0, 0, 0, 0, 0}, {0, 1, 0, 0, 0, 0, 0},
  {0, 0, 1, 0, 0, 0, 0}, {0, 0, 0, 1, 0, 0, 0},
  {0, 0, 0, 0, 1, 0, 0}}
```

In[∘]:= `Det[B[5, 7] - x IdentityMatrix[7]]`

Out[∘]= $1 - x^7$

```
In[•]:= B[3, 7] := {{0, 0, 0, 1, 0, 0, 0}, {0, 0, 0, 0, 1, 0, 0},
           {0, 0, 0, 0, 0, 1, 0}, {0, 0, 0, 0, 0, 0, 1},
           {1, 0, 0, 0, 0, 0, 0}, {0, 1, 0, 0, 0, 0, 0},
           {0, 0, 1, 0, 0, 0, 0}}

In[•]:= Det[B[3, 7] - x IdentityMatrix[7]]
```

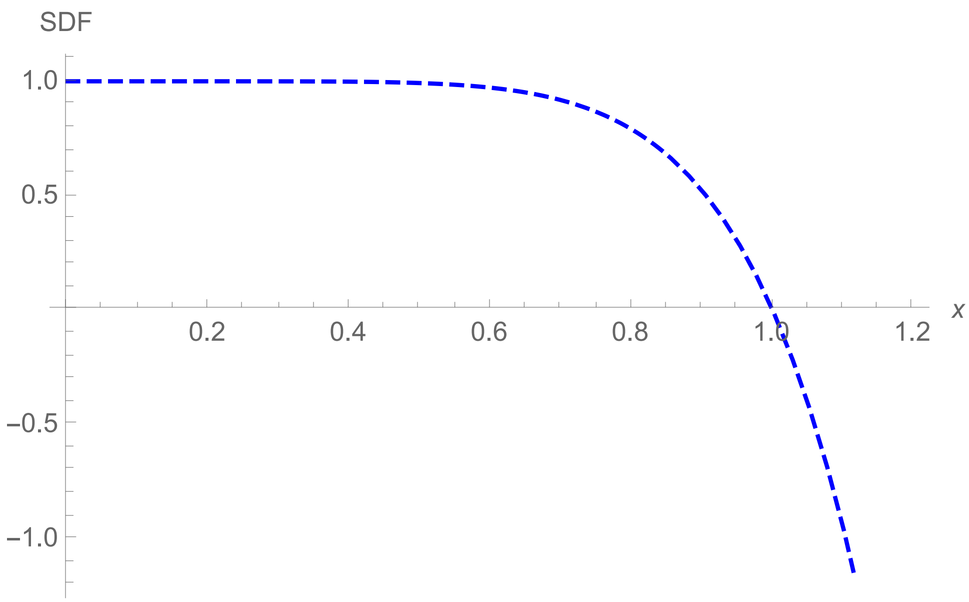
Out[•]= $1 - x^7$

```
In[•]:= Band[1, 7] := {{1, 2}, {2, 3}, {3, 4}, {4, 5}, {5, 6},
           {6, 7}, {7, 1}}

In[•]:= B[1, 7] := {{0, 1, 0, 0, 0, 0, 0}, {0, 0, 1, 0, 0, 0, 0},
           {0, 0, 0, 1, 0, 0, 0}, {0, 0, 0, 0, 1, 0, 0},
           {0, 0, 0, 0, 0, 1, 0}, {0, 0, 0, 0, 0, 0, 1},
           {1, 0, 0, 0, 0, 0, 0}}

In[•]:= Det[B[1, 7] - x IdentityMatrix[7]]
```

Out[•]= $1 - x^7$

```
Plot[1 - x^7, {x, 0, 1.2}, PlotStyle → {Blue, Dashed},
 AxesLabel → {x, BCF}]
```

Out[∘]=




F[1, 7] = F[3, 7]= F[5,7]

Nonagon

In[∘]:=
```
makeA[9]
```

In[∘]:=
```
A
```

In[∘]:=
```
B[1, 9] := {{0, 1, 0, 0, 0, 0, 0, 0, 0},
   {0, 0, 1, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 1, 0, 0, 0, 0, 0},
   {0, 0, 0, 0, 1, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 1, 0, 0, 0},
   {0, 0, 0, 0, 0, 0, 1, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 1, 0},
   {0, 0, 0, 0, 0, 0, 0, 0, 1}, {1, 0, 0, 0, 0, 0, 0, 0, 0}}
```

In[∘]:=
```
Det[B[1, 9] - x IdentityMatrix[9]]
```

Out[∘]= $1 - x^9$

```
In[•]:= B[3, 9] := {{0, 0, 0, 1, 0, 0, 0, 0, 0},
   {0, 0, 0, 0, 1, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 1, 0, 0, 0},
   {0, 0, 0, 0, 0, 0, 1, 0, 0}, {0, 0, 0, 0, 0, 8, 0, 1, 0},
   {0, 0, 0, 0, 0, 0, 0, 0, 1}, {1, 0, 0, 0, 0, 0, 0, 0, 0},
   {0, 1, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 1, 0, 0, 0, 0, 0, 0}}
```

In[•]:= `Det[B[3, 9] - x IdentityMatrix[9]]`

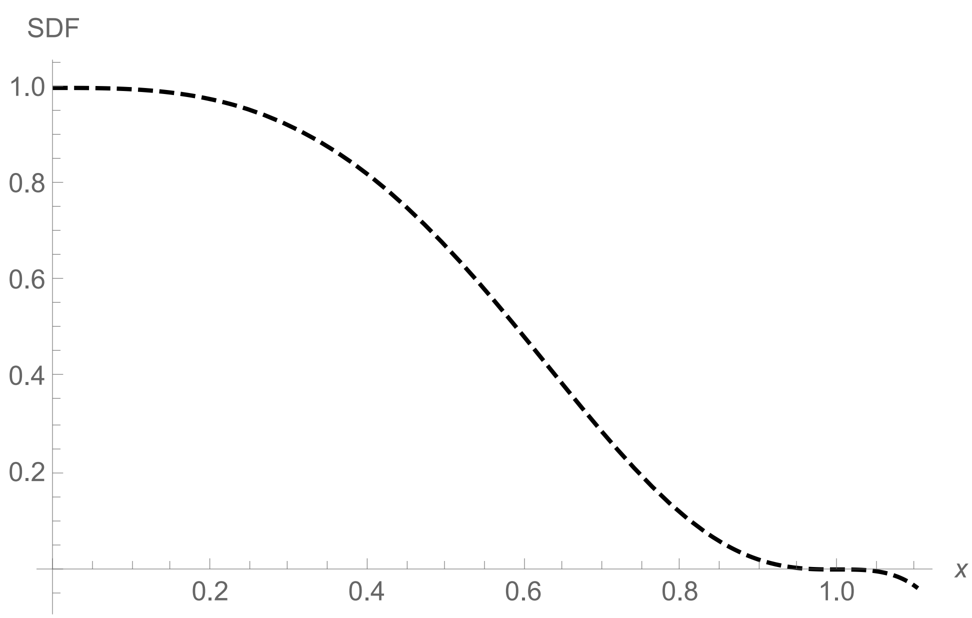
Out[•]= $1 - 3 x^3 + 3 x^6 - x^9$

In[•]:= `Factor[1 - 3 x^3 + 3 x^6 - x^9]`

Out[•]= $-(-1 + x)^3 (1 + x + x^2)^3$

```
Plot[1 - 3 x^3 + 3 x^6 - x^9, {x, 0, 1.1},
 PlotStyle → {Black, Dashed}, AxesLabel → {x, BCF}]
```

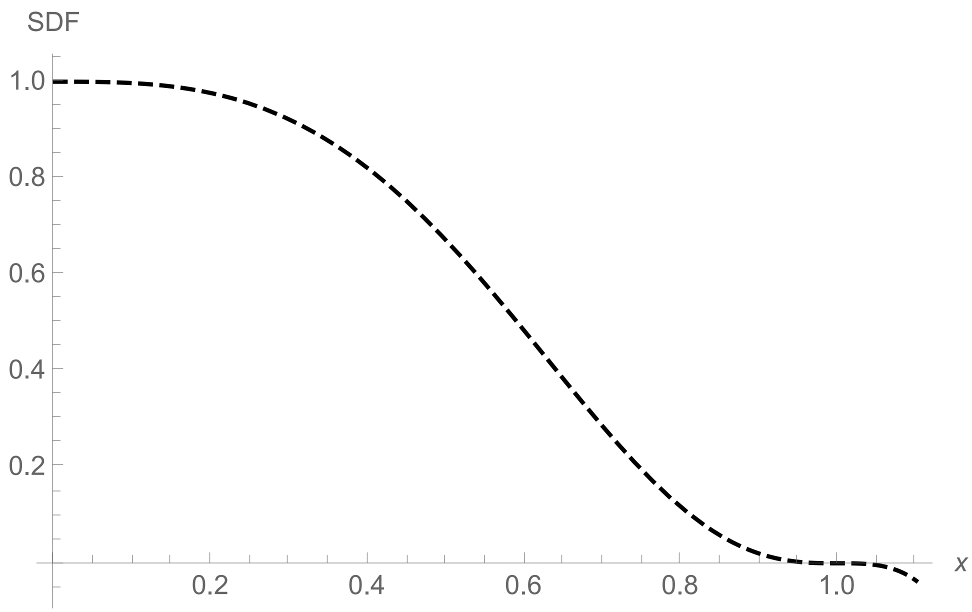
Out[•]=




f[x, 3, 9] = $(1 - x^3)^3$

```
Plot[(1 - x^3)^3 , {x, 0, 1.1}, PlotStyle → {Black, Dashed},
 AxesLabel → {x, BCF}]
```

Out[•]=




In[•]:=
```
bnd[1, 9] := {{1, 2}, {2, 3}, {3, 4}, {4, 5}, {5, 6},
   {6, 7}, {7, 8}, {8, 9}, {9, 1}}
bnd[3, 9] := {{{1, 4}, {4, 7}, {7, 1}}, {{5, 8}, {8, 2}, {2, 5}},
   {{3, 6}, {6, 9}, {9, 3}}}
bnd[5, 9] := {{1, 6}, {6, 2}, {2, 7}, {7, 3}, {3, 8},
   {8, 4}, {4, 9}, {9, 5}, {5, 1}}
bnd[7, 9] := {{1, 8}, {8, 6}, {6, 4}, {4, 2}, {2, 9},
   {9, 7}, {7, 5}, {5, 3}, {3, 1}}
```

In[•]:=
```
B[5, 9] := {{0, 0, 0, 0, 0, 1, 0, 0, 0},
   {0, 0, 0, 0, 0, 0, 1, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 1, 0},
   {0, 0, 0, 0, 0, 0, 0, 0, 1}, {1, 0, 0, 0, 0, 0, 0, 0, 0},
   {0, 1, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 1, 0, 0, 0, 0, 0, 0},
   {0, 0, 0, 1, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 1, 0, 0, 0, 0}}
```

In[•]:=
```
Det[B[5, 9] - x IdentityMatrix[9]]
```

Out[•]= $1 - x^9$

```
In[•]:= B[7, 9] := {{0, 0, 0, 0, 0, 0, 0, 1, 0},
          {0, 0, 0, 0, 0, 0, 0, 0, 1}, {1, 0, 0, 0, 0, 0, 0, 0, 0},
          {0, 1, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 1, 0, 0, 0, 0, 0, 0},
          {0, 0, 0, 1, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 1, 0, 0, 0, 0},
          {0, 0, 0, 0, 0, 1, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 1, 0, 0}}

In[•]:= Det[B[7, 9] - x IdentityMatrix[9]]
```

Out[•]= $1 - x^9$

## 15 - Polygon

```
In[•]:= bnd[1, 15] := {{1, 2}, {2, 3}, {3, 4}, {4, 5},
          {5, 6}, {6, 7}, {7, 8}, {8, 9}, {9, 10}, {10, 11},
          {11, 12}, {12, 13}, {13, 14}, {14, 15}, {15, 1}}

        bnd[3, 15] := {{1, 4}, {2, 5}, {3, 6}, {4, 7}, {5, 8},
          {6, 9}, {7, 10}, {8, 11}, {9, 12}, {10, 13},
          {11, 14}, {12, 15}, {13, 1}, {14, 2}, {15, 3}}

        bnd[9, 15] := {{1, 10}, {2, 11}, {3, 12}, {4, 13},
          {5, 14}, {6, 15}, {7, 1}, {8, 2}, {9, 3}, {10, 4},
          {11, 5}, {12, 6}, {13, 7}, {14, 8}, {15, 9}}
```

```
In[•]:= B[1, 15] := {{0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
   {0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
   {0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
   {0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
   {0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0},
   {0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0},
   {0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0},
   {0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0},
   {0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0},
   {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0},
   {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0},
   {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0},
   {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0},
   {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1},
   {1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}

In[•]:= Det[B[1, 15] - x IdentityMatrix[15]]
```

Out[•]= $1 - x^{15}$

```
In[•]:= bnd[3, 15] := {{1, 4}, {2, 5}, {3, 6}, {4, 7}, {5, 8},
   {6, 9}, {7, 10}, {8, 11}, {9, 12}, {10, 13},
   {11, 14}, {12, 15}, {13, 1}, {14, 2}, {15, 3}}
```

In[•]:=
```
B[3, 15] := {{0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
   {0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
   {0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0},
   {0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0},
   {0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0},
   {0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0},
   {0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0},
   {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0},
   {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0},
   {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0},
   {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0},
   {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1},
   {1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
   {0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
   {0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
```

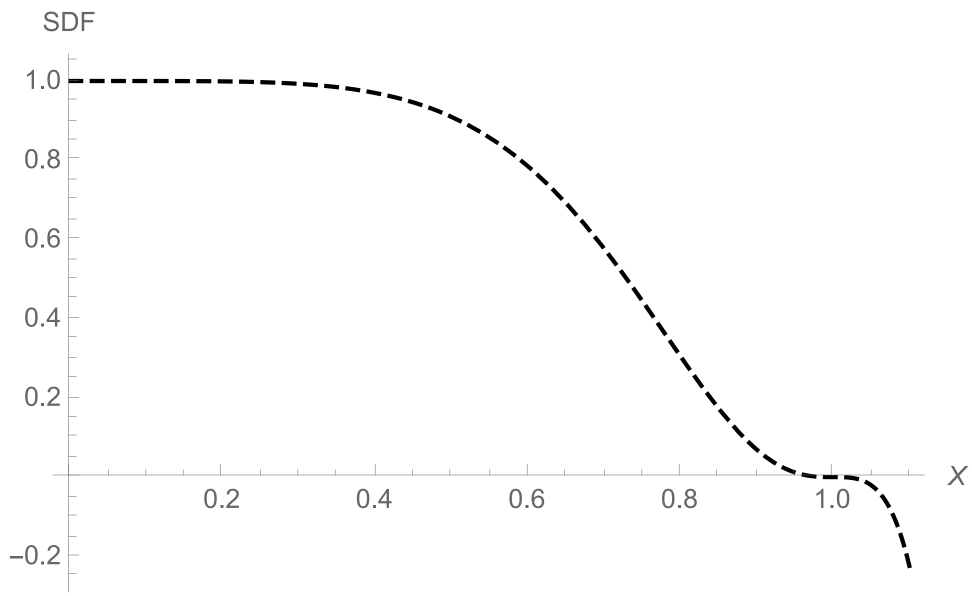
In[•]:=
```
MatrixForm[B[3, 15]]
```

In[•]:=
```
Det[B[3, 15] - x IdentityMatrix[15]]
```

Out[•]= $\left(1 - x^5\right)^3$

