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Quantum gravitational corrections to the nonrelativistic scattering potential of two masses

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We treat general relativity as an effective field theory, obtaining the full nonanalytic component of the scattering matrix potential to one-loop order. The lowest order vertex rules for the resulting effective field theory are presented and the one-loop diagrams which yield the leading nonrelativistic post-Newtonian and quantum corrections to the gravitational scattering amplitude to second order in G are calculated in detail. The Fourier transformed amplitudes yield a nonrelativistic potential and our result is discussed in relation to previous calculations. The definition of a potential is discussed as well, and we show that the ambiguity of the potential under coordinate changes is resolved.

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I. INTRODUCTION

The idea that a field theory need not be strictly renormalizable in the traditional sense yet can still yield useful quantum predictions when treated as an effective field theory [1] has been clearly demonstrated in chiral perturbation theory and in other applications [2]. Quantum loop calculations lead to well defined results in the low energy limit. Interestingly, such methods can also be applied to general relativity. As an effective field theory, the quantization of general relativity can be carried out in a consistent way, since the troublesome singularities which occur for various types of matter sources in traditional renormalization schemes [3–7] can be absorbed into phenomenological constants which characterize the effective action of the theory. This effective field theory approach offers then a possible way around the familiar renormalization difficulties of general relativity in the low energy regime and, using this approach with background field quantization [8] of general relativity, one of us [9,10] some years ago derived the leading quantum and classical corrections to the Newtonian potential of two large nonrelativistic masses. This calculation has since been the focus of a number of publications [11–14], and this work continues, most recently in the paper [15]. Unfortunately, because of the difficulty of the calculation and its myriad of tensor indices there has been little agreement among these various authors. The classical component of the correction has previously been discussed in Refs. [16–20], and here there is general agreement although, as we shall discuss, there exists an unavoidable ambiguity in defining the potential. The basic disagreements lie rather in the quantum corrections, and in the present paper we shall present what we believe to be the *definitive* result for the leading classical and quantum corrections of order G^2, using the full scattering amplitude as the definition of the nonrelativistic potential.

We note that, as a prelude to this effort, in a recent paper [24] two of us have dealt with the quantum and classical corrections to the Reissner-Nordström and Kerr-Newman metrics of charged scalars and fermions. Such quantum and classical corrections have also been considered from the viewpoint of a scattering potential in a paper [25] by one of us. Recently we have also calculated the full classical and quantum corrections to the Schwarzshild and Kerr metrics of scalars and fermions [26] and have shown in detail how the higher order gravitational contributions to these metrics emerge from loop calculations. In the present paper then we consider the corresponding calculation of the full scattering amplitude.

Of course, treating general relativity as an effective field theory is carried out at the cost of introducing a never ending set of additional higher derivative couplings into the theory. In this sense Einstein's general relativity is still a perfectly valid theory for gravitational interactions—although now it represents only the *minimal* theory. At some stage additional derivative couplings must be appended to the Einstein action, signifying manifestations of the higher energy component of the effective field theory. However, the low energy scattering potential is free from these new couplings and represents a model-independent result for quantum gravity.

This calculation is possible because the post-Newtonian and quantum corrections which we consider are determined fully by the *nonanalytic* pieces of the one loop amplitude generated by the lowest order Einstein action. Of course, in order to deal with the ultraviolet divergencies which arise at one loop, one must renormalize the parameters of higher derivative terms in the action. However, such pieces will only affect the analytic parts of the one-loop amplitude, and will not contribute to our potential.

We will employ the same conventions as in our previous papers, namely, $(\hbar=c=1)$ as well as the Minkowski metric convention $(+1,-1,-1,-1)$. We will begin in Sec. II with a very short introduction to the effective field theory quantization of general relativity and focus here especially on the distinction between nonanalytic and analytic contributions to the scattering amplitude. We also include a discussion of the definition of the nonrelativistic potential.

Next in Sec. III we evaluate the diagrams which contribute to the scattering and examine in detail the results for the

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various components. The resulting nonanalytic piece of the scattering amplitude is then used in order to construct the leading corrections to the nonrelativistic gravitational potential. We also discuss our result in relation to previous calculations and attempt to sort out the various inconsistencies in the published numbers. Finally in a concluding section we summarize our findings.

II. REVIEW OF GENERAL RELATIVITY AS AN EFFECTIVE FIELD THEORY

We begin with a brief review of general relativity, the Lagrangian of which (not including a cosmological term) is

$$\mathcal{L}=\sqrt{-g}\left[\frac{2R}{\kappa^2}+\mathcal{L}_{\text{matter}}\right], \tag{1}$$

where $\kappa^2=32\pi G$ is the gravitational coupling, $R^{\mu}_{\nu\alpha\beta}\equiv\partial_\alpha\Gamma^{\mu}_{\nu\beta}-\partial_\beta\Gamma^{\mu}_{\nu\alpha}+\Gamma^{\mu}_{\sigma\alpha}\Gamma^{\sigma}_{\nu\beta}-\Gamma^{\mu}_{\sigma\beta}\Gamma^{\sigma}_{\nu\alpha}$ is the curvature tensor, and g denotes the determinant of the metric field $g_{\mu\nu}$. Here $\sqrt{-g}\mathcal{L}_{\text{matter}}$ is a covariant Lagrangian for the matter fields, and, in principle, any type of matter field could be included. This action defines the classical theory of general relativity.

In order to treat Eq. (1) as an *effective* field theory one must include all possible higher derivative couplings of the fields in the gravitational Lagrangian. In this way any field singularities generated by loop diagrams can be associated with some component of the action and hence can be absorbed via a simple redefinition of the coupling constants of the theory. Treating all such coupling coefficients as experimentally determined quantities, the effective field theory is then finite and contains no singularities at any finite order of the loop expansion.

We can consequently write an effective action for pure general relativity as

$$\mathcal{L}=\sqrt{-g}\left\{\frac{2R}{\kappa^2}+c_1R^2+c_2R^{\mu\nu}R_{\mu\nu}+\ldots\right\}, \tag{2}$$

where the ellipsis denotes that the effective action is in fact an infinite series—at each new loop order additional higher derivative terms must be taken into account. This Lagrangian includes all possible higher derivative couplings, and every coupling constant in the Lagrangian is considered to be determined empirically. Similarly one must include higher derivative contributions to the matter Lagrangian in order to treat this piece of the Lagrangian as an effective field theory. Details of such considerations can be found in Refs. [9,10].

In our calculations we will consider only the nonanalytic contributions, which are generated by the propagation of two or more massless particles in the Feynman diagrams. Such nonanalytic effects are long ranged and, in the low energy limit of the effective field theory, they dominate over the analytic contributions which arise from the propagation of massive modes. The typical nonanalytic terms we will consider are of the type $1/\sqrt{-q^2}$ and $\ln q^2$—while the typical analytic contribution is a power series in q. The feature from which the analytic contributions originate is the propagation of massive particles while the nonanalytic effect comes from massless propagation and can be seen directly by Taylor expanding the two types of propagators. Indeed the massless propagator $1/q^2$ cannot be expanded in a series, while we have the obvious representation for the massive propagator as $1/(q^2-m^2)=-1/m^2(1-q^2/m^2+\ldots)$. Thus the massive parts of the diagrams will always be expandable and hence analytic—while the massless contributions have the possibility of generating nonanalytic components. It should be noted that such nonanalytic pieces of the scattering amplitude are essential to the unitarity of the S matrix.

A. Definition of the potential

Before proceeding to the actual calculations, it is important to note that the precise definition of a potential in a relativistic quantum field theory such as general relativity is not obvious. In the original papers, the one-particle-irreducible potential was calculated [9,10]. However, additional diagrams are required in order to relate this quantity to physical observables. Subsequent work has considered alternative definitions. Clearly a gravitational potential should be gauge invariant in order to make physical sense, but one can choose between various definitions of the potential depending on the physical situation, how the total energy is defined, etc. In QCD one can always turn to the venerable Wilson loop description but there exists no standard Wilson loop description for the gravitational potential, although some work has been done in this direction [21] using the Arnowitt-Deser-Misner formula for the total energy of the gravitational system. Such an approach to the gravitational potential has been taken in Ref. [11]. For yet another approach to the Newtonian potential, see Ref. [22]. An alternative path is to use the scattering amplitude itself to define the potential. This description of the potential seems to us to be the simplest and most intuitive picture and has been employed by a number of authors [17,18,16,12,23,15]. Herein then we shall also use the full scattering amplitude in order to represent the potential, defining

$$\begin{aligned}\langle f|T|i\rangle&\equiv(2\pi)^4\delta^{(4)}(p-p')[\mathcal{M}(q)]\\&=-(2\pi)\delta(E-E')\langle f|\tilde{V}(\mathbf{q})|i\rangle,\end{aligned} \tag{3}$$

where p,p' is the incoming, outgoing four-momentum. The corresponding coordinate space representation can be found by taking the nonrelativistic limit and Fourier transforming, yielding the result

$$V(\mathbf{x})=\frac{1}{2m_1}\frac{1}{2m_2}\int\frac{d^3q}{(2\pi)^3}e^{i\mathbf{q}\cdot\mathbf{x}}\mathcal{M}(\vec{q}), \tag{4}$$

which will serve as our definition of the nonrelativistic potential.

It should be noted, however, that this is not the only way in which to define the potential $V(\mathbf{q})$ in terms of the scatter-

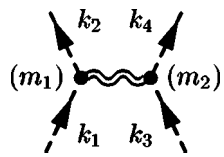

FIG. 1. The tree diagram giving Newton's law.

ing amplitude. One could, for example, subtract off the second order Born contributions, which would lead to the nonrelativistic potential used in bound state quantum mechanics and would be equivalent to using the prescription

$$i\langle f|T|i\rangle = -2\pi i\,\delta(E-E')\left[\langle f|\tilde{V}_{bs}(\mathbf{q})|i\rangle + \sum_n \frac{\langle f|\tilde{V}_{bs}(\mathbf{q})|n\rangle\langle n|\tilde{V}_{bs}(\mathbf{q})|i\rangle}{E-E_n+i\epsilon} + \ldots\right]. \quad (5)$$

The definition of the bound state potential is discussed in detail in Ref. [17]. In particular, in a Hamiltonian treatment there are also terms in the Hamiltonian involving Gp^2/r that contribute at the same order. The relation of the bound state potential $\tilde{V}_{bs}(\mathbf{q})$, in Einstein-Infeld-Hoffmann coordinates, to the lowest order scattering potential is

$$\tilde{V}_{bs}(r) = V(r) + \frac{7Gm_1m_2(m_1+m_2)}{2c^2r^2}. \quad (6)$$

B. The diagrams contributing to the nonanalytic component of the scattering matrix

We will consider here only the nonanalytic contributions from the one-loop diagrams. Since many diagrams yield purely analytic contributions to the S matrix, such diagrams need not be considered and will be omitted from the beginning. The diagrams which *do* yield nonanalytic contributions to the S matrix amplitude are those containing two or more massless propagating particles. Such a typical amplitude will be of the form

$$\mathcal{M} \sim \left[A + Bq^2 + \ldots + \alpha\kappa^4\frac{1}{q^2} + \beta_1\kappa^4\ln(-q^2) + \beta_2\kappa^4\frac{m}{\sqrt{-q^2}} + \ldots\right]. \quad (7)$$

Here the coefficients $A, B, \ldots$ correspond to analytic pieces which are of no interest to us, since these terms will only dominate in the high energy regime of the effective theory. Rather, the $\alpha, \beta_1, \beta_2, \ldots$ coefficients correspond to the nonlocal, nonanalytic contributions to the amplitude and are the ones which we seek. In particular, the β_1, β_2 terms will yield the leading post-Newtonian and quantum corrections to the potential.

III. RESULTS FOR THE FEYNMAN DIAGRAMS

In this section we will present our results. All diagrams have been performed both by hand and by computer. In order to evaluate the diagrams by computer, an algorithm for MAPLE VII (TM)[1] was developed. This program contracts the various indices and performs the loop integrations. All results obtained this way were confirmed results obtained by hand. The resulting amplitudes were then Fourier transformed to produce the scattering potential, and only the nonanalytic pieces of the amplitude were retained. For this part of the calculation, the following Fourier integrals are useful:

$$\int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{r}} \frac{1}{|\mathbf{q}|^2} = \frac{1}{4\pi r},$$

$$\int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{r}} \frac{1}{|\mathbf{q}|} = \frac{1}{2\pi^2 r^2},$$

$$\int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{r}} \ln(\mathbf{q}^2) = \frac{-1}{2\pi r^3}. \quad (8)$$

After this brief introduction, we proceed to give the results for each diagram in turn. Note that the basic vertices needed for our calculation are given in Appendix A.

A. The tree diagram

The result for Fig. 1 in the nonrelativistic limit is the well-known lowest order tree-level result which yields the Newtonian potential. We define the diagram using the Feynman rules as

$$i\mathcal{M}_{1(a)}(\vec{q}) = \tau_1^{\mu\nu}(k_1,k_2,m_1)\left[\frac{i\mathcal{P}_{\mu\nu\alpha\beta}}{q^2}\right]\tau_1^{\alpha\beta}(k_3,k_4,m_2), \quad (9)$$

where $q = k_1 - k_2 = k_4 - k_3$. We will find it most convenient to quote our results in the nonrelativistic limit. Relativistic amplitudes are denoted by $\mathcal{M}$ and nonrelativistic amplitudes by M, with the latter being defined in a normalization given by

$$M_i(\vec{q}) = \frac{1}{2m_1}\frac{1}{2m_2}\mathcal{M}_i(\vec{q}). \quad (10)$$

By contracting all indices for the tree level amplitude and taking the nonrelativistic limit, we find

[1] MAPLE and MAPLE V are registered trademarks of Waterloo Maple Inc.

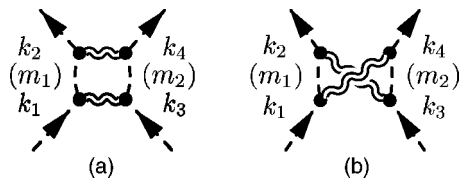

FIG. 2. The box (a) and crossed box (b) diagrams which contribute to the nonanalytic component of the potential.

$$M_{1(a)}(\vec{q}) = -\frac{4\pi G m_1 m_2}{\vec{q}^2}, \tag{11}$$

whose Fourier transform produces the scattering potential

$$V_{1(a)}(r) = -\frac{G m_1 m_2}{r}, \tag{12}$$

which is indeed the familiar Newtonian form.

B. The box and crossed box diagrams

We can write the contributions of Figs. 2(a) and 2(b) as

$$\begin{aligned}\mathcal{M}_{2a} = \int & \frac{d^4 l}{(2\pi)^4} \tau_1^{\mu\nu}(k_1, k_1+l, m_1)\, \tau_1^{\rho\sigma}(k_1+l, k_2, m_1) \\ & \times \tau_1^{\alpha\beta}(k_3, k_3-l, m_2)\, \tau_1^{\gamma\delta}(k_3-l, k_4, m_2) \\ & \times \left[\frac{i}{(k_1+l)^2 - m_1^2}\right]\left[\frac{i}{(k_3-l)^2-m_2^2}\right]\left[\frac{i\mathcal{P}_{\mu\nu\alpha\beta}}{l^2}\right] \\ & \times \left[\frac{i\mathcal{P}_{\rho\sigma\gamma\delta}}{(l+q)^2}\right]\end{aligned} \tag{13}$$

for the box and

$$\begin{aligned}\mathcal{M}_{2b} = \int & \frac{d^4 l}{(2\pi)^4} \tau_1^{\mu\nu}(k_1, k_1+l, m_1)\, \tau_1^{\rho\sigma}(k_1+l, k_2, m_1) \\ & \times \tau_1^{\gamma\delta}(k_3, l+k_4, m_2)\, \tau_1^{\alpha\beta}(l+k_4, k_4, m_2) \\ & \times \left[\frac{i}{(k_1+l)^2 - m_1^2}\right]\left[\frac{i}{(l+k_4)^2-m_2^2}\right]\left[\frac{i\mathcal{P}_{\mu\nu\alpha\beta}}{l^2}\right] \\ & \times \left[\frac{i\mathcal{P}_{\rho\sigma\gamma\delta}}{(l+q)^2}\right]\end{aligned} \tag{14}$$

for the cross box. These diagrams are among the most challenging that we will encounter, because of the rather complicated integrals—containing *four* propagators—which must be evaluated. However, those pieces of the amplitude which are loop momentum-dependent simplify, because the external particles are on shell and we are only seeking the nonanalytic pieces of the scattering amplitude. This allows reduction of parts of the amplitude initially having four propagators to pieces where effectively only three or two propagators remain. An example of this simplification can be seen by the replacement of the integral

$$\begin{aligned}\int & \frac{d^4 l}{(2\pi)^2} \frac{l\cdot k_1}{l^2(l+q)^2[(l+k_1)^2-m_1^2][(l-k_3)^2-m_2^2]} \\ & = \frac{1}{2}\int \frac{d^4 l}{(2\pi)^2} \frac{[(l+k_1)^2]-l^2-m_1^2}{l^2(l+q)^2[(l+k_1)^2-m_1^2][(l-k_3)^2-m_2^2]}\end{aligned} \tag{15}$$

by

$$\frac{1}{2}\int \frac{d^4 l}{(2\pi)^2} \frac{1}{l^2(l+q)^2[(l-k_3)^2-m_2^2]} \tag{16}$$

because the l^2 part will not contribute to the nonanalytic component. The integrals with three or two propagator terms are explicitly given in Appendix B. Another simplification arises when the momentum q^μ contracts with a loop momentum l^μ in the numerator. An example is

$$\begin{aligned}\int & \frac{d^4 l}{(2\pi)^2} \frac{l\cdot q}{l^2(l+q)^2[(l+k_1)^2-m_1^2][(l-k_3)^2-m_2^2]} \\ & = \frac{1}{2}\int \frac{d^4 l}{(2\pi)^2} \frac{(l+q)^2-l^2-q_1^2}{l^2(l+q)^2[(l+k_1)^2-m_1^2][(l-k_3)^2-m_2^2]},\end{aligned} \tag{17}$$

which simplifies to

$$-\frac{1}{2}\int \frac{d^4 l}{(2\pi)^2} \frac{q^2}{l^2(l+q)^2[(l-k_3)^2-m_2^2]}. \tag{18}$$

Via these simplifications one can reduce the box and cross box amplitudes to a reduced piece consisting only of integrals with two or three propagators and a component with the basic form of the box and crossed box integrals, i.e., with no loop momentum terms in the numerator. Then, using the integrals presented in Appendix B, performing the above-described contractions in the two diagrams, and taking the nonrelativistic limit we end up with the result

$$M^{\text{red}}_{2(a)+2(b)}(\vec{q}) = \frac{46}{3} G^2 m_1 m_2 \ln \vec{q}^2 \tag{19}$$

for the momentum-reduced component of the box plus cross box and

$$M^{\text{irred}}_{2(a)+2(b)}(\vec{q}) = 16 G^2 m_1 m_2 \ln \vec{q}^2 \tag{20}$$

for the irreducible piece. Fourier transforming, we find then the scattering potential contributions

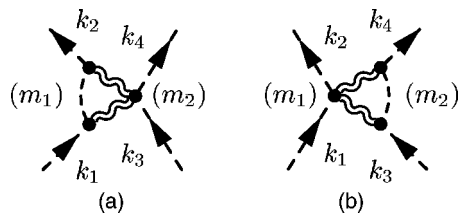

FIG. 3. The set of triangle diagrams contributing to the scattering potential.

$$V^{\rm red}_{2(a)+2(b)}(r)=-\frac{23}{3}\frac{m_1m_2G^2}{\pi r^3} \tag{21}$$

for the reducible component and

$$V^{\rm irred}_{2(a)+2(b)}(r)=-8\frac{m_1m_2G^2}{\pi r^3} \tag{22}$$

for the irreducible piece so that the total result for the box and cross box contribution to the potential is

$$V^{\rm tot}_{2(a)+2(b)}(r)=-\frac{47}{3}\frac{m_1m_2G^2}{\pi r^3}. \tag{23}$$

These results are in agreement with those of Ref. [15].

C. The triangle diagrams

The next piece we will consider is that of Fig. 3, for which we find

$$\mathcal{M}_{3(a)}(q)=\int\frac{d^4l}{(2\pi)^4}\tau_1^{\mu\nu}(k_1,l+k_1,m_1)\tau_1^{\alpha\beta}(l+k_1,k_2,m_1)\times\tau_2^{\sigma\rho\gamma\delta}(k_3,k_4,m_2)\left[\frac{i\mathcal{P}_{\alpha\beta\gamma\delta}}{(l+q)^2}\right]\left[\frac{i\mathcal{P}_{\mu\nu\sigma\rho}}{l^2}\right]\times\left[\frac{i}{(l+k_1)^2-m_1^2}\right], \tag{24}$$

$$\mathcal{M}_{3(b)}(q)=\int\frac{d^4l}{(2\pi)^4}\tau_1^{\sigma\rho}(k_3,k_3-l,m_2)\tau_1^{\gamma\delta}(k_3-l,k_4,m_2)\times\tau_2^{\mu\nu\alpha\beta}(k_1,k_2,m_1)\left[\frac{i\mathcal{P}_{\mu\nu\sigma\rho}}{l^2}\right]\left[\frac{i\mathcal{P}_{\alpha\beta\gamma\delta}}{(l+q)^2}\right]\times\left[\frac{i}{(l-k_3)^2-m_2^2}\right]. \tag{25}$$

The calculation Figs. 3(a) and 3(b) yields no real complications—the integrals needed are quite straightforward and are presented in Appendix B. However, a significant simplification results from the use of the identity

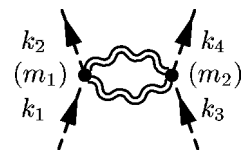

FIG. 4. The double-seagull diagram contribution to the scattering potential.

$$\mathcal{P}_{\gamma\delta\sigma\rho}\mathcal{P}_{\alpha\beta\mu\nu}\tau^{\sigma\rho\mu\nu}(k_1,k_2,m_1)=\tau_{\gamma\delta\alpha\beta}(k_1,k_2,m_1) \tag{26}$$

and, taking the nonrelativistic limit, we find for these two pieces

$$M_{3(a)}(\vec{q})=-8G^2m_1m_2\left(\frac{7}{2}\ln\vec{q}^2+\frac{\pi^2m_1}{|\vec{q}|}\right),$$

$$M_{3(b)}(\vec{q})=-8G^2m_1m_2\left(\frac{7}{2}\ln\vec{q}^2+\frac{\pi^2m_2}{|\vec{q}|}\right). \tag{27}$$

Our results for these diagrams agree with those of Ref. [15], and the Fourier transformed result is

$$V_{3(a)+3(b)}(r)=-4\frac{G^2m_1m_2(m_1+m_2)}{r^2}+28\frac{m_1m_2G^2}{\pi r^3}. \tag{28}$$

D. The double-seagull diagram

We have for the double-seagull term (see Fig. 4)

$$\mathcal{M}_{4(a)}(q)=\frac{1}{2!}\int\frac{d^4l}{(2\pi)^4}\tau_2^{\alpha\beta\gamma\delta}(k_1,k_2,m_1)\tau_2^{\sigma\rho\mu\nu}(k_3,k_4,m_2)\times\left[\frac{i\mathcal{P}_{\alpha\beta\mu\nu}}{(l+q)^2}\right]\left[\frac{i\mathcal{P}_{\gamma\delta\sigma\rho}}{l^2}\right]. \tag{29}$$

The double-seagull loop diagram is quite straightforward, and is simplified by use of the identity in Eq. (26). Note, however, that there exists a symmetry factor of 1/2!. The resulting amplitude is found to be

$$M_{4(a)}(\vec{q})=44G^2m_1m_2\ln\vec{q}^2, \tag{30}$$

whose Fourier transform yields the double-seagull contribution to the potential

$$V_{4(a)}(r)=-22\frac{m_1m_2G^2}{\pi r^3}, \tag{31}$$

which agrees with the result presented in Ref. [15].

E. The vertex correction diagrams

There exist two classes of vertex correction diagrams: For the massive loop diagrams, shown in Figs. 5(a) and 5(b), we have

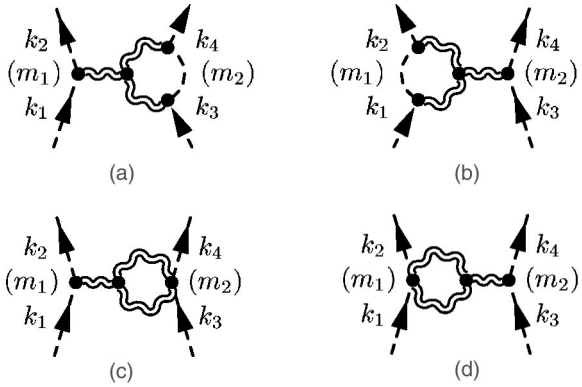

FIG. 5. The class of the graviton vertex corrections which yield nonanalytic corrections to the potential.

$$\mathcal{M}_{5(a)}(q)=\frac{d^4l}{(2\pi)^4}\tau_1^{\alpha\beta}(k_1,k_2,m_1)\tau_1^{\mu\nu}(k_3,k_3-l,m_2)\times\tau_1^{\rho\sigma}(k_3-l,k_4,m_2)\tau_3^{\lambda\kappa\phi\epsilon(\gamma\delta)}(l,-q)\left[\frac{i\mathcal{P}_{\lambda\kappa\mu\nu}}{l^2}\right]\times\left[\frac{i\mathcal{P}_{\phi\epsilon\rho\sigma}}{(l+q)^2}\right]\left[\frac{i\mathcal{P}_{\alpha\beta\gamma\delta}}{q^2}\right]\left[\frac{i}{(l-k_3)^2-m_2^2}\right], \tag{32}$$

$$\mathcal{M}_{5(b)}(q)=\int\frac{d^4l}{(2\pi)^4}\tau_1^{\alpha\beta}(k_1,l+k_1,m_1)\times\tau_1^{\mu\nu}(l+k_1,k_2,m_1)\tau_1^{\lambda\kappa}(k_3,k_4,m_2)\times\tau_3^{\gamma\delta\rho\sigma(\phi\epsilon)}(-l,q)\left[\frac{i\mathcal{P}_{\alpha\beta\gamma\delta}}{l^2}\right]\left[\frac{i\mathcal{P}_{\mu\nu\rho\sigma}}{(l+q)^2}\right]\times\left[\frac{i\mathcal{P}_{\phi\epsilon\lambda\kappa}}{q^2}\right]\left[\frac{i}{(l+k_1)^2-m_1^2}\right], \tag{33}$$

while for the pure graviton loop diagrams shown in Figs. 5(c) and 5(d) we have

$$\mathcal{M}_{5(c)}(q)=\frac{1}{2!}\int\frac{d^4l}{(2\pi)^4}\tau_2^{\lambda\kappa\epsilon\phi}(k_3,k_4,m_2)\tau_1^{\alpha\beta}(k_1,k_2,m_1)\times\tau_3^{\mu\nu\rho\sigma(\gamma\delta)}(l,-q)\left[\frac{i\mathcal{P}_{\mu\nu\lambda\kappa}}{l^2}\right]\left[\frac{i\mathcal{P}_{\rho\sigma\epsilon\phi}}{(l+q)^2}\right]\times\left[\frac{i\mathcal{P}_{\alpha\beta\gamma\delta}}{q^2}\right], \tag{34}$$

$$\mathcal{M}_{5(d)}(q)=\frac{1}{2!}\int\frac{d^4l}{(2\pi)^4}\tau_2^{\rho\sigma\mu\nu}(k_1,k_2,m_1)\tau_1^{\lambda\kappa}(k_3,k_4,m_2)\times\tau_3^{\alpha\beta\gamma\delta(\epsilon\phi)}(-l,q)\left[\frac{i\mathcal{P}_{\mu\nu\gamma\delta}}{l^2}\right]\left[\frac{i\mathcal{P}_{\rho\sigma\alpha\beta}}{(l+q)^2}\right]\times\left[\frac{i\mathcal{P}_{\lambda\kappa\epsilon\phi}}{q^2}\right]. \tag{35}$$

The vertex correction diagrams are certainly the most challenging to perform, disregarding the box and crossed box diagrams, and the results go back to the original calculation of Refs. [9,10]—however, due to an algebraic error, the original result quoted for such diagrams was in error. Since that time, the results for the gravitational vertex corrections have been checked at length in various publications [13,15]; however, until Ref. [26] the correct forms have not been given. Using the results of Ref. [26] and taking the nonrelativistic limit, we find the amplitudes

$$M_{5(a)+5(b)}(\vec{q})=2G^2m_1m_2\left[\frac{\pi^2(m_1+m_2)}{|\vec{q}|}+\frac{5}{3}\ln\vec{q}^2\right],$$

$$M_{5(c)+5(d)}(\vec{q})=-\frac{52}{3}G^2m_1m_2\ln\vec{q}^2, \tag{36}$$

whose Fourier transform yields the corrected results for the vertex modifications to the scattering potential:

$$V_{5(a)+5(b)}(r)=\frac{G^2m_1m_2(m_1+m_2)}{r^2}-\frac{5}{3}\frac{m_1m_2G^2}{\pi r^3}, \tag{37}$$

$$V_{5(c)+5(d)}(r)=\frac{26}{3}\frac{m_1m_2G^2}{\pi r^3}, \tag{38}$$

where again we note the presence of a symmetry factor 1/2! in the case of the pure graviton loop diagram.[2] Our results for these diagrams are *not* in agreement with previous calculations. For the vertex correction diagrams in Figs. 5(a) and 5(b), there is total disagreement with our result. However, we have performed a very detailed analysis of the vertex diagrams in our companion paper [26], showing how the classical ingredients match in detail those required by the classical Schwartzschild metric. The quantum vertex result of Ref. [15] is the same as that of Ref. [13] and the latter yields an incorrect *classical* metric (the authors of Ref. [15] do not display their classical vertex correction). On the other hand, our detailed evaluation of the metric correction in Ref. [26] gives us confidence in the correctness of our result.

F. The vacuum polarization diagram

The result for the vacuum polarization contribution (see Fig. 6) can be found from the amplitude

[2]It is correctly pointed out in Ref. [15] that the coefficient of the two-graviton vertex quoted in Refs. [9,10] is too small by a factor of 2. However, the numerical result for the loop integrals given therein is correct because of the presence of this symmetry factor.

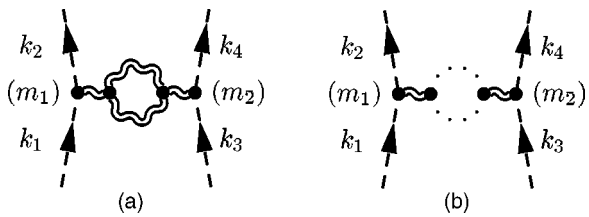

FIG. 6. The vacuum polarization diagrams which contribute to the potential. Note that there exists a ghost diagram along with the graviton loop.

$$\mathcal{M}_{6(a)+6(b)}(q)=\tau_{\rho\sigma}(k_1,k_2,m_1)\frac{i\mathcal{P}^{\rho\sigma\lambda\xi}}{q^2}\Pi_{\lambda\xi\mu\nu}(q)\times\frac{i\mathcal{P}^{\mu\nu\gamma\delta}}{q^2}\tau_{\gamma\delta}(k_3,k_4,m_2), \tag{39}$$

where the vacuum polarization tensor is found from the effective Lagrangian obtained by 't Hooft and Veltman [3–5]

$$\mathcal{L}=-\frac{1}{16\pi^2}\ln\vec{q}^2\left(\frac{1}{120}R^2+\frac{7}{20}R_{\mu\nu}R^{\mu\nu}\right) \tag{40}$$

and is given by

$$\hat{\Pi}_{\alpha\beta,\gamma\delta}=-\frac{2G}{\pi}\ln(-q^2)\left[\frac{21}{120}q^4I_{\alpha\beta,\gamma\delta}+\frac{23}{120}q^4\eta_{\alpha\beta}\eta_{\gamma\delta}-\frac{23}{120}q^2(\eta_{\alpha\beta}q_\gamma q_\delta+\eta_{\gamma\delta}q_\alpha q_\beta)-\frac{21}{240}\times q^2(q_\alpha q_\delta\eta_{\beta\gamma}+q_\beta q_\delta\eta_{\alpha\gamma}+q_\alpha q_\gamma\eta_{\beta\delta}+q_\beta q_\gamma\eta_{\alpha\delta})+\frac{11}{30}q_\alpha q_\beta q_\gamma q_\delta\right]. \tag{41}$$

Contracting the various indices, we find the result

$$M_{6(a)+6(b)}(\vec{q})=\frac{43}{15}G^2m_1m_2\ln\vec{q}^2, \tag{42}$$

which is equivalent to that originally derived in Refs. [9,10]. After Fourier transforming we find a contribution to the scattering potential,

$$V_{6(a)+6(b)}(r)=-\frac{43}{30}\frac{m_1m_2G^2}{\pi r^3}, \tag{43}$$

which is in agreement with that given by Ref. [15].

IV. THE RESULT FOR THE GRAVITATIONAL POTENTIAL

Adding up all the corrections to the nonrelativistic potential we have our final result,

$$V(r)=-\frac{Gm_1m_2}{r}\left[1+3\frac{G(m_1+m_2)}{r}+\frac{41}{10\pi}\frac{G\hbar}{r^2}\right]. \tag{44}$$

The classical term in this potential agrees with Eq. (2.5) of Iwasaki [17].

However, we note that the quantum component of the potential is not equivalent to *any* previous published result [9–13,15]. As described above, we believe this result to be the definitive form of the nonrelativistic scattering matrix potential. Our results agree with Ref. [15] for all diagrams except the vertex corrections.[3] However, we performed multiple checks on the vertex correction, since we have seen that our version leads exactly to the required form of the classical Schwarzschild metric, for both fermions and bosons.

A. A potential ambiguity

There is an important issue concerning the potential that has not been thus far much discussed in the literature—that both the classical and quantum corrections to the potential are in some sense ambiguous. For the classical correction, this realization goes back to Refs. [18–20], and these arguments generalize readily to the quantum component. In this section we discuss this ambiguity and argue that at one loop order our calculation of the quantum correction does have a well defined meaning.

As explored in Ref. [18] the classical post-Newtonian potential is not invariant under a coordinate transformation of the form

$$r\to r\left[1+\alpha\frac{G(m_1+m_2)}{r}\right]. \tag{45}$$

We are always free to make such a coordinate change, and that given in Eq. (45) modifies the classical correction to the potential via

$$\frac{Gm_1m_2}{r}\left[1+c\frac{G(m_1+m_2)}{r}\right]\to\frac{Gm_1m_2}{r}\left[1+(c-\alpha)\frac{G(m_1+m_2)}{r}\right]. \tag{46}$$

Following Refs. [18,20] the ambiguity can be seen to arise in a field theory calculation through a modification of the graviton propagator

$$\frac{1}{q^2}=\frac{1}{q_0^2-\vec{q}^2}, \tag{47}$$

accomplished by using the energy conservation in the following identical way which holds true for general x:

[3]To be precise we are comparing to the third version of Ref. [15], available in the electronic archive. We had several disagreements with the original version; all except the vertex correction have been corrected in the third version.

$$\frac{1}{\frac{1}{2}(1-x)[(p_{20}-p_{10})^2+(p_{40}-p_{30})^2]-x(p_{20}-p_{10})(p_{40}-p_{30})-\vec{q}^2}. \tag{48}$$

Using this propagator in order to derive the gravitational potential, the result will in general depend on x, with the relation to the coordinate change being $\alpha=-\frac{1}{4}(1-x)$. Not only will the terms of order G^2/r^2 depend on x, but so will the corrections of order $(G/r)(v/c)$. This indicates that the post-Newtonian part of the static gravitational potential is *not* well defined in and of itself.

The coordinate ambiguity also generalizes to the quantum part of the potential. There exists a coordinate redefinition

$$r\to r\left[1+\beta\frac{G\hbar}{r^2}\right], \tag{49}$$

which changes the coefficient of the quantum term in the potential

$$\frac{Gm_1m_2}{r}\left[1+d\frac{G\hbar}{r^2}\right]\to\frac{Gm_1m_2}{r}\left[1+(d-\beta)\frac{G\hbar}{r^2}\right]. \tag{50}$$

Therefore, we need to address the uniqueness of the quantum correction to the potential.

Since general relativity is invariant under coordinate shifts, if the potentials are changed there must exist other modifications that compensate for these changes, leaving the resulting physics invariant. Within a Hamiltonian formulation, these modifications take the form of momentum-dependent pieces—in order to make the potential well defined, one must also specify the momentum terms in the Hamiltonian.

First we examine the classical ambiguity. Classically there are two dimensionless variables available for the expansion away from the Newtonian limit,

$$\frac{p^2}{m^2},\quad \frac{Gm}{r}. \tag{51}$$

In bound state problems these quantities appear at the same order, as can be seen by use of the virial theorem. In the Hamiltonian the leading terms are order

$$H_1\sim\frac{p^2}{m},\quad \frac{Gm^2}{r} \tag{52}$$

and are unambiguous. At next order we have pieces of the form

$$H_2\sim\frac{p^4}{m^3},\quad \frac{Gm^2}{r}\frac{Gm}{r},\quad \left(\frac{Gm}{r}\right)\left(\frac{p^2}{m}\right). \tag{53}$$

There exists an ambiguity between the last two of these terms arising from a coordinate change, but this effect cancels for physical observables, as shown explicitly in Ref. [18]. Writing the Hamiltonian in the center of mass frame to post-Newtonian order, we have

$$H=\left(\frac{\mathbf{p}^2}{2m_1}+\frac{\mathbf{p}^2}{2m_2}\right)-\left(\frac{\mathbf{p}^4}{8m_1^3}+\frac{\mathbf{p}^4}{8m_2^3}\right)-\frac{Gm_1m_2}{r}\left[1+a\frac{\mathbf{p}^2}{m_1m_2}+b\frac{(\mathbf{p}\cdot\hat{\mathbf{r}})^2}{m_1m_2}+c\frac{G(m_1+m_2)}{r}\right]. \tag{54}$$

In the standard Einstein-Infeld-Hoffmann coordinates, the coefficients a,b,c have the values

$$a=\frac{1}{2}\left[1+3\frac{(m_1+m_2)^2}{m_1m_2}\right],$$

$$b=\frac{1}{2},$$

$$c=-\frac{1}{2}, \tag{55}$$

but under the coordinate transformation of Eq. (45), one finds the modifications

$$a\to\frac{1}{2}\left[1+(3+2\alpha)\frac{(m_1+m_2)^2}{m_1m_2}\right],$$

$$b\to\frac{1}{2}-\alpha\frac{(m_1+m_2)^2}{m_1m_2},$$

$$c\to-\frac{1}{2}-\alpha. \tag{56}$$

Therefore, in order to specify the correction to the static potential, one needs also to identify the momentum coordi-

nates. In particular, the classical potential calculated by Iwasaki is appropriate for Einstein-Infeld-Hoffmann coordinates.

This classical ambiguity has not been worked out explicitly at the following order in the expansion, but the general pattern is apparent. Specifically, at the next order one has four types of terms,

$$H_3 \sim \frac{p^6}{m^5}, \quad \frac{p^4}{m^3}\left(\frac{Gm}{r}\right), \quad \frac{p^2}{m}\left(\frac{Gm}{r}\right)^2, \quad \frac{Gm^2}{r}\left(\frac{Gm}{r}\right)^2. \tag{57}$$

The last term here is noteworthy because it goes like $1/r^3$,—i.e., like the quantum correction in the potential—but it is distinguishable by its mass dependence plus the fact that it is order G^3. Such terms will also be ambiguous under a coordinate transformation, but the effects will cancel among the effects terms of this order. So the ''classical'' coordinate change above does change the $1/r^3$ term in the potential, but has a specific form and the effects cancel among other classical terms in the Hamiltonian.

Now let us examine the quantum effects, keeping only one power of the quantum expansion parameter $G\hbar/r^2$.

The quantum potential will enter the Hamiltonian at order

$$H_q = \frac{Gm^2}{r}\frac{G\hbar}{r^2} \tag{58}$$

and the other term of this order is

$$H_{qp} \sim \frac{p^2}{m}\frac{G\hbar}{r^2}. \tag{59}$$

If one makes the ''quantum'' coordinate change of Eq. (49), this will generate such a term in the Hamiltonian, which for physical observables cancels the effect of the change in the potential. Therefore in order to make the quantum correction to the potential well defined, one must specify the value of the terms contained in H_{qp}. However, the important point is that in *any* coordinates in which one calculates quantum corrections, one will *not* generate a term such as H_{qp}. This is because all quantum effects that involve the interparticle separation r arise from loop diagrams of order G^2 while H_{qp} has only a single power of G. Quantum effects on a single particle line are of order G but do not involve r, while those involving two lines are of order G^2. Since all calculations yield no term of the form H_{qp}, they must give the same quantum potential. The fully specified quantum potential is that one in the coordinates in which H_{qp} vanishes. This, of course, is then the one that we have calculated. We conclude that, despite the expected general coordinate invariance, the calculated quantum potential is a well defined quantity to this order.

V. DISCUSSION

The scattering amplitude has been used to provide a solid definition of the quantum corrections to the Newtonian potential. Our basic calculation is of certain nonanalytic terms in momentum space, the $\sqrt{-q^2}$ and $q^2 \ln -q^2$ terms. A transformation of these terms to coordinate space allows us to interpret these as long distance corrections to the potential. Our result is displayed in Eq. (44).

We have done the full calculation in harmonic gauge. It may be possible to verify the gauge invariance of the result by working in a more general covariant gauge, but this is considerably more involved computationally and we have not attempted it. However, because the scattering amplitude is an S-matrix element, we expect on general principles that the result should be gauge invariant.

The quantum corrections that we find for the scattering potential hold equally for the bound state potential. This is because the transformation between the two has no quantum component. The difference between the two is a classical correction that comes from iterating the lowest order potential. Dimensional analysis reveals that in the nonrelativistic limit this iteration has the ability to generate only a classical effect and not a quantum correction.

We have found a result for the nonrelativistic potential which we believe is the final and complete result for this quantity. The potential matches the expectations from dimensional analysis as discussed previously [9,10] and the known ambiguity of the form of the classical correction has been seen to originate from the possibility of rewriting a potential energy term in the Hamiltonian in terms of kinetic energy and vice versa, as also discussed in [18,20,19]. Such rewritings are not possible for the quantum term at the order to which we work. Therefore the quantum correction to the potential is a definite exact quantity.

The quantum corrections are too small to be observed experimentally. However, the fact that these are reliably predicted is important for our understanding of quantum gravity. These effects are due to the low energy propagation of massless degrees of freedom and hence are uniquely predicted for any quantum theory of gravity that reduces to general relativity in the low energy limit.[4] In this sense, these are low energy theorems of quantum gravity.

ACKNOWLEDGMENTS

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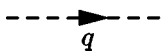
APPENDIX A: VERTICES AND PROPAGATORS

We begin by listing the Feynman rules which are employed in our calculation. For a derivation of these forms, see Ref. [26].

[4]Indeed, related effects have been found within the context of M theory [27] although the precise coefficients have not been calculated.

1. Scalar propagator

The massive scalar propagator is



$$= \frac{i}{q^2 - m^2 + i\epsilon} \ .$$

2. Graviton propagator

The graviton propagator in harmonic gauge can be written in the form

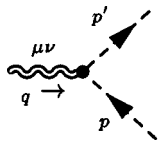
$$= \frac{i\mathcal{P}^{\alpha\beta\gamma\delta}}{q^2 + i\epsilon}$$

where

$$\mathcal{P}^{\alpha\beta\gamma\delta} = \frac{1}{2}[\eta^{\alpha\gamma}\eta^{\beta\delta} + \eta^{\beta\gamma}\eta^{\alpha\delta} - \eta^{\alpha\beta}\eta^{\gamma\delta}].$$

3. 2-scalar-1-graviton vertex

The 2-scalar-1-graviton vertex is discussed in the literature. We write it as



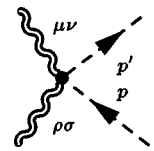
$$= \tau_1^{\mu\nu}(p, p', m)$$

where

$$\tau_1^{\mu\nu}(p,p',m) = -\frac{i\kappa}{2}\{p^\mu p'^\nu + p^\nu p'^\mu - \eta^{\mu\nu}[(p\cdot p') - m^2]\}.$$

4. 2-scalar-2-graviton vertex

The 2-scalar-2-graviton vertex is also discussed in the literature. We write it here with the full symmetry of the two gravitons:



$$= \tau_2^{\eta\lambda\rho\sigma}(p, p', m)$$

where

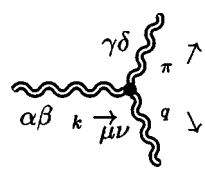
$$\tau_2^{\eta\lambda\rho\sigma}(p,p') = i\kappa^2\Bigg\{\Bigg[I^{\eta\lambda\alpha\delta}I^{\rho\sigma\beta}_{\delta} - \frac{1}{4}(\eta^{\eta\lambda}I^{\rho\sigma\alpha\beta} + \eta^{\rho\sigma}I^{\eta\lambda\alpha\beta})\Bigg] \times (p_\alpha p'_\beta + p'_\alpha p_\beta) - \frac{1}{2}\left(I^{\eta\lambda\rho\sigma} - \frac{1}{2}\eta^{\eta\lambda}\eta^{\rho\sigma}\right) \times [(p\cdot p') - m^2]\Bigg\} \tag{A1}$$

with

$$I_{\alpha\beta\gamma\delta} = \frac{1}{2}(\eta_{\alpha\gamma}\eta_{\beta\delta} + \eta_{\alpha\delta}\eta_{\beta\gamma}).$$

5. 3-graviton vertex

The 3-graviton vertex can be derived via the background field method and has the form [9,10]



$$= \tau_{3\,\alpha\beta\gamma\delta}^{\mu\nu}(k, q)$$

where

$$\begin{aligned}\tau_{3\,\alpha\beta\gamma\delta}^{\mu\nu}(k,q) = -\frac{i\kappa}{2}\times\Bigg(&\mathcal{P}_{\alpha\beta\gamma\delta}\left[k^\mu k^\nu + (k-q)^\mu(k-q)^\nu + q^\mu q^\nu - \frac{3}{2}\eta^{\mu\nu}q^2\right] + 2q_\lambda q_\sigma[I^{\sigma\lambda}_{\alpha\beta}I^{\mu\nu}_{\gamma\delta} + I^{\sigma\lambda}_{\gamma\delta}I^{\mu\nu}_{\alpha\beta} - I^{\mu\sigma}_{\alpha\beta}I^{\nu\lambda}_{\gamma\delta} - I^{\mu\sigma}_{\gamma\delta}I^{\nu\lambda}_{\alpha\beta}]\\
&+[q_\lambda q^\mu(\eta_{\alpha\beta}I^{\nu\lambda}_{\gamma\delta} + \eta_{\gamma\delta}I^{\nu\lambda}_{\alpha\beta}) + q_\lambda q^\nu(\eta_{\alpha\beta}I^{\mu\lambda}_{\gamma\delta} + \eta_{\gamma\delta}I^{\mu\lambda}_{\alpha\beta}) - q^2(\eta_{\alpha\beta}I^{\mu\nu}_{\gamma\delta} - \eta_{\gamma\delta}I^{\mu\nu}_{\alpha\beta}) - \eta^{\mu\nu}q_\sigma q_\lambda(\eta_{\alpha\beta}I^{\sigma\lambda}_{\gamma\delta}\\
&+\eta_{\gamma\delta}I^{\sigma\lambda}_{\alpha\beta})] + \{2q_\lambda[I^{\lambda\sigma}_{\alpha\beta}I^{\nu}_{\gamma\delta\sigma}(k-q)^\mu + I^{\lambda\sigma}_{\alpha\beta}I^{\mu}_{\gamma\delta\sigma}(k-q)^\nu - I^{\lambda\sigma}_{\gamma\delta}I^{\nu}_{\alpha\beta\sigma}k^\mu - I^{\lambda\sigma}_{\gamma\delta}I^{\mu}_{\alpha\beta\sigma}k^\nu] + q^2(I^{\mu}_{\alpha\beta\sigma}I^{\nu\sigma}_{\gamma\delta} + I^{\nu\sigma}_{\alpha\beta}I^{\mu}_{\gamma\delta\sigma})\\
&+\eta^{\mu\nu}q_\sigma q_\lambda(I^{\lambda\rho}_{\alpha\beta}I^{\sigma}_{\gamma\delta\rho} + I^{\lambda\rho}_{\gamma\delta}I^{\sigma}_{\alpha\beta\rho})\} + \Bigg\{[k^2 + (k-q)^2]\left[I^{\mu\sigma}_{\alpha\beta}I^{\nu}_{\gamma\delta\sigma} + I^{\mu\sigma}_{\gamma\delta}I^{\nu}_{\alpha\beta\sigma} - \frac{1}{2}\eta^{\mu\nu}\mathcal{P}_{\alpha\beta\gamma\delta}\right]\\
&-[I^{\mu\nu}_{\gamma\delta}\eta_{\alpha\beta}k^2 + I^{\mu\nu}_{\alpha\beta}\eta_{\gamma\delta}(k-q)^2]\Bigg\}\Bigg).\end{aligned} \tag{A2}$$

