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Weierstrass traveling wave solutions for dissipative Benjamin, Bona, and Mahony (BBM) equation

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In this paper the effect of a small dissipation on waves is included to find exact solutions to the modified Benjamin, Bona, and Mahony (BBM) equation by viscosity. Using Lyapunov functions and dynamical systems theory, we prove that when viscosity is added to the BBM equation, in certain regions there still exist bounded traveling wave solutions in the form of solitary waves, periodic, and elliptic functions. By using the canonical form of Abel equation, the polynomial Appell invariant makes the equation integrable in terms of Weierstrass $\wp$ functions. We will use a general formalism based on Ince's transformation to write the general solution of dissipative BBM in terms of $\wp$ functions, from which all the other known solutions can be obtained via simplifying assumptions. Using ODE (ordinary differential equations) analysis we show that the traveling wave speed is a bifurcation parameter that makes transition between different classes of waves. © *2013 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4817342]

I. INTRODUCTION

In 1895, Korteweg and de Vries, under the assumption of small wave amplitude and large wavelength of inviscid and incompressible fluids, derived an equation for the water waves, now known as the KdV equation,[14] which also serves as a justifiable model for long waves in a wide class of nonlinear dispersive systems. KdV equation has been also used to account adequately for observable phenomena such as the interaction of solitary waves and dissipationless undular shocks. For the water wave problem Eq. (1.1) is nondimensionalized since the physical parameters $u = \frac{3\eta}{2H}$, $x = \frac{\sqrt{6}x^*}{H}$, and $t = \sqrt{\frac{6g}{H}}t^*$, where η is the vertical displacement of the free surface, H is the depth of the undisturbed water, x^* is dimensional distance, and t^* is the dimensional time are all scaled into the definition of nondimensional space x, time t, and water velocity $u(x, t)$. When the physical parameters and scaling factors are appropriately absorbed into the definitions of u, x, and t, the KdV equation is obtained in the tidy form

$$u_t + u_x + uu_x + u_{xxx} = 0. \tag{1.1}$$

Although (1.1) shows remarkable properties it manifests non-physical properties; the most noticeable being unbounded dispersion relation. It is helpful to recognize that the main difficulties presented by (1.1) arise from the dispersion term u_{xxx}. The linearized version of (1.1), which does not include the nonlinear term, is

$$u_t + u_x + u_{xxx} = 0. \tag{1.2}$$

First, as noted in Ref. 5, when the solution of (1.2) is expressible as a summation of Fourier components in the form $F(k)e^{i(kx - \omega t)}$, the dispersion relation is $\omega(k) = k - k^3$, where $\omega(k)$ is the

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frequency, and k is the wave number. The phase velocity $C_p = \frac{\omega(k)}{k} = 1 - k^2$ becomes negative for $k^2 > 1$, contrary to the original assumption of the forward traveling waves. More significantly, the group velocity $C_g = \frac{d\omega}{dk} = 1 - 3k^2$ has no lower bound.

To circumvent this feature, it has been shown by Peregrine,[21] and later by Benjamin *et al.*[5] that (1.1) has an alternative format, which is called the regularized long-wave or Benjamin, Bona, