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In Integers We Trust

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Abstract

Integers and their offspring, the rationals, are the only reliable numbers. This essay presents the case for my claim.

Preface

Building on the work of his great predecessor, Weierstrass, Dedekind states that space as a continuum is a belief not a demonstrable result. It stands on the same footing as the parallel axiom of Euclidian geometry that puzzled mathematicians for more than two millennia starting with Euclid himself. Dedekind believed that all the propositions of geometry are the same whether space is a continuum or not a continuum. Today some accept his belief by noting that the rationals are dense in a non continuum space and, of course, in a continuum. For example, non Euclidian geometry without the parallel axiom is the framework for the general theory of relativity.

Weierstrass also presented an example of a continuous function nowhere differentiable on its domain. Knopp (1951, p. 379), describes this function. It is a uniformly convergent, trigonometric series so it is continuous for all x but nowhere differentiable. Graves (1945, pp. 125-6) exposition is superb. He shows that the upper and lower right-hand and left-hand derivates are ± infinity. Zygmund (1955, pp. 175-83) describes an even more remarkable result, Kolmogoroff's Example, that proves there exists an integrable function f(x) such that $\mathcal{Z}$[f] [trigonometric series] diverges everywhere.

These suggest a useful empirical application in time series. Observing a continuous random walk at its current position cannot predict where it was or where it is going. Its path can be described by the continuous nowhere differentiable function of Weierstrass trigonometric series. A space whose functions are all continuous exclude functions of Weierstrass. Uncertainty cannot enter it. Our world has uncertainty. Countably infinite sequences of rationals suffice to study it. As Dedekind shows they even suffice to yield irrationals. The Dedekind cut defines real numbers. It uses sequences of rationals in the definition. This essay describes some insurmountable obstacles especially with irrationals.

Opening Argument

1. The ancient Greeks discovered that $\sqrt{2}$ is irrational.
The proof of this result by Pythagoras described by Hawking (2007, p. 2) is geometric. It uses an isosceles right triangle with two legs equal to 1 and demonstrates its hypotenuse is not rational .
2. e and π are irrational. Neither is an algebraic number.
3. $\left(1+\frac{1}{n}\right)^n \to e$ as n $\to \infty$
4. $\left(1+\frac{1}{n}\right) > \left(1+\frac{1}{n+1}\right) \Longrightarrow \left(1+\frac{1}{n}\right)^n > \left(1+\frac{1}{n+1}\right)^n$
5. $F[n] = \left(1+\frac{1}{n}\right)^n - \left(1+\frac{1}{n+1}\right)^n > 0$ for all positive numbers.
6. The terms in the sequence $\{ \left(1+\frac{1}{n}\right)^n | \; n = 1,2, \ldots ,\}$ are rationals;

$\{(1+1), (1+1/2)^2, (1+1/3)^3, (1+1/4)^4, (1+1/5)^5, \ldots\}$

$\{2, 9/4, 64/27, (5/4)^4, (6/5)^5, \ldots \}$

7. There is a number σ, the Dedekind cut, such that

$$\left(1+\frac{1}{n}\right)^n > \sigma > \left(1+\frac{1}{n+1}\right)^n \tag{1}$$

i. A Dedekind cut is unique.
Proof. If there were 2 Dedekind cuts, $\sigma_1 \neq \sigma_2$ for the same sequences of rationals then there would be an a countably infinite sequence of rationals between them. This would contradict their status as least upper bounds and greatest lower bounds.
ii. If the Dedekind cut σ were equal to a term in the low sequence of rationals then it would be rational. If it could not equal any of these numbers then it would be an irrational number.
Proof. By hypothesis

$\sigma < l_n$ for every $l_n \in$ low. $l_n - \sigma > 1/n$ for all $l_n \in$ low.

By contradiction. Suppose $l_n - \sigma \leq 1/n$ for all $l_n \in$ low. Hence

$\sigma < l_n - 1/n < l_n < \sigma$ because σ is the sup for low giving a contradiction.

My Case

The sole reality underlying the rules of grammar is verbal communication. The rules of grammar permit any grammatical statement no matter how ridiculous. The assertion X is not provable means either one of two alternatives. Either X contradicts a Peano axiom or no proof of X is known. Therefore,
Y = {X or $\overline{X}$} is not provable and $\overline{Y}$ is not provable. Gödel shows that *any* finite set of consistent axioms contains propositions that are neither provable nor non provable. Adding such propositions as axioms to a consistent set of axioms does not affect this result. Hawking (2007) offers a different interpretation of Gödel's result that I fail to understand. Constructive proofs are safer than are reductio ad absurdum. Next are two simple examples of propositions on primes.
First, if p is an odd prime, then p+1 is composite because it is divisible by 2.
Second, slightly harder, the interval between successive primes is divisible by 2.
Proof. By hypothesis p_n = 2 k + 1 because it is odd. Also k ≥ 1. p_{n+1} = 2 q + 1 and q > k.

$$p_{n+1} - p_n = 2\ q + 1 - (2k + 1) = 2\ (q-k) > 0.$$

The three alternatives for the numbers a and b, a > b, b > a and b = a can be settled unambiguously when they are rationals. Dedekind excludes the alternative, equality, for his cuts when they are irrationals (Dedekind, Continuity and Irrational Numbers).

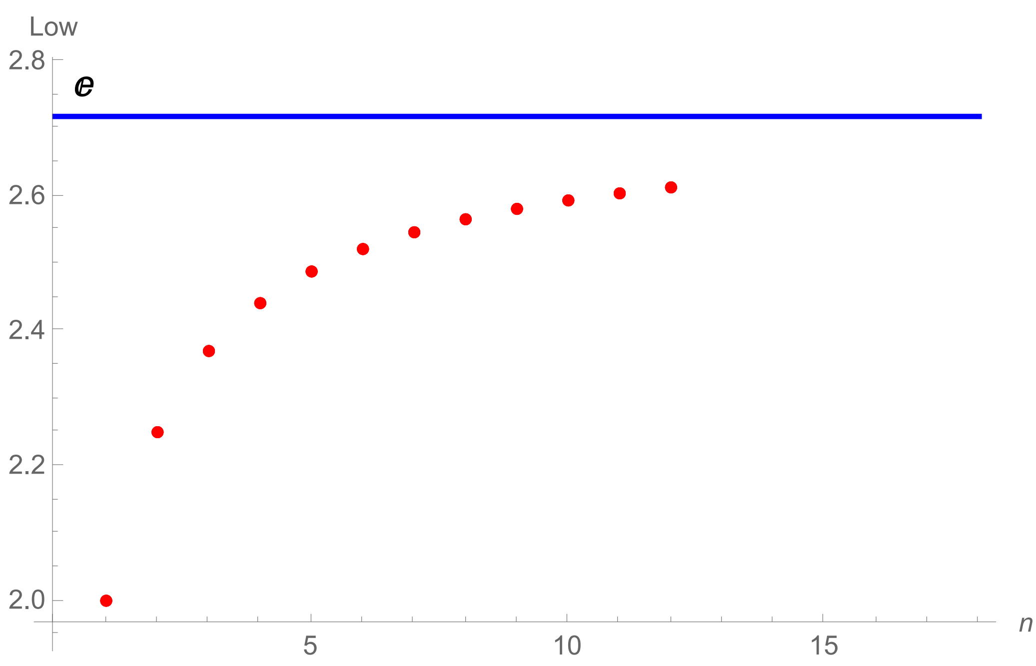
Out[•]=


Figure 1

Figure 1 shows that the sequence { $\left(1+\frac{1}{n}\right)^n$ } on the y-axis approaches e from below.

Figure 2 shows the difference between the two series, low and high.

low = { $\left(1+\frac{1}{n}\right)^n$} and high={ $\left(1+\frac{1}{n+1}\right)^n$}.

Their difference is positive for all n. The unique irrational σ, a Dedekind cut, separates them. Every term in the high sequence is above σ so σ is the infimum of this sequence. Every term in the low sequence is below σ so σ is the sup of the low sequence.

Out[•]=

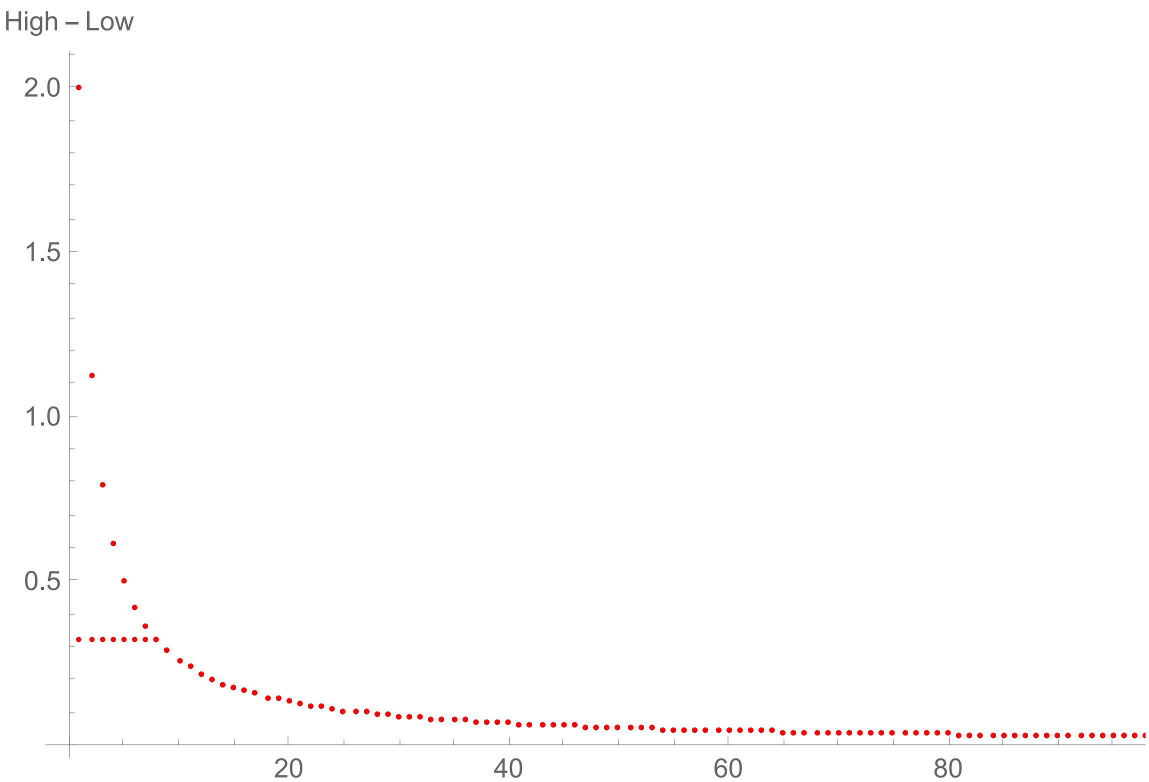


Figure 2

Out[•]=

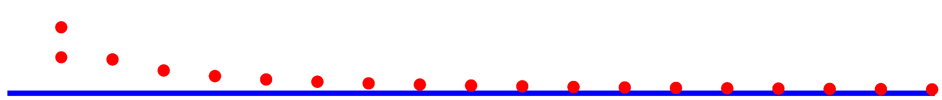


Figure 3

Figure 3 shows that the terms in the sequence high, $\{\left(1+\frac{1}{i}\right)^{i+1}\}$, approach e from above.

The following square matrix shows 4 pertinent sequences.

In[∘]:= **ded** := $\begin{pmatrix} \left(1+\frac{1}{n}\right)^{n} & \left(1+\frac{1}{n}\right)^{n+1} \\ \left(1+\frac{1}{n+1}\right)^{n} & \left(1+\frac{1}{n+1}\right)^{n+1} \end{pmatrix}$

Propositions.

$$\text{i. } \left(1+\frac{1}{n}\right)^{n+1} > \left(1+\frac{1}{n}\right)^{n} \to e \text{ from below} \tag{2}$$

$$\text{ii. } \left(1+\frac{1}{n}\right)^{n}\left(\left(1+\frac{1}{n}\right)-1\right) > 0 \text{ for all } n \tag{3}$$

Question. Do $\left\{\left(1+\frac{1}{n}\right)^{n+1}\right\}$ and $\left\{\left(1+\frac{1}{n}\right)^{n}\right\}$ yield the unique Dedekind cut $\gamma = e$?
The diagrams suggest the answer is yes.

Dedekind Cuts

A Dedekind cut is unique but the countably infinite sequences of rationals that define it are not unique. The conditions that can decide equality between irrational Dedekind cuts depend on sequences with countably infinite number of terms. Let α denote the Dedekind cut for the sequences (A_1, A_2). Let β denote the Dedekind cut for the sequences (B_1, B_2). The 4 possible relations among them that would yield equality between their Dedekind cuts are:
(1) $(A_1, A_2) \Longleftrightarrow \sup\{A_1\} < \inf\{A_2\}$,
(2) $(B_1, B_2) \Longleftrightarrow \sup\{B_1\} < \inf\{B_2\}$,
(3) $(A_1, B_2) \Longleftrightarrow \sup\{A_1\} < \inf\{B_2\}$,
(4) $(B_1, A_2) \Longleftrightarrow \sup\{B_1\} < \inf\{A_2\}$.
A_1 in (3) replaces B_1 in (2) and B_1 in (4) replaces A_1 in (1). These conditions pose no obstacles in principle but do so in practice. Finding the inf and sup for a countably infinite sequence of rationals is hard unless the terms obey known suitable conditions such as monotonicity. In practice given 4 countably infinite sequences of rationals I know of no practical method to solve these problems. Moreover, the obstacle to determine the relation between any pair of Dedekind

cuts for irrational numbers is no different for inequality than it is for equality. Invoking the Axiom of Choice is not practical advice for this purpose.