APPENDIX B: USEFUL INTEGRALS

In the evaluation of the various diagrams we employ the following integrals

$$J=\int \frac{d^4l}{(2\pi)^4}\frac{1}{l^2(l+q)^2}=\frac{i}{32\pi^2}[-2L]+\ldots, \tag{B1}$$

$$J_\mu=\int \frac{d^4l}{(2\pi)^4}\frac{l_\mu}{l^2(l+q)^2}=\frac{i}{32\pi^2}[q_\mu L]+\ldots, \tag{B2}$$

$$J_{\mu\nu}=\int \frac{d^4l}{(2\pi)^4}\frac{l_\mu l_\nu}{l^2(l+q)^2}=\frac{i}{32\pi^2}\left[q_\mu q_\nu\left(-\frac{2}{3}L\right)-q^2\eta_{\mu\nu}\left(-\frac{1}{6}L\right)\right]+\ldots, \tag{B3}$$

as well as

$$I=\int \frac{d^4l}{(2\pi)^4}\frac{1}{l^2(l+q)^2[(l+k)^2-m^2]}=\frac{i}{32\pi^2m^2}[-L-S]+\ldots, \tag{B4}$$

$$I_\mu=\int \frac{d^4l}{(2\pi)^4}\frac{l_\mu}{l^2(l+q)^2[(l+k)^2-m^2]}=\frac{i}{32\pi^2m^2}\left\{k_\mu\left[\left(-1-\frac{1}{2}\frac{q^2}{m^2}\right)L-\frac{1}{4}\frac{q^2}{m^2}S\right]+q_\mu\left(L+\frac{1}{2}S\right)\right\}+\ldots, \tag{B5}$$

$$I_{\mu\nu}=\int \frac{d^4l}{(2\pi)^4}\frac{l_\mu l_\nu}{l^2(l+q)^2[(l+k)^2-m^2]}=\frac{i}{32\pi^2m^2}\left\{q_\mu q_\nu\left(-L-\frac{3}{8}S\right)+k_\mu k_\nu\left(-\frac{1}{2}\frac{q^2}{m^2}L-\frac{1}{8}\frac{q^2}{m^2}S\right)+(q_\mu k_\nu+q_\nu k_\mu)\left[\left(\frac{1}{2}+\frac{1}{2}\frac{q^2}{m^2}\right)L+\frac{3}{16}\frac{q^2}{m^2}S\right]+q^2\eta_{\mu\nu}\left(\frac{1}{4}L+\frac{1}{8}S\right)\right\}+\ldots, \tag{B6}$$

$$\begin{aligned}I_{\mu\nu\alpha}=&\int \frac{d^4l}{(2\pi)^4}\frac{l_\mu l_\nu l_\alpha}{l^2(l+q)^2[(l+k)^2-m^2]}\\=&\frac{i}{32\pi^2m^2}\left\{q_\mu q_\nu q_\alpha\left(L+\frac{5}{16}S\right)+k_\mu k_\nu k_\alpha\left(-\frac{1}{6}\frac{q^2}{m^2}\right)+(q_\mu k_\nu k_\alpha+q_\nu k_\mu k_\alpha+q_\alpha k_\mu k_\nu)\left(\frac{1}{3}\frac{q^2}{m^2}L+\frac{1}{16}\frac{q^2}{m^2}S\right)\right.\\&+(q_\mu q_\nu k_\alpha+q_\mu q_\alpha k_\nu+q_\nu q_\alpha k_\mu)\left[\left(-\frac{1}{3}-\frac{1}{2}\frac{q^2}{m^2}\right)L-\frac{5}{32}\frac{q^2}{m^2}S\right]+(\eta_{\mu\nu}k_\alpha+\eta_{\mu\alpha}k_\nu+\eta_{\nu\alpha}k_\mu)\left(\frac{1}{12}q^2L\right)\\&\left.+(\eta_{\mu\nu}q_\alpha+\eta_{\mu\alpha}q_\nu+\eta_{\nu\alpha}q_\mu)\left(-\frac{1}{6}q^2L-\frac{1}{16}q^2S\right)\right\}+\ldots,\end{aligned} \tag{B7}$$

where we have defined $L=\ln(-q^2)$ and $S=(\pi^2m)/\sqrt{-q^2}$. Only the lowest order nonanalytic terms are included in the above forms. Higher order nonanalytic contributions as well as the neglected analytic terms are denoted by the ellipsis. The following identity can be verified on shell, $k\cdot q=q^2/2$, where $k-k'=q$ and $k^2=m^2=k'^2$. In some cases the integrals are used with k replaced by $-k'$, where $k'\cdot q=-q^2/2$.

In order to do evaluate the box diagrams the following lowest order integrals are needed. The higher order contributions of nonanalytic terms as well as neglected analytic terms are again denoted by the ellipsis [28]:

$$K=\int \frac{d^4l}{(2\pi)^4}\frac{1}{l^2(l+q)^2[(l+k_1)^2-m_1^2][(l-k_3)^2-m_2^2]}$$

$$=\frac{i}{16\pi^2 m_1 m_2 q^2}\left[\left(1-\frac{w}{3m_1m_2}\right)L\right]+\ldots, \tag{B8}$$

$$K'=\int \frac{d^4l}{(2\pi)^4}\frac{1}{l^2(l+q)^2[(l+k_1)^2-m_1^2][(l+k_4)^2-m_2^2]}$$

$$=\frac{i}{16\pi^2 m_1 m_2 q^2}\left[\left(-1+\frac{W}{3m_1m_2}\right)L\right]+\ldots. \tag{B9}$$

Here $k_1\cdot q=q^2/2$, $k_2\cdot q=-q^2/2$, $k_3\cdot q=-q^2/2$, and $k_4\cdot q=q^2/2$, where $k_1-k_2=k_4-k_3=q$ and $k_1^2=m_1^2=k_2^2$ together with $k_3^2=m_2^2=k_4^2$. Also, we have defined $w=(k_1\cdot k_3)-m_1m_2$ and $W=(k_1\cdot k_4)-m_1m_2$.

The following constraints for the nonanalytic terms of the above integrals hold true on shell:

$$I_{\mu\nu\alpha}\eta^{\alpha\beta}=I_{\mu\nu}\eta^{\mu\nu}=J_{\mu\nu}\eta^{\mu\nu}=0, \tag{B10}$$

$$I_{\mu\nu\alpha}q^\alpha=-\frac{q^2}{2}I_{\mu\nu},\quad I_{\mu\nu}q^\nu=-\frac{q^2}{2}I_\mu,\quad I_\mu q^\mu=-\frac{q^2}{2}I,$$

$$J_{\mu\nu}q^\nu=-\frac{q^2}{2}J_\mu,\quad J_\mu q^\mu=-\frac{q^2}{2}J, \tag{B11}$$

$$I_{\mu\nu\alpha}k^\alpha=\frac{1}{2}J_{\mu\nu},\quad I_{\mu\nu}k^\nu=\frac{1}{2}J_\mu,\quad I_\mu k^\mu=\frac{1}{2}J. \tag{B12}$$

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