```
Plot[(1 - x^5)^3, {x, 0, 1.1}, PlotStyle → {Black, Dashed},
 AxesLabel → {X, BCF}]
```

Out[•]=

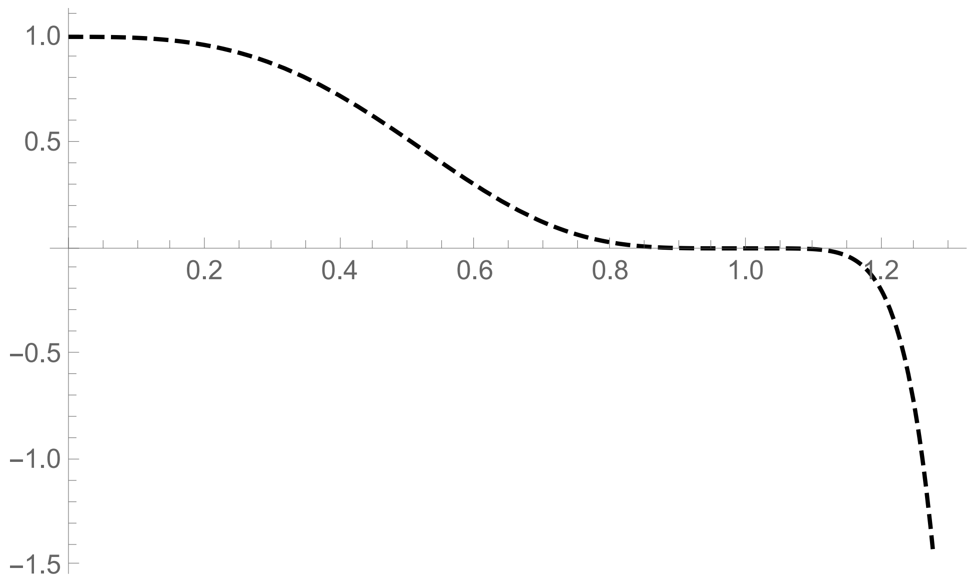



In[•]:=
```
B[5, 15] := {{0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0},
   {0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0},
   {0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0},
   {0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0},
   {0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0},
   {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0},
   {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0},
   {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0},
   {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0},
   {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1},
   {1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
   {0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
   {0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
   {0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
   {0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
```

In[•]:=
```
Det[B[5, 15] - x IdentityMatrix[15]]
```

Out[•]= $\left(1 - x^3\right)^5$

In[•]:=
```
Plot[(1 - x^3)^5, {x, 0, 1.3}, PlotStyle → {Dashed, Black}]
```

Out[•]=



```
bnd[9, 15] := {{1, 10}, {2, 11}, {3, 12}, {4, 13},
  {5, 14}, {6, 15}, {7, 1}, {8, 2}, {9, 3}, {10, 4},
  {11, 5}, {12, 6}, {13, 7}, {14, 8}, {15, 9}}
```

```
In[∘]:= B[9, 15] := {{0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0},
    {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0},
    {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0},
    {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0},
    {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0},
    {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1},
    {1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
    {0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
    {0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
    {0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
    {0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
    {0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0},
    {0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0},
    {0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0},
    {0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0}}