The existence of irrational numbers implied by the Peano axioms and the definition of rational numbers are not in doubt. However, the definition of irrational numbers given by the Dedekind cut is not constructive because it appeals to countably infinite sequences of rationals. They cannot be written out explicitly as a list. The sequence of prime numbers shows the complications. First, the number of primes is not finite. If the number were finite, n, so that $p_1, p_2, \ldots, p_n$ were all the primes then $\prod_{i=1}^{n} p_i + 1$ is a number not divisible by any of these n primes so it would be a prime bigger than p_n. Yet there is no known formula with a finite of variable that yields a prime. The next example illustrates another complication. A simple continued fraction with an infinite number of terms, $\sum_{i=1}^{\infty} a_i / \sum_{i=1}^{\infty} b_i$, where a_i and b_i are positive integers is an irrational number.
Theorem 7.7. The value of any simple continued fraction with an infinite number of positive integers is irrational (Niven and Zuckerman ,1966. p.157). The ratio would be rational for a finite number of terms or for a repeated sequence of a finite number of terms.

The second example that appeals to an infinite sequence of terms is typical. Constructive infinite sequences define a_n in terms of n such as

$$a_n = \left(1 + \frac{1}{n}\right)^n, \text{ n = 1, 2,..., .}$$

This gives any term in the infinite sequence. There are many other famous examples, thanks to Euler, the pioneer in developing the theory of continued fractions.

The careful analysis of continuity by Weierstrass laid the foundation for Dedekind. A careful reader of Weierstrass will note that he defines ϵ-δ neighborhoods using only rationals and nowhere mentions irrationals.

The assertion that the combination of rationals and irrationals yields a continuum is an axiom, not a theorem. Thus (Dedekind,1963, p. 12) tells us,

"If space has at all a real existence it is *not* necessary for it to be continuous; many of its properties would remain the same even were it discontinuous. And if we knew for certain that space was discontinuous there would be nothing to prevent us, in case we so desired, from filling up the gaps, in thought, and thus making it continuous; this filling up would consist in a creation of new-point individuals and would have to be effected in accordance with the above principle.”

Theorem. The smallest interval between rationals is positive.
Proof. Take k > 1 and ≤ n. Hence $1/n^k - 1/n^{k+1} = (n-1)/n^k n^{k+1} > 0$ for all finite n and k ≤ n. QED.
For k = n the gap, $(n-1)/n^{2n+1}$, remains positive for all finite n, no matter how big.

It seems the axiom of continuity stands on the same footing as the axiom on parallel lines in Euclidean geometry. Because $y = e^x$ is irrational for all real x, it follows that $e^{ix} = \text{Cos } x + i \text{ Sin } x$ is also irrational for all real x apart from x=0 and x = π. Hence Cos x is irrational for all real $x \in [-2\pi, 2\pi]$. Although the rationals are dense in the space of real numbers, this shows curves in this space without any rational numbers.

Open Problems

1. Does every irrational have a simple continued fraction.
2. Although the Dedekind cut of an irrational is unique, the relation between two cuts, > , <, =, cannot be inferred unless they are given by constructive countably infinite sequences of rationals. Examples of such information are the formulas for e or π.

References

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Hawking, Stephen. 2007. God Created the Integers. Philadelphia: Running Press.

Knopp, Konrad. 1951. Theory and Application of Infinite Series. Trans. 2 nd German Edition by R. G. H. Young. London : Blackie.

Niven, Ivan and Zuckerman, Herbert S. 1966. An Introduction to the Theory of Numbers. 2d ed. New York: Wiley

Zygmund, Antoni. 1955. Trigonometrical Series. New York ?: Dover

Programs for Diagrams

In[•]:=
```
eul := Line[{{0, ⅇ}, {18, ⅇ}}]
```

In[•]:=
```
lg := Graphics[{Blue, Thick, eul}]
```

In[•]:=
```
lbl := Graphics[Text["ⅇ", {.6, 2.76}]]
```

In[•]:= high[n_Integer] := Table$\left[\left(1+\frac{1}{i}\right)^{i}, \{i, 1, n\}\right]$

In[•]:= vhigh[n_Integer] := Table$\left[\left(1+\frac{1}{i}\right)^{i+1}, \{i, 1, n\}\right]$

In[•]:= tst[n_Integer] := Table$\left[\frac{1}{i}\left(1+\frac{1}{i}\right)^{i}, \{i, 1, n\}\right]$

In[•]:=
```
vhigh[1]
```

Out[•]= {4}

In[•]:=
```
high[1]
```

Out[•]= {2}

In[•]:= low[n_Integer] := Table$\left[\left(1+\frac{1}{1+i}\right)^{i}, \{i, 1, n\}\right]$