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The temperature measured by a uniformly accelerated observer

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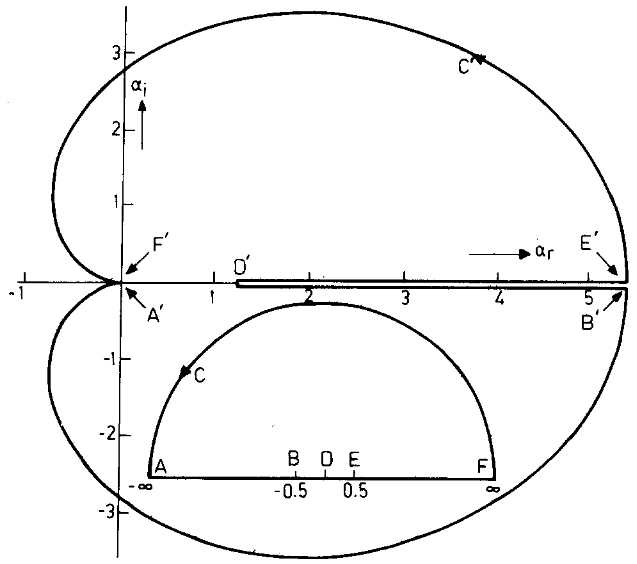

Fig. 4. Plots of the contour C in the ω-plane and of its mapping C' on the α-plane.

where N is the number of zeros of the denominator in the interior of C.[14] The integral will be evaluated in the α-plane. Through (32) and (C1) $\alpha_r(\omega)$ and $\alpha_i(\omega)$ are known everywhere on C. This makes the mapping of C simple. The image C' is shown in Fig. 4. The mapping is conformal except at the points $\omega = 0$, $\pm \frac{1}{2}$, and ∞. The integral (C2) transforms to

$$\int_{C'} \frac{d\alpha}{\alpha - a} = 2\pi i N$$

in the α-plane. Now, if a lies in the interior of C', N equals unity. On the other hand, if a is a point outside of C', N equals zero. So, in the upper ω half-plane the function $\alpha(\omega)$ has one and only one a-point if C' encircles the point a and nowhere takes the value of any a outside of C'. As $(-2\omega \pm 1)^{1/2} = \mp i(2\omega \mp 1)^{1/2}$, $\alpha(\omega)$ in (32) is an even function. Consequently C' is the boundary of the total domain of values of α over the whole ω-plane.

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Temperature measured by a uniformly accelerated observer

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By use of elementary quantum-mechanical arguments we demonstrate how an observer with uniform acceleration a perceives himself as immersed in a heat bath of temperature $kT = a\hbar/2\pi c$.

There has been much interest of late in the subject of black holes and the existence of an intrinsic thermodynamic temperature.[1] This phenomenon can also be demonstrated more simply in the case of a uniformly accelerating observer, who in his rest frame sees himself as being immersed in a heat bath of temperature[2]

$$kT = a\hbar/2\pi c, \tag{1}$$

where a is the proper acceleration. Although this result is generally derived and discussed in the context of quantum field theory,[3] we show in this note that this temperature dependence may also be demonstrated via simple quantum-mechanical arguments which are accessible to advanced undergraduates—or at least to any student who has had exposure to the quantized radiation field.

We consider first the electromagnetic decay of a one electron atomic state $|A\rangle$ at zero temperature. We write the Hamiltonian as

$$H = H_0 + V_I, \tag{2}$$

where

$$H_0 = -\frac{\hbar^2\nabla^2}{2m} + e\phi(r) \tag{3}$$

is the free Hamiltonian for the atomic system and[4]

$$V_I = \frac{i\hbar e}{mc}\mathbf{A}(\mathbf{x},t)\cdot\nabla + \frac{e^2}{2mc^2}\mathbf{A}(\mathbf{x},t)\cdot\mathbf{A}(\mathbf{x},t) \tag{4}$$

represents the interaction between the atom and the photon field. Here

$$\mathbf{A}(\mathbf{x},t) = \int \frac{d^3k}{(2\pi)^{3/2}}\sqrt{\frac{\hbar c^2}{2\omega}}\sum_\lambda \left(\hat{\epsilon}(\mathbf{k},\lambda)e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t}a_{\mathbf{k},\lambda} + \hat{\epsilon}(\mathbf{k},\lambda)e^{-i\mathbf{k}\cdot\mathbf{x} + i\omega t}a^\dagger_{\mathbf{k},\lambda}\right) \tag{5}$$

is the radiation field operator in the usual Coulomb gauge

$$\nabla\cdot\mathbf{A} = 0 \tag{6}$$

and $a_{\mathbf{k},\lambda}$, $a^\dagger_{\mathbf{k},\lambda}$ are the usual annihilation, creation operators for photons—if $|n_{\mathbf{k},\lambda}\rangle$ is a state with n photons in the mode $\mathbf{k},\lambda$ then[5]

$$\begin{aligned} a_{\mathbf{k},\lambda}|n_{\mathbf{k}',\lambda'}\rangle &= \sqrt{n_{k,\lambda}}\,\delta_{\lambda\lambda'}\delta^3(\mathbf{k}-\mathbf{k}')|(n-1)_{\mathbf{k},\lambda}\rangle, \\ a^\dagger_{\mathbf{k},\lambda}|n_{\mathbf{k},\lambda}\rangle &= \sqrt{n_{\mathbf{k},\lambda}+1}\,|(n+1)_{\mathbf{k},\lambda}\rangle. \end{aligned} \tag{7}$$

For simplicity suppose that our atomic system has only two levels—an excited state $|A\rangle$ and a ground state $|B\rangle$—and we require that at $t=0$ the system is in the state

$$|\Psi(t=0)\rangle = |A\rangle|\text{vac}\rangle, \tag{8}$$

where $|\text{vac}\rangle$ represents the state with no photons present. The state $|\Psi(t=0)\rangle$ is, of course, an eigenstate of H_0:

$$H_0|\Psi(t=0)\rangle = E_A^{(0)}|\Psi(t=0)\rangle, \tag{9}$$

where $E_A^{(0)}$ is the energy of state A in the absence of any radiation field coupling

$$\left(-\frac{\hbar^2\nabla^2}{2m} + e\phi(r)\right)|A\rangle = E_A^{(0)}|A\rangle. \tag{10}$$

We now calculate the amplitude that the system remains in the excited state $|A\rangle|\text{vac}\rangle$ at later time t. Using time-dependent perturbation theory this is

$$F_A(t) = \langle\text{vac}|\langle A|U(t)|A\rangle|\text{vac}\rangle, \tag{11}$$

where[6]

$$\begin{aligned} U(t) &= e^{-(i/\hbar)H_0t} - \frac{i}{\hbar}\int_0^t dt'\,e^{-(i/\hbar)H_0(t-t')}V_I(t')e^{-(i/\hbar)H_0t'} \\ &\quad + \left(-\frac{i}{\hbar}\right)^2\int_0^t dt''\int_0^{t''}dt'\,e^{-i(1/\hbar)H_0(t-t'')} \\ &\quad \times V_I(t'')e^{-(i/\hbar)H_0(t''-t')}V_I(t')e^{-(i/\hbar)H_0t'} + \cdots \end{aligned} \tag{12}$$

is the usual time evolution operator. Neglecting the higher-order term $e^2/2mA^2$ in the interaction[7] we have

$$\langle\text{vac}|\langle A|V_I(t')|A\rangle|\text{vac}\rangle = 0 \tag{13}$$

and so must go to second order in the interaction.

$$\langle\text{vac}|\langle A|V_I(t'')e^{-i/\hbar H_0(t''-t')}V_I(t')|A\rangle|\text{vac}\rangle. \tag{14}$$

We find then, in the electric dipole approximation—$e^{i\mathbf{k}\cdot\mathbf{x}} \approx 1$—

$$\begin{aligned} F_A(t) &= e^{-(i/\hbar)E_A^{(0)}t}\Big[1 + \left(\frac{e\hbar}{mc}\right)^2\int_0^t dt''\int_0^{t''}dt' \\ &\quad \times e^{(i/\hbar)E_A^{(0)}(t''-t)}\langle A|\nabla_i|B\rangle e^{-(i/\hbar)E_B^{(0)}(t''-t')} \\ &\quad \times \langle B|\nabla_j|A\rangle G_{ij}(t'',t',0,0) + \cdots\Big], \end{aligned} \tag{15}$$

where

$$\begin{aligned} G_{ij}(t_2,t_1,\mathbf{x}_2,\mathbf{x}_1) &= \langle\text{vac}|A_i(t_2,\mathbf{x}_2)A_j(t_1,\mathbf{x}_1)|\text{vac}\rangle \\ &= \int\frac{d^3k}{(2\pi)^3}\left(\frac{\hbar c^2}{2\omega_k}\right)\sum_\lambda \hat{\epsilon}_i(\mathbf{k},\lambda)\hat{\epsilon}_j(\mathbf{k},\lambda) \\ &\quad \times e^{-i\omega(t_2-t_1) + i\mathbf{k}\cdot(\mathbf{x}_2-\mathbf{x}_1)}. \end{aligned} \tag{16}$$

The time integration is most easily accomplished via a change of variables

$$\int_o^t dt''\int_0^{t''}dt' = \int_0^{t/2}dS\int_0^S ds + \int_{t/2}^t dS\int_0^{2t-2S}ds, \tag{17}$$

where $S=(t''+t')/2$, $s=t''-t'$. Then

$$\begin{aligned} &\int_0^t dt''\int_0^{t''}dt'\,e^{i/\hbar(E_A^{(0)}-E_B^{(0)}-\hbar\omega)(t''-t')} \\ &= \frac{i\hbar}{E_A^{(0)}-E_B^{(0)}-\hbar\omega}t + \left(\frac{-i\hbar}{E_A^{(0)}-E_B^{(0)}-\hbar\omega}\right)^2 \\ &\quad \times\left(e^{i/\hbar(E_A^{(0)}-E_B^{(0)}-\hbar\omega)t/2} + \tfrac{1}{2}e^{i/\hbar(E_A^{(0)}-E_B^{(0)}-\hbar\omega)t} - \tfrac{3}{2}\right). \end{aligned} \tag{18}$$

The term involving the exponentials is rapidly oscillating and can be neglected for reasonable values of t.[8] Finally then the integration over $\mathbf{k}$ may be performed using the usual convergence factor $\omega \to \omega + i\epsilon$ and the identity[9]

$$\frac{1}{\omega-\omega_0+i\epsilon} = P\left(\frac{1}{\omega-\omega_0}\right) - i\pi\delta(\omega-\omega_0). \tag{19}$$

We have then

$$\begin{aligned} F_A(t) &= e^{-(i/\hbar)E_A^{(0)}t}\left[1 - \frac{i}{\hbar}\left(\Delta E_A - i\frac{\hbar\Gamma_A}{2}\right)t + \cdots\right] \\ &\approx e^{(i/\hbar)(E_A^{(0)}+\Delta E_A)t}e^{-\Gamma_A t/2}, \end{aligned} \tag{20}$$

where

$$\Delta E_A = \frac{e^2(E_A^{(0)}-E_B^{(0)})^2}{\hbar}\int\frac{d^3k}{(2\pi)^3 2\omega}\sum_\lambda P \times\frac{|\langle B|\hat{\epsilon}(\mathbf{k},\lambda)\cdot\mathbf{r}|A\rangle|^2}{E_A^{(0)}-E_B^{(0)}-\hbar\omega} \tag{21}$$

is the usual energy shift predicted via second-order, time-independent perturbation theory and

$$\Gamma_A = \frac{e^2(E_A^{(0)}-E_B^{(0)})^2}{\hbar^2}\int\frac{d^3k}{(2\pi)^3 2\omega}\sum_\lambda 2\pi\delta(E_A^{(0)}-E_B^{(0)}-\hbar\omega) \times|\langle B|\hat{\epsilon}(\mathbf{k},\lambda)\cdot\mathbf{r}|\text{A}\rangle|^2 \tag{22}$$

is the width of the excited state $|A\rangle$ as given by Fermi's golden rule. Here we have used the identity

$$[H_0,\mathbf{r}] = -\frac{\hbar^2}{m}\nabla. \tag{23}$$

Note also that if we apply similar techniques to the ground state $|B\rangle|\text{vac}\rangle$ we find

$$\begin{aligned} F_B(t) &= \langle\text{vac}|\langle B|U(t)|B\rangle|\text{vac}\rangle \\ &\simeq e^{-i(1/\hbar)}(E_B^{(0)}+\Delta E_B)t, \end{aligned} \tag{24}$$

$$\Delta E_B = \frac{e^2(E_A^{(0)}-E_B^{(0)})^2}{\hbar}\int\frac{d^3k}{(2\pi)^3 2\omega}\sum_\lambda P \times\frac{|\langle A|\hat{\epsilon}(\mathbf{k},\lambda)\cdot\mathbf{r}|B\rangle|^2}{E_B^{(0)}-E_A^{(0)}-\hbar\omega} \tag{25}$$

is the energy shift of state B given by its interaction with the radiation field and there is no decay width Γ_B since

$$E_B^{(0)} - E_A^{(0)} - \hbar\omega < 0. \tag{26}$$

Therefore then we have merely derived familiar results by a somewhat cumbersome procedure. However, this method may easily now be generalized to the situation that the atom is immersed in a heat bath at temperature T and to the case of finite acceleration. In the former case it is well known that the "vacuum" state $|\mathrm{vac}\rangle_T$ is no longer a state with no photons present but rather is a state having[10]

$$n(\omega) = 1/[\exp(\hbar\omega/kT) - 1] \tag{27}$$

photons in *each* mode $\mathbf{k}, \lambda$. We can now repeat the previous analysis, defining

$$F_S^T(t) = {}_T\langle \mathrm{vac}|\langle S\,|U(t)|S\rangle|\mathrm{vac}\rangle_T \qquad \mathrm{S = A,B}. \tag{28}$$

The derivation goes through as before, but now with

$$\begin{aligned} G_{ij}^T(t_2,t_1,\mathbf{x}_2,\mathbf{x}_1) &= {}_T\langle \mathrm{vac}|A_i(\mathbf{x}_2,t_2)A_j(\mathbf{x}_1,t_1)|\mathrm{vac}\rangle_T \\ &= \int \frac{d^3k}{(2\pi)^3 2\omega}\hbar c^2 \sum_\lambda \hat{\epsilon}_i(\mathbf{k},\lambda)\hat{\epsilon}_j(\mathbf{k},\lambda)[n(\omega)e^{i\omega(t_2-t_1)-i\mathbf{k}\cdot(\mathbf{x}_2-\mathbf{x}_1)} + [n(\omega)+1]e^{-i\omega(t_2-t_1)+i\mathbf{k}\cdot(\mathbf{x}_2-\mathbf{x}_1)}] \\ &= \int \frac{d^3k}{(2\pi)^3 2\omega}\hbar c^2 \sum_\lambda \hat{\epsilon}_i(\mathbf{k},\lambda)\hat{\epsilon}_j(\mathbf{k},\lambda)\left[\frac{1}{e^{\hbar\omega/kT}-1}e^{i\omega(t_2-t_1)-i\mathbf{k}\cdot(\mathbf{x}_2-\mathbf{x}_1)} + \left(\frac{1}{e^{\hbar\omega/kT}-1}+1\right)e^{-i\omega(t_2-t_1)+i\mathbf{k}\cdot(\mathbf{x}_2-\mathbf{x}_1)}\right], \end{aligned} \tag{29}$$

where we have used Eq. (7). The terms in this expression are easy to interpret—the piece involving simply $e^{-i\omega\Delta t}$ is the spontaneous emission considered previously, the term involving $n(\omega)e^{-i\omega\Delta t}$ represents stimulated emission while $n(\omega)e^{+i\omega\Delta t}$ accounts for absorption. Substitution into Eq. (15) then gives

$$F_S^T(t) = e^{-(i/\hbar)(E_S^{(0)}+\Delta E_S)t}e^{-\Gamma_S t/2} \qquad S = A,B \tag{30}$$

with

$$\begin{aligned} \Delta E_A &= \frac{e^2(E_A^{(0)}-E_B^{(0)})^2}{\hbar}P\int \frac{d^3k}{(2\pi)^3 2\omega}\sum_\lambda |\langle B\,|\hat{\epsilon}(\mathbf{k},\lambda)\cdot\mathbf{r}|A\rangle|^2 \\ &\quad\times\left(\frac{1}{E_A^{(0)}-E_B^{(0)}-\hbar\omega} + 2\frac{(E_A^{(0)}-E_B^{(0)})}{(E_A^{(0)}-E_B^{(0)})^2-\hbar^2\omega^2}\frac{1}{e^{\hbar\omega/kT}-1}\right), \\ \Delta E_B &= -\frac{e^2(E_A^{(0)}-E_B^{(0)})^2}{\hbar}P\int \frac{d^3k}{(2\pi)^3 2\omega}\sum_\lambda |\langle B\,|\hat{\epsilon}(\mathbf{k},\lambda)\cdot\mathbf{r}|A\rangle|^2\left(\frac{1}{E_A^{(0)}-E_B^{(0)}+\hbar\omega} + 2\frac{(E_A^{(0)}-E_B^{(0)})}{(E_A^{(0)}-E_B^{(0)})^2-\hbar^2\omega^2}\frac{1}{e^{\hbar\omega/kT}-1}\right) \\ \Gamma_B &= e^{-(E_A^{(0)}-E_B^{(0)})/kT}\Gamma_A = \frac{e^2(E_A^{(0)}-E_B^{(0)})^2}{\hbar^2}\int \frac{d^3k}{(2\pi)^3 2\omega}\sum_\lambda |\langle B\,|\hat{\epsilon}(\mathbf{k},\lambda)\cdot\mathbf{r}|A\rangle|^2 \\ &\quad\times 2\pi\delta(E_A^{(0)}-E_B^{(0)}-\hbar\omega)(e^{\hbar\omega/kT}-1)^{-1}. \end{aligned} \tag{31}$$

We see then that there exist temperature dependent components to the radiation field energy shifts which change E_A and E_B by identical amounts but in opposite directions. Also, we note that state B has now picked up a nonzero decay width Γ_B corresponding to absorption. Of course, if the system is in thermal equilibrium, then transitions from A to B must proceed at the same rate as those from B to A, so the populations of these two states must be related by

$$n_A\Gamma_A = n_B\Gamma_B, \tag{32}$$

$$\frac{n_A}{n_B} = \frac{\Gamma_B}{\Gamma_A} = e^{-(E_A^{(0)}-E_B^{(0)})/kT}, \tag{32}$$

as expected.

Finally, consider now that the atom is traveling through Minkowski space at fixed acceleration a. We may then represent its motion as[11]

$$t = \frac{c}{a}\sinh\frac{a\tau}{c}, \qquad z = \frac{c^2}{a}\left(\cosh\frac{a\tau}{c} - 1\right), \qquad x = y = 0, \tag{33}$$

in terms of the proper time τ measured by an observer traveling with the atom. As seen by this observer riding with the atom, the amplitude to remain in the same state A or B after time τ is

$$\begin{aligned} F_S^a(\tau) &= {}_a\langle \mathrm{vac}|\langle S\,|U(\tau)|S\rangle|\mathrm{vac}\rangle_a \\ &= e^{-(1/\hbar)E_S^{(0)}\tau}\left[1 + \left(\frac{e\hbar}{mc}\right)^2\int_0^\tau d\tau''\int_0^{\tau''} d\tau'\, e^{(i/\hbar)(E_S^{(0)}-E_{S'}^{(0)})(\tau''-\tau')}\langle S\,|\nabla_i|S'\rangle\langle S'|\nabla_j|S\rangle G_{ij}^a(\tau'',\tau',0,0) + \cdots\right] \end{aligned} \tag{34}$$

where $|\mathrm{vac}\rangle_a$ is the Minkowski vacuum as perceived by the accelerated observer and

$$G_{ij}^a(\tau_2,\tau_1,\mathbf{x}_2,\mathbf{x}_1) = {}_a\langle \mathrm{vac}|A_i(\mathbf{x}_2,\tau_2)A_j(\mathbf{x}_1,\tau_1)|\mathrm{vac}\rangle_a. \tag{35}$$

Now, however, we cannot expand as in Eq. (16) since at constant acceleration we no longer have simple plane waves as solutions of the Maxwell equations. In principle one could solve the Maxwell equations for constant acceleration and write G^a_{ij} as a sum over such solutions.

However, it is simpler to perform a coordinate transformation. We note that, temporarily forgetting about the polarization terms, the $\mathbf{k}$ integration in Eq. (16) can be performed[12]

$$\delta^{ij}G_{ij}(t_2, t_1, \mathbf{x}_2, \mathbf{x}_1) = 2\int \frac{d^3k}{(2\pi)^3}\left(\frac{\hbar c^2}{2\omega}\right)e^{-i\omega(t_2-t_1)+i\mathbf{k}\cdot(\mathbf{x}_2-\mathbf{x}_1)}$$
$$= \frac{\hbar c}{2\pi^2}\int_0^\infty dk\, e^{-ikc(t_2-t_1)}\frac{\sin k|\mathbf{x}_2-\mathbf{x}_1|}{|\mathbf{x}_2-\mathbf{x}_1|} = -\frac{\hbar c}{2\pi^2}\frac{1}{c^2(t_2-t_1)^2-(\mathbf{x}_2-\mathbf{x}_1)^2}. \tag{36}$$

Now considering the atom moving through Minkowski space at finite acceleration we have from Eq. (33)

$$c^2(t_2-t_1)^2-(\mathbf{x}_2-\mathbf{x}_1)^2 = \frac{c^4}{a^2}\left(\sinh\frac{a\tau_2}{c}-\sinh\frac{a\tau_1}{c}\right)^2 - \frac{c^4}{a^2}\left(\cosh\frac{a\tau_2}{c}-\cosh\frac{a\tau_1}{c}\right)^2 = \frac{c^4}{a^2}\sinh^2\left(a\frac{\tau_2-\tau_1}{2c}\right). \tag{37}$$

This coordinate change yields the correlation function $\widetilde{G}$ in the moving system.

$$\delta^{ij}\widetilde{G}_{ij}(\tau_2,\tau_1,0,0) = -\frac{\hbar a^2}{2\pi^2c^3}\operatorname{csch}^2\left(a\frac{\tau_2-\tau_1}{2c}\right). \tag{38}$$

It is convenient to rewrite this function as[13]

$$-\frac{\hbar a^2}{2\pi^2c^3}\operatorname{csch}^2\left(a\frac{\tau-i\epsilon}{2c}\right) = -\frac{\hbar}{2\pi^2c}\left\{\frac{1}{(\tau-i\epsilon)^2}+\sum_{n=1}^{\infty}\left[\left(\tau-i\frac{2\pi c}{a}n\right)^2\right]^{-1}+\left[\left(\tau+i\frac{2\pi c}{a}n\right)^2\right]^{-1}\right\}, \tag{39}$$

so that

$$-\frac{\hbar a^2}{2\pi^2c^3}\operatorname{csch}^2\left(a\frac{\tau-i\epsilon}{2c}\right) = 2\int\frac{d^3k}{(2\pi)^3}\left(\frac{\hbar c^2}{2\omega}\right)\left[e^{-i\omega\tau}+\sum_{n=1}^{\infty}\left(e^{-i\omega[\tau-i(2\pi c/a)n]}+e^{i\omega[\tau+i(2\pi c/a)n]}\right)\right]$$
$$= 2\int\frac{d^3k}{(2\pi)^3}\left(\frac{\hbar c^2}{2\omega}\right)\left(e^{-i\omega\tau}+\frac{1}{e^{2\pi\omega c/a}-1}\left(e^{-i\omega\tau}+e^{i\omega\tau}\right)\right). \tag{40}$$

Denoting then symbolically[14]

$$\sum_\lambda \hat{\epsilon}_i(\mathbf{k},\lambda)\hat{\epsilon}_j(\mathbf{k},\lambda) \underset{a\neq o}{\longrightarrow} \sum_\lambda \hat{\epsilon}^a_i(\mathbf{k},\lambda)\hat{\epsilon}^a_j(\mathbf{k},\lambda) \tag{41}$$

we have, in complete analogy to the derivation of Eq. (20),

$$F^a_S(\tau) = e^{(-i/\hbar)(E^{(0)}_S+\Delta E_S)\tau}e^{-\Gamma_S\tau/2}, \tag{42}$$

where

$$\Delta E^a_A = \frac{e^2(E^{(0)}_A-E^{(0)}_B)^2}{\hbar}\int\frac{d^3k}{(2\pi)^3 2\omega}\sum_\lambda|\langle B|\hat{\epsilon}^a(\mathbf{k},\lambda)\cdot\mathbf{r}|A\rangle|^2$$
$$\times P\left(\frac{1}{E^{(0)}_A-E^{(0)}_B-\hbar\omega}+\frac{1}{e^{2\pi\omega c/a}-1}2\frac{(E^{(0)}_A-E^{(0)}_B)}{(E^{(0)}_A-E^{(0)}_B)^2-\hbar^2\omega^2}\right),$$
$$\Delta E^a_B = -\frac{e^2(E^{(0)}_A-E^{(0)}_B)^2}{\hbar}\int\frac{d^3k}{(2\pi)^3 2\omega}\sum_\lambda|\langle B|\hat{\epsilon}^a(\mathbf{k},\lambda)\cdot\mathbf{r}|A\rangle|^2$$
$$\times P\left(\frac{1}{E^{(0)}_A-E^{(0)}_B+\hbar\omega}+\frac{1}{e^{2\pi\omega c/a}-1}2\frac{(E^{(0)}_A-E^{(0)}_B)}{(E^{(0)}_A-E^{(0)}_B)^2-\hbar\omega^2}\right), \tag{43}$$
$$\Gamma^a_B = e^{-2\pi c(E^{(0)}_A-E^{(0)}_B)/a\hbar}\Gamma^a_A = \frac{e^2(E^{(0)}_A-E^{(0)}_B)^2}{\hbar^2}\int\frac{d^3k}{(2\pi)^3 2\omega}\sum_\lambda$$
$$\times 2\pi\delta(E^{(0)}_A-E^{(0)}_B-\hbar\omega)|\langle B|\hat{\epsilon}^a(\mathbf{k},\lambda)\cdot\mathbf{r}|A\rangle^2\left(\frac{1}{e^{2\pi\omega c/a}-1}\right).$$

These terms then have the same form as their finite temperature counterparts—Eq. (31)—but with

$$kT = a\hbar/2\pi c, \tag{44}$$

which is the correspondence we were attempting to demonstrate.

Although mathematically we have shown then this correspondence between temperature and acceleration, it is also possible to understand physically what is taking place. It is well known that in the state we have called $|\text{vac}\rangle$—i.e., the state at temperature zero with no photons present—there exists an energy $\frac{1}{2}\hbar\omega$ per mode which is associated with the so-called zero-point motion of a quantum field.[15] That is, we have for each mode

$$\langle\text{vac}|\tfrac{1}{2}[E^2(\mathbf{k},\lambda)+B^2(\mathbf{k},\lambda)]|\text{vac}\rangle = \tfrac{1}{2}\hbar\omega. \tag{45}$$

Since $|E| = |B|$ for the radiation field we have

$$\langle\text{vac}|E^2(\mathbf{k},\lambda)|\text{vac}\rangle = \tfrac{1}{2}\hbar\omega, \tag{46}$$

so that there exists an average strength of the electric field strength even in the vacuum state. This field strength may be considered to be the reason for spontaneous emission. This can be demonstrated by using Fermi's golden rule and the relation[16]

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} \tag{47}$$

between the electric field and the vector potential (at $T=0$)

$$\begin{aligned}\Gamma_A &= \int \frac{d^3k}{(2\pi)^3\hbar} 2\pi\delta(E_A^{(0)} - E_B^{(0)} - \hbar\omega)e^2 \\ &\quad\times \sum_\lambda \langle A|r_i|B\rangle\langle B|r_j|A\rangle \\ &\quad\times \langle \mathrm{vac}|E_i(\mathbf{k},\lambda)E_j(\mathbf{k},\lambda)|\mathrm{vac}\rangle \\ &= \int \frac{d^3k}{(2\pi)^3}\frac{2\pi}{\hbar}\delta(E_A^{(0)} - E_B^{(0)} - \hbar\omega)e^2\frac{\hbar\omega}{2} \\ &\quad\times \sum_\lambda |\langle B|\hat{\epsilon}(\mathbf{k},\lambda)\cdot\mathbf{r}|A\rangle|^2 \end{aligned} \tag{48}$$

in agreement with Eq. (22). Thus it is the interaction of the electron with these *virtual* photons associated with the zero-point motion which is responsible for spontaneous emission. When the electron is accelerated it still sees this ensemble of virtual photons, of course, but now because of the particular form of its motion—Eq. (33)—we see by comparison of Eq. (43) with Eq. (31) that some of these virtual states appear as if they were *real* photon states with a thermal distribution.

$$n(\omega) = 1/(e^{2\pi\omega c/a} - 1) \tag{49}$$

associated with the temperature $kT = a\hbar/2\pi c$.

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[7]Although this term gives a contribution

$$A = \frac{e^2}{2mc^2}\int \frac{d^3k}{(2\pi)^3 2\omega}$$

to the energy of each level, it does not contribute to energy differences or decay rate at order e^2 and thus will be dropped hereafter.

[8]Note that for t small we can expand the exponentials and find

$$F_A(t) \cong e^{-iE_A^{(0)}t}\left[1 - \frac{3}{8}\frac{\hbar^2 e^2}{m^2c^2}\int \frac{d^3k}{(2\pi)^3}\left(\frac{\hbar c^2}{2\omega}\right)\times\sum_\lambda |\langle B|\hat{\epsilon}(\mathbf{k},\lambda)\cdot\nabla|A\rangle|^2 t^2 + \cdots\right]$$

so that the decay rate is nonexponential. This behavior is well known and disappears at any measurable point in time so that only the term linear in t need be considered.

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[12]The integral over ω can be performed directly using a convergence factor $t_2 - t_1 - i\epsilon$ or as given in *Table of Integrals, Series and Products*, edited by I. Gradshteyn and I. Rhzhik (Academic, New York, 1965), 3.893, No. 1.

[13]M. Abramowitz and J. Stegun, *Handbook of Mathematical Functions* (Natl. Bur. Stand., Washington, DC, 1968). Here we have used Eq. 4.3.92 to write

$$\begin{aligned}\mathrm{csch}^2 x &= -\csc^2 ix = \sum_{k=-\infty}^{\infty}\frac{1}{(x+ik\pi)^2} \\ &= \frac{1}{x^2} + \sum_{k=1}^{\infty}\left(\frac{1}{(x+ik\pi)^2} + \frac{1}{(x-ik\pi)^2}\right).\end{aligned}$$

[14]In the uniformly accelerated frame, we no longer necessarily have the simple result

$$\sum_\lambda \hat{\epsilon}_i(\mathbf{k},\lambda)\hat{\epsilon}_j(\mathbf{k},\lambda) = \delta_{ij} - \hat{k}_i\hat{k}_j.$$

In fact there is some debate at present as to whether the angular distribution remains isotropic. Cf. U. Gerlach, Phys.Rev. D **15**, 2310 (1983); K. Hinton, P. C. W. Davies, and J. Pfautsch, Phys. Lett. **120** B, 88 (1983).

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[16]We are working in a transverse gauge so that $\phi = 0$ and the usual result

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$$

reduces to Eq. (47).