In[∘]:= Det[B[9, 15] - x IdentityMatrix[15]]
```

Out[∘]= $\left(1 - x^5\right)^3$

BCF for composite bands in 15 - polygon

f[x, 3, 15] := $\left(1 - x^3\right)^5$

f[x, 5, 15] := $\left(1 - x^5\right)^3$

f[x, 9, 15] := $\left(1 - x^5\right)^3$

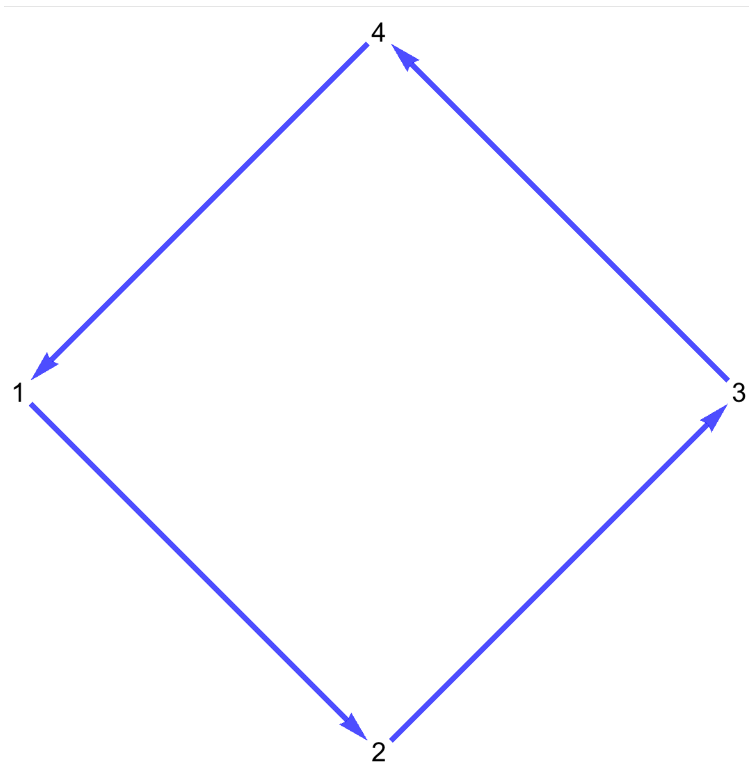
## Eulerian Cycles Exhibited by the Fair Pentagon, Sexagon & Octagon

Fair even m-polygons do not have bands, partitions or permutation matrixes. In contrast to fair odd-m polygons that do have bands, partitions and permutation matrixes for their bands. The latter also have perimeters made by simple m-circuits but this is not the case for fair *even* m-polygons. This is because vertexes in fair even m-polygons have an *odd* number of arrows unlike fair odd m-polygons whose vertexes have an *even* number of arrows. This difference is illustrated by the characteristic function. Every simple circuit has a positive characteristic function on the semi-open interval (0,1]. For the fair *even* m-polygon the characteristic function is negative for all x in the semi-open interval (0,1].

## Square

In[•]:=
```
makeGrph[B1, 2]
```

Out[•]=




## Graph of the Perimeter of a Square

Note the direction of the arrow from vertex 4 to vertex 1

In[•]:=
```
C1 := {{0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1},
   {1, 0, 0, 0}}
```

In[•]:=
```
MatrixForm[C1 - x IdentityMatrix[4]]
```

Out[•]//MatrixForm=

$$\begin{pmatrix} -x & 1 & 0 & 0 \\ 0 & -x & 1 & 0 \\ 0 & 0 & -x & 1 \\ 1 & 0 & 0 & -x \end{pmatrix}$$

$$F[x, 1, 4] = \begin{pmatrix} -x & 1 & 0 & 0 \\ 0 & -x & 1 & 0 \\ 0 & 0 & -x & 1 \\ 1 & 0 & 0 & -x \end{pmatrix}.$$

$f[x, 1, 4] = Det[F[x, 1, 4]] = -1 + x^4$.

In[•]:=
```
B1 := {{0, 0, 0, 1}, {0, 0, 1, 0}, {1, 0, 0, 0},
   {0, 1, 0, 0}}
```

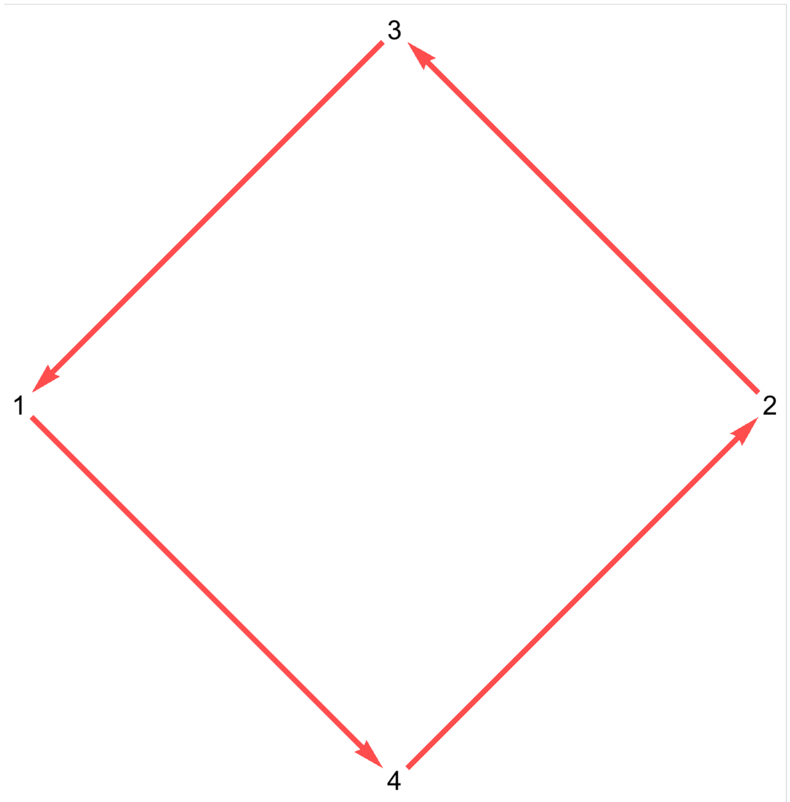
In[•]:=
```
MatrixForm[B1]
```

Out[•]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

In[•]:= makeGrph[B1, 3]

Out[•]=



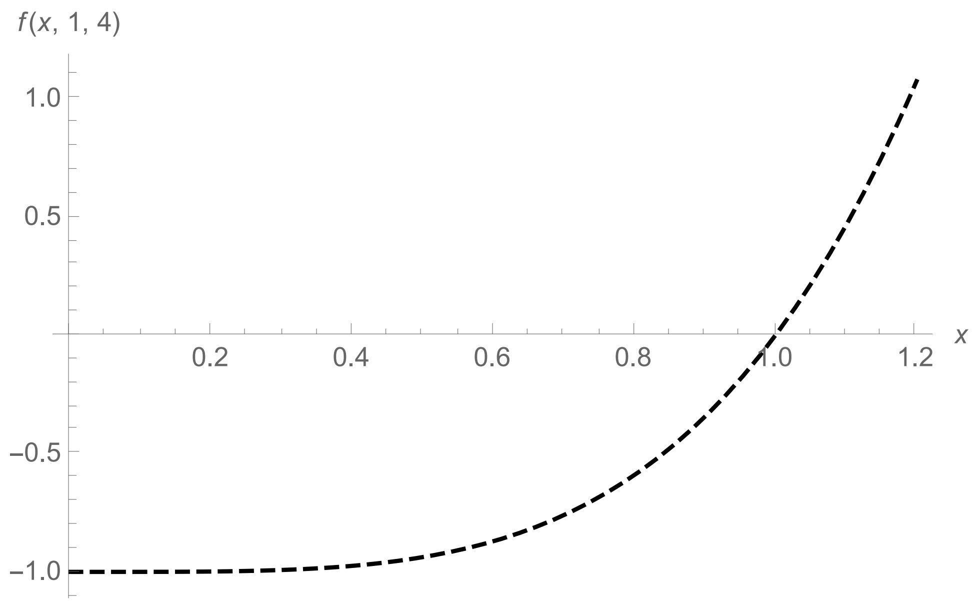

## Graph of the Perimeter of a *Fair* Square

Note the direction of the arrow from vertex 1 to vertex 4

In[•]:= Det[B1 - x IdentityMatrix[4]]

Out[•]= $-1 + x^4$

In[•]:= 
```
Plot[-1 + x^4, {x, 0, 1.2}, PlotStyle → {Dashed, Black},
 AxesLabel → {x, f[x, 1, 4]}]
```

Out[•]=



## Sexagon [Latin]

Its four 6 - circuits are negative for $0 < x \leq 1$.

In[•]:= 
```
makeA[6]
```

In[•]:= 
```
arrw
```

In[•]:= 
```
Apply[DirectedEdge, arrw, 1]
```

Out[•]= 
```
{1 → 2, 1 → 4, 1 → 6, 2 → 3, 2 → 5, 3 → 1, 3 → 4, 3 → 6,
 4 → 2, 4 → 5, 5 → 1, 5 → 3, 5 → 6, 6 → 2, 6 → 4}
```

| | | | | |
|---|---|---|---|---|
| 1, 2 | 1, 4 | 3, 1 | 5, 1 | 1, 6 |
| 2, 3 | 2, 5 | 4, 2 | 6, 2 | - |
| 3, 4 | 3, 6 | 5, 3 | - | - |
| 4, 5 | - | 6, 4 | - | - |
| 5, 6 | - | - | - | - |

| | | | |
|---|---|---|---|
| - | 1, 6 | 5, 6 | 6, 2 |
| 1, 2 | 6, 4 | 3, 6 | 1, 4 |
| 2, 3 | 4, 2 | - | - |
| 3, 4 | 2, 5 | - | - |
| 4, 5 | 5, 3 | □ | - |
| 5, 1 | 3, 1 | □ | - |

In[∘]:=
```
c62 := {{0, 1, 0, 0, 0, 1}, {0, 0, 1, 0, 1, 0},
   {1, 0, 0, 1, 0, 0}, {0, 1, 0, 0, 1, 0},
   {1, 0, 1, 0, 0, 1}, {0, 1, 0, 1, 0, 0}}
```

In[∘]:=
```
Det[c62 - x IdentityMatrix[6]]
```

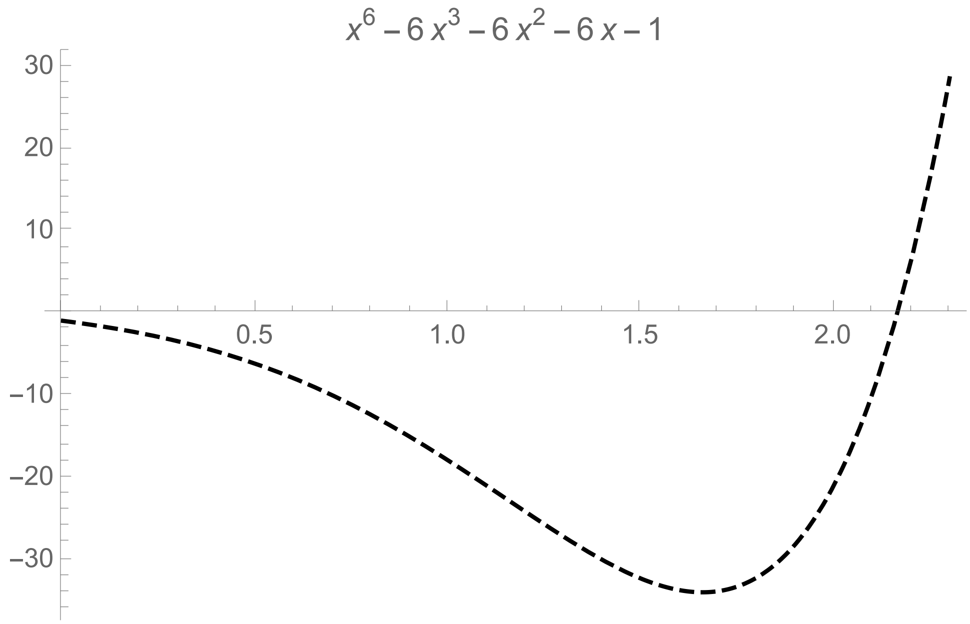
Out[∘]= $-1 - 6x - 6x^2 - 6x^3 + x^6$

In[∘]:= Factor$\left[-1 - 6x - 6x^2 - 6x^3 + x^6\right]$

Out[∘]= $\left(1 + x + x^2\right)\left(-1 - 5x - x^3 + x^4\right)$

In[•]:=
```
Plot[-1 - 6 x - 6 x^2 - 6 x^3 + x^6, {x, 0, 2.3},
 PlotStyle → {Dashed, Black},
 PlotLabel → -1 - 6 x - 6 x^2 - 6 x^3 + x^6]
```

Out[•]=




## Graph EulerCycle Sexagon 11 Arrows

Column 1 is c16; Column 2 is c26

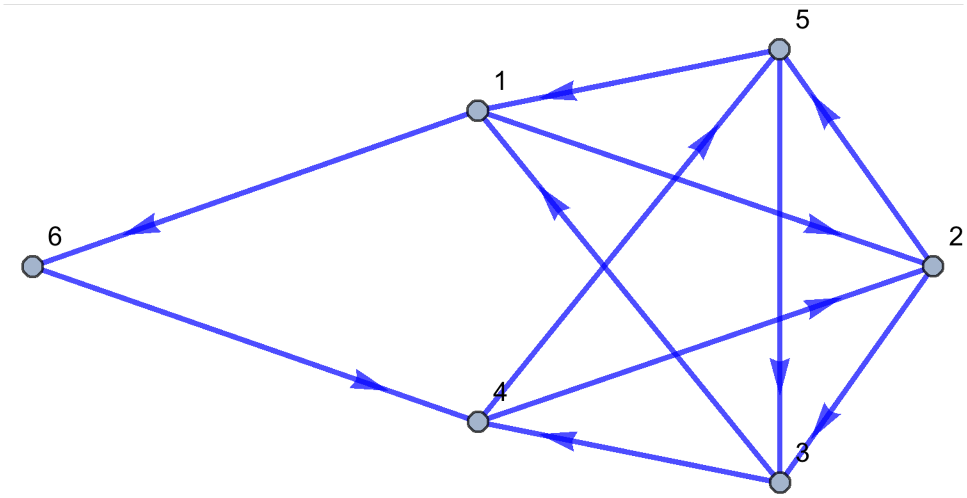
In[•]:=
```
c16 := {1 ⟷ 2, 2 ⟷ 3, 3 ⟷ 4, 4 ⟷ 5, 5 ⟷ 1}
```

In[•]:=
```
c26 := {1 ⟷ 6, 6 ⟷ 4, 4 ⟷ 2, 2 ⟷ 5, 5 ⟷ 3, 3 ⟷ 1}
```

In[•]:=
```
fc16 := fig[c16 ⋃ c26, 2]
```

In[•]:= fc16

Out[•]=




In[•]:= FindEulerianCycle[fc16]

Out[•]= {{1 ⧴ 2, 2 ⧴ 3, 3 ⧴ 1, 1 ⧴ 6, 6 ⧴ 4,
4 ⧴ 2, 2 ⧴ 5, 5 ⧴ 3, 3 ⧴ 4, 4 ⧴ 5, 5 ⧴ 1}}

This EulerCycle uses 11 of the 15 one-way arrows in the sexagon.

```
In[•]:= par1 := {{0, 1, 0, 0, 0, 1}, {0, 0, 1, 0, 1, 0},
   {1, 0, 0, 1, 0, 0}, {0, 0, 0, 0, 1, 0},
   {1, 0, 1, 0, 0, 0}, {0, 0, 0, 1, 0, 0}}
```

In[•]:= MatrixForm[par1]

Out[•]//MatrixForm=

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

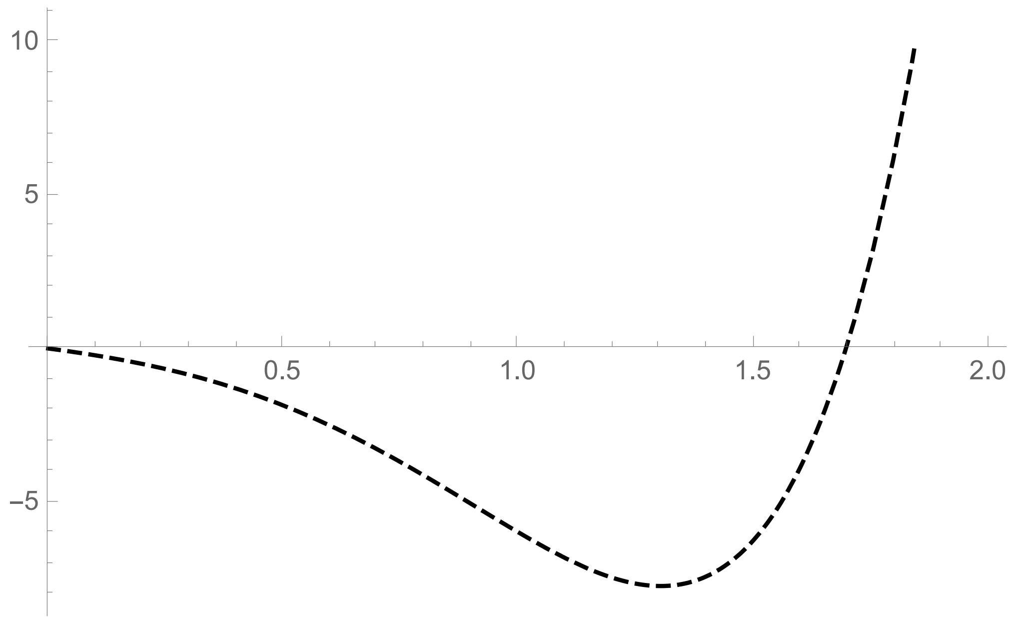
In[•]:= Det[par1 - x IdentityMatrix[6]]

Out[•]= $-2\,x - 2\,x^2 - 3\,x^3 + x^6$

In[•]:=
```
Factor[-2 x - 2 x^2 - 3 x^3 + x^6]
```

Out[•]= $x\left(1+x+x^2\right)\left(-2-x^2+x^3\right)$

In[•]:=
```
Plot[-2 x - 2 x^2 - 3 x^3 + x^6, {x, 0, 2},
 PlotStyle → {Dashed, Black}, AxesOrigin → {0, 0}]
```

Out[•]=



In[•]:=
```
h := makeGrph[A, 3]
```

In[•]:=
```
FindEulerianCycle[h]
```

Out[•]= {{1 ⟼ 2, 2 ⟼ 3, 3 ⟼ 1, 1 ⟼ 4,
 4 ⟼ 2, 2 ⟼ 5, 5 ⟼ 3, 3 ⟼ 4, 4 ⟼ 5, 5 ⟼ 1}}

Voila! An odd fair m-polygon has a Eulerian Cycle that covers all its arrows. An even fair m-polygon has an Eulerian Cycle that does not cover all its arrows. The fair pentagon and sexagon show this.

## Octagon

Mathematica could not find an Eulerian Cycle. But I did and as I expected, it excludes 3 arrows.

In[•]:=
```
makeA[7]
```

In[•]:=
```
g7 := makeGrph[A, 2]
```

In[•]:=
```
FindEulerianCycle[g7]
```

Out[•]=
```
{{1 ↔ 2, 2 ↔ 3, 3 ↔ 1, 1 ↔ 4, 4 ↔ 2, 2 ↔ 5, 5 ↔ 1,
  1 ↔ 6, 6 ↔ 2, 2 ↔ 7, 7 ↔ 3, 3 ↔ 4, 4 ↔ 5, 5 ↔ 3,
  3 ↔ 6, 6 ↔ 4, 4 ↔ 7, 7 ↔ 5, 5 ↔ 6, 6 ↔ 7, 7 ↔ 1}}
```

| | | | | | | |
|---|---|---|---|---|---|---|
| 1, 2 | 1, 4 | 1, 6 | 3, 1 | 5, 1 | 7, 1 | 1, 8 |
| 2, 3 | 2, 5 | 2, 7 | 4, 2 | 6, 2 | 8, 2 | - |
| 3, 4 | 3, 6 | 3, 8 | 5, 3 | 7, 3 | - | - |
| 4, 5 | 4, 7 | - | 6, 4 | 8, 4 | - | - |
| 5, 6 | 5, 8 | - | 7, 5 | - | - | - |
| 6, 7 | - | - | 8, 6 | - | - | - |
| 7, 8 | - | - | - | - | - | - |

| | | | | | |
|---|---|---|---|---|---|
| 1, 2 | 1, 4 | 1, 6 | 8, 6 | 5, 8 | 1, 8 |
| 2, 3 | 4, 2 | 6, 4 | 6, 2 | - | 8, 2 |
| 3, 4 | 2, 5 | 4, 7 | 2, 7 | - | - |
| 4, 5 | 5, 3 | 7, 5 | 7, 8 | - | - |
| 5, 6 | 3, 1 | 5, 1 | - | - | - |
| 6, 7 | - | - | - | - | - |
| 7, 1 | - | - | - | - | - |

In[•]:=
```
makeA[8]
```

In[•]:=
```
Apply[DirectedEdge, arrw, 1]
```

Out[•]=
```
{1 ↔ 2, 1 ↔ 4, 1 ↔ 6, 1 ↔ 8, 2 ↔ 3, 2 ↔ 5, 2 ↔ 7,
 3 ↔ 1, 3 ↔ 4, 3 ↔ 6, 3 ↔ 8, 4 ↔ 2, 4 ↔ 5, 4 ↔ 7,
 5 ↔ 1, 5 ↔ 3, 5 ↔ 6, 5 ↔ 8, 6 ↔ 2, 6 ↔ 4, 6 ↔ 7,
 7 ↔ 1, 7 ↔ 3, 7 ↔ 5, 7 ↔ 8, 8 ↔ 2, 8 ↔ 4, 8 ↔ 6}
```

In[•]:=
```
A
```

In[•]:=
```
eul := {{0, 1, 0, 1, 0, 1, 0, 0}, {0, 0, 1, 0, 1, 0, 1, 0},
   {1, 0, 0, 1, 0, 1, 0, 1}, {0, 1, 0, 0, 1, 0, 1, 0},
   {1, 0, 1, 0, 0, 1, 0, 0}, {0, 1, 0, 1, 0, 0, 1, 0},
   {1, 0, 1, 0, 1, 0, 0, 1}, {0, 0, 0, 1, 0, 1, 0, 0}}
```

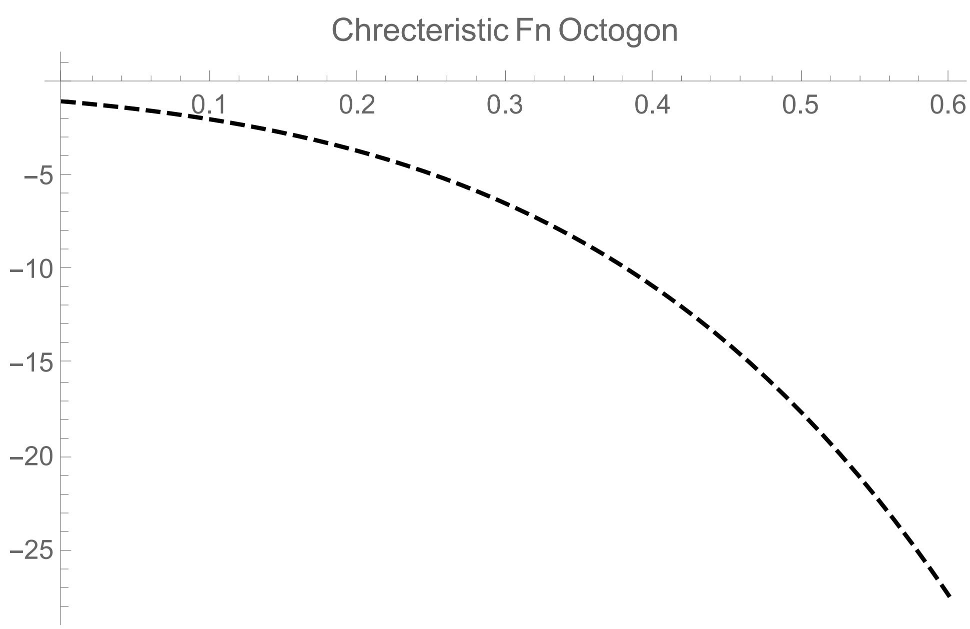
In[•]:=
```
Det[eul - x IdentityMatrix[8]]
```

Out[•]= $-1-7x-22x^2-39x^3-36x^4-16x^5+x^8$

In[•]:= Factor$\left[-1-7x-22x^2-39x^3-36x^4-16x^5+x^8\right]$

Out[•]= $-1-7x-22x^2-39x^3-36x^4-16x^5+x^8$

In[•]:= Plot$\left[-1-7x-22x^2-39x^3-36x^4-16x^5+x^8\right.$, {x, 0, .6},
 PlotStyle → {Dashed, Black}, AxesOrigin → {0, 0},
 PlotLabel → Chrecteristic Fn Octogon ]

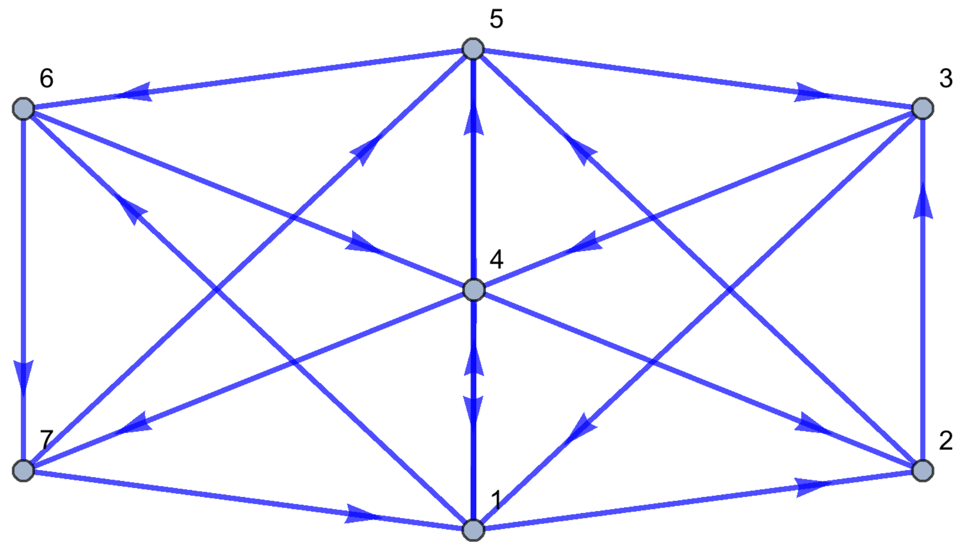
Out[•]=



In[•]:=
```
c4 := {8 ⟷ 6, 6 ⟷ 2, 2 ⟷ 7, 7 ⟷ 8}
```

In[•]:=
```
c3 := {1 ⟷ 6, 6 ⟷ 4, 4 ⟷ 7, 7 ⟷ 5, 5 ⟷ 1}
```

```
In[∘]:= c1 := {1 ⟷ 2, 2 ⟷ 3, 3 ⟷ 4, 4 ⟷ 5, 5 ⟷ 6, 6 ⟷ 7, 7 ⟷ 1}

In[∘]:= c2 := {1 ⟷ 4, 4 ⟷ 2, 2 ⟷ 5, 5 ⟷ 3, 3 ⟷ 1}

In[∘]:= fig[c1 ⋃ c2 ⋃ c3, 2]
```

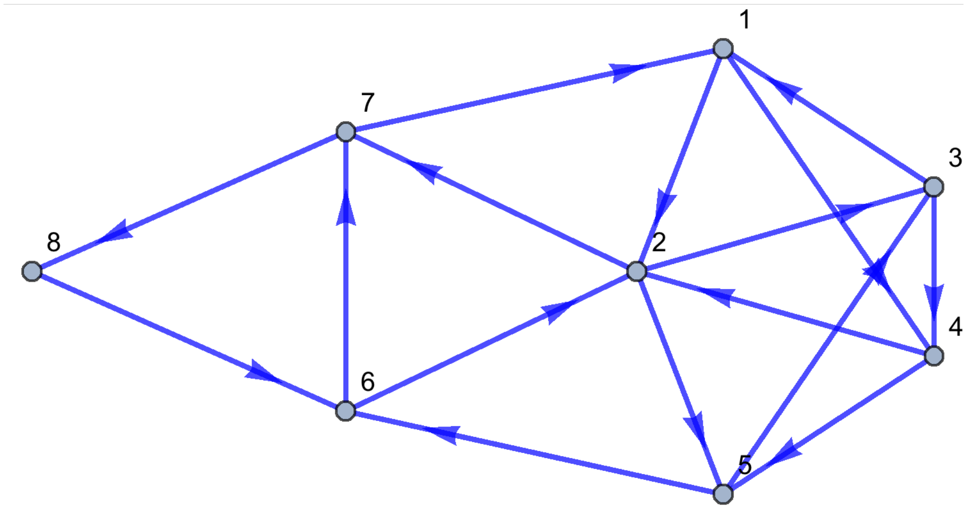
Out[∘]=




```
In[∘]:= feu4 := fig[c1 ⋃ c2 ⋃ c3 ⋃ c4, 2]

In[∘]:= FindEulerianCycle[feu4]

Out[∘]= {{1 ⟷ 2, 2 ⟷ 3, 3 ⟷ 1, 1 ⟷ 4, 4 ⟷ 2, 2 ⟷ 5, 5 ⟷ 1,
   1 ⟷ 6, 6 ⟷ 2, 2 ⟷ 7, 7 ⟷ 5, 5 ⟷ 6, 6 ⟷ 4, 4 ⟷ 5,
   5 ⟷ 3, 3 ⟷ 4, 4 ⟷ 7, 7 ⟷ 8, 8 ⟷ 6, 6 ⟷ 7, 7 ⟷ 1}}

In[∘]:= feu124 := fig[c1 ⋃ c2 ⋃ c4, 2]
```

In[•]:= 
```
feu124
```

Out[•]=




Decagon

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1, 2 | 1, 4 | 1, 6 | 1, 8 | 1, 10 | 3, 1 | 5, 1 | 7, 1 | 9, 1 |
| 2, 3 | 2, 5 | 2, 7 | 2, 9 | - | 4, 2 | 6, 2 | 8, 2 | 10, 2 |
| 3, 4 | 3, 6 | 3, 8 | 3, 10 | - | 5, 3 | 7, 3 | 9, 3 | - |
| 4, 5 | 4, 7 | 4, 9 | - | - | 6, 4 | 8, 4 | 10, 4 | - |
| 5, 6 | 5, 8 | 5, 10 | - | - | 7, 5 | 9, 8 | - | - |
| 6, 7 | 6, 9 | - | - | - | 8, 6 | 10, 6 | - | - |
| 7, 8 | 7, 10 | - | - | - | 9, 7 | - | - | - |
| 8, 9 | - | - | - | - | 10, 8 | - | - | - |
| 9, 10 | - | - | - | - | - | - | - | - |

In[•]:= 
```
makeA[10]
```

In[•]:= 
```
Apply[DirectedEdge, arrw, 1]
```

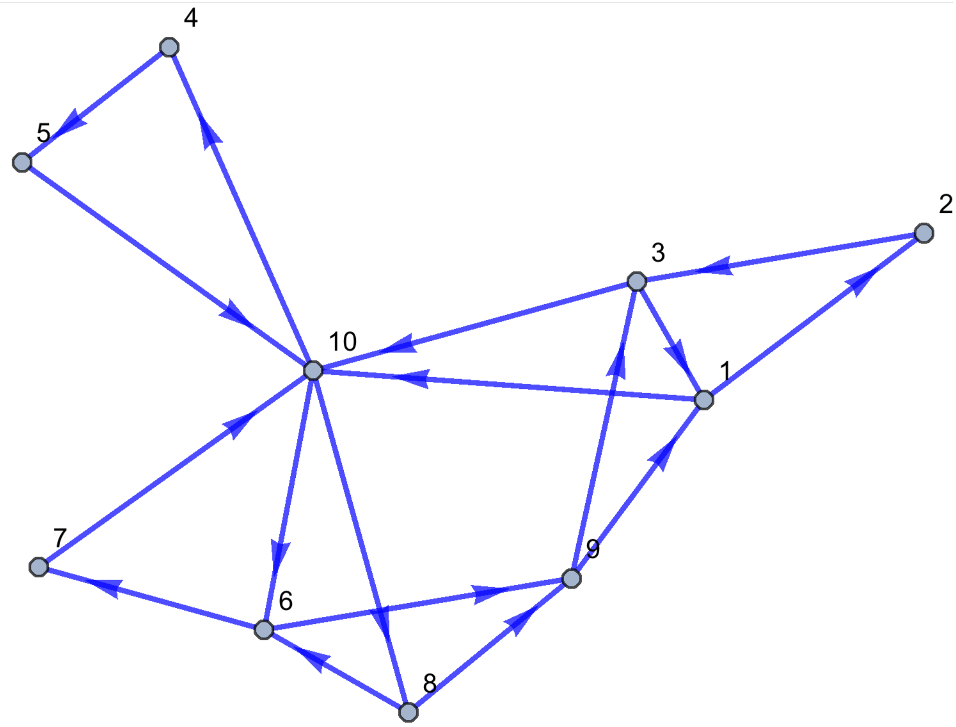
```
{9 ↦ 10, 7 ↦ 8, 5 ↦ 6, 3 ↦ 4, 1 ↦ 2}
```

In[•]:= 
```
c4 := {1 ↦ 8, 8 ↦ 6, 6 ↦ 9, 9 ↦ 3, 3 ↦ 1}
```

In[•]:= 
```
c3 := {1 ↦ 6, 6 ↦ 2, 2 ↦ 7, 7 ↦ 3, 3 ↦ 8, 8 ↦ 4,
   4 ↦ 9, 9 ↦ 5, 5 ↦ 1}
```

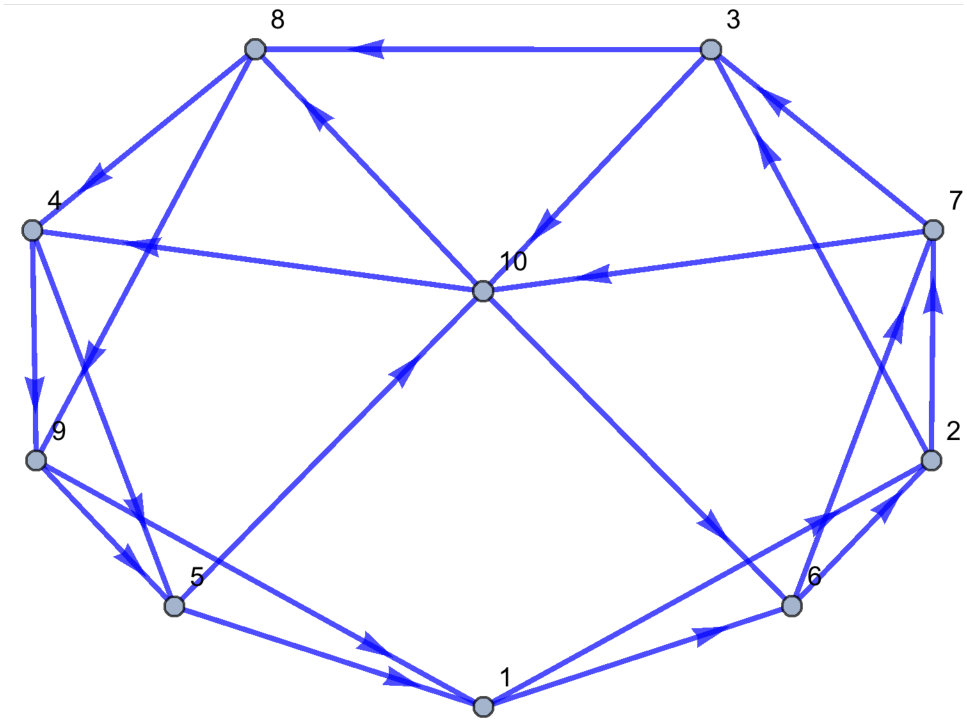
```
In[∘]:= c2 := {1 ↔ 4, 4 ↔ 2, 2 ↔ 5, 5 ↔ 3, 3 ↔ 6, 6 ↔ 4,
          4 ↔ 7, 7 ↔ 5, 5 ↔ 8, 8 ↔ 2, 2 ↔ 9, 9 ↔ 7, 7 ↔ 1}

In[∘]:= c1 := {1 ↔ 10, 10 ↔ 2, 2 ↔ 3, 3 ↔ 10, 10 ↔ 4, 4 ↔ 5,
          5 ↔ 10, 10 ↔ 6, 6 ↔ 7, 7 ↔ 10, 10 ↔ 8, 8 ↔ 9, 9 ↔ 1}

In[∘]:= fig[c1 ⋃ c4, 2]
```

Out[∘]=



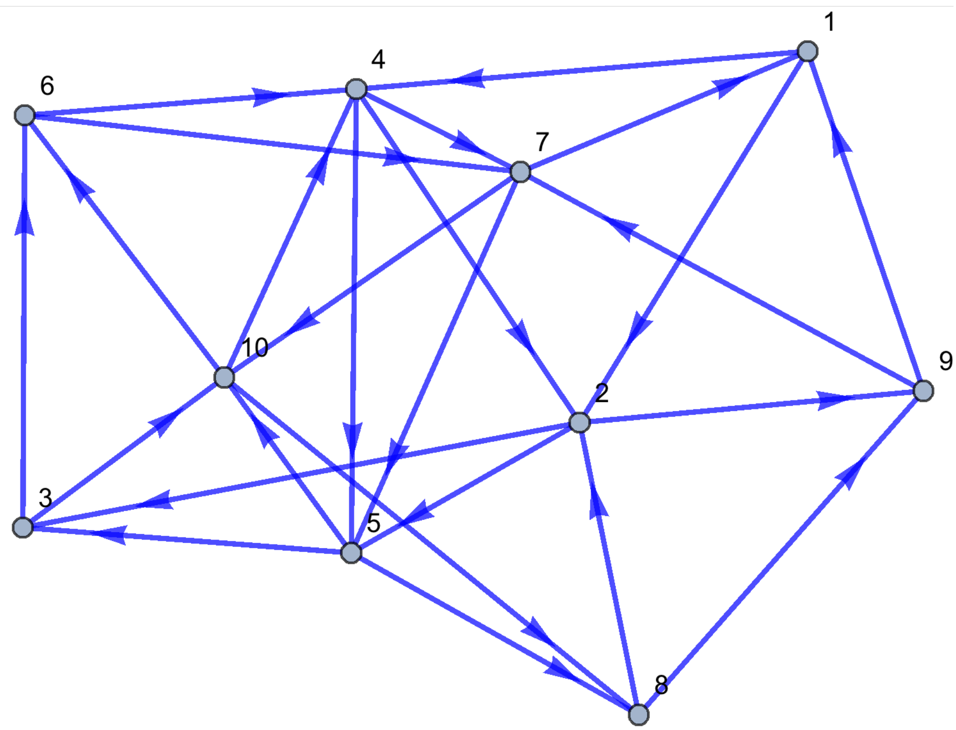
In[•]:= fig[c1 ⋃ c3, 2]

Out[•]=



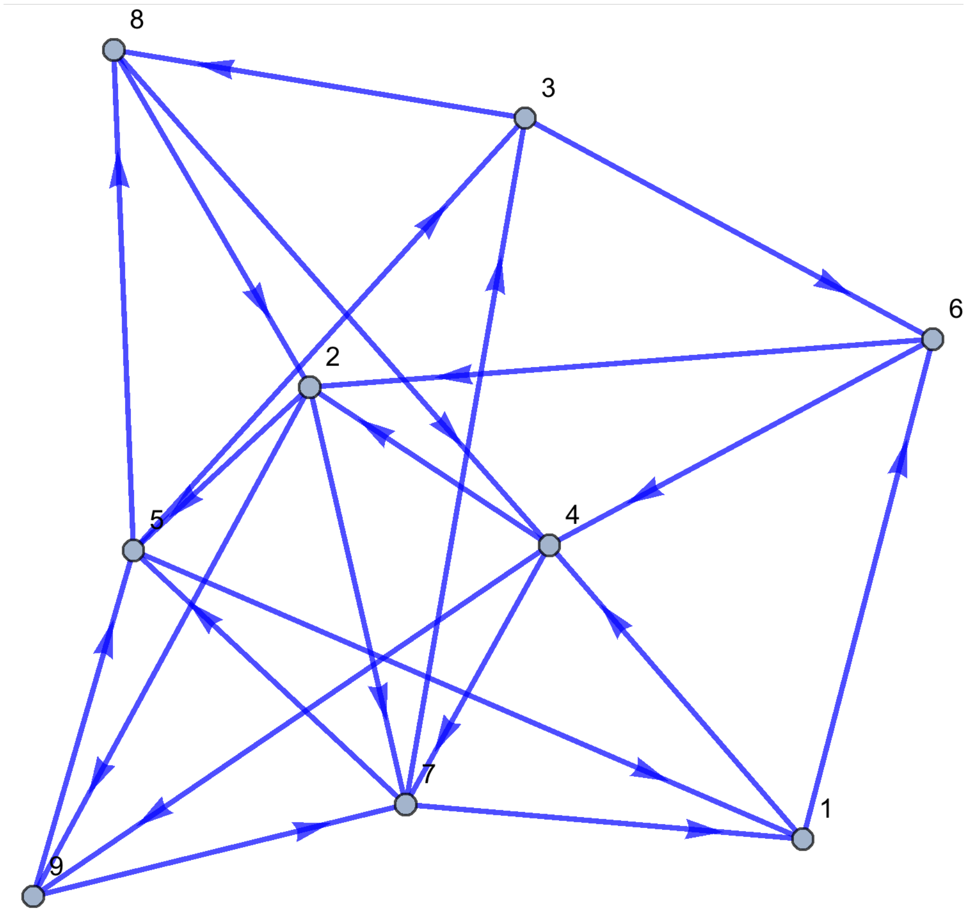

In[•]:= fig[c1 ⋃ c2, 2]

Out[•]=



In[•]:= fig[c3 ⋃ c2, 2]

Out[•]=




In[•]:= fabcd := fig[c1 ⋃ c2 ⋃ c3 ⋃ c4, 3]

In[•]:= FindEulerianCycle[fabcd]

Out[•]= {{1 ⧴ 4, 4 ⧴ 2, 2 ⧴ 3, 3 ⧴ 1, 1 ⧴ 6, 6 ⧴ 4, 4 ⧴ 5, 5 ⧴ 1, 1 ⧴ 8, 8 ⧴ 4, 4 ⧴ 7, 7 ⧴ 1, 1 ⧴ 10, 10 ⧴ 4, 4 ⧴ 9, 9 ⧴ 3, 3 ⧴ 6, 6 ⧴ 2, 2 ⧴ 5, 5 ⧴ 8, 8 ⧴ 6, 6 ⧴ 7, 7 ⧴ 10, 10 ⧴ 6, 6 ⧴ 9, 9 ⧴ 5, 5 ⧴ 10, 10 ⧴ 8, 8 ⧴ 2, 2 ⧴ 7, 7 ⧴ 3, 3 ⧴ 8, 8 ⧴ 9, 9 ⧴ 7, 7 ⧴ 5, 5 ⧴ 3, 3 ⧴ 10, 10 ⧴ 2, 2 ⧴ 9, 9 ⧴ 1}}

## Conjectures

Fair even m-polygons cannot have partitions. The maximal Eulerian cycle in these polygons are not partitions. The arrows left out relative to the total number of arrows measure the rate of structural unemployment. Since odd fair polygons have partitions, they have full employment. Hence the model shows structural  unemployment displays cyclical behavior. How it relates to economic growth poses a problem to the model for future study.

| | | | | |
|---|---|---|---|---|
| unemployed | 2/6 | 4/15 | 3/28 | 5/45 |
| m | 4 | 6 | 8 | 10 |

| | | | | |
|---|---|---|---|---|
| unemployed | 1/3 | 0.267 | 0.107 | 1/9 |
| m | 4 | 6 | 8 | 10 |

## Programs

## Some Geometric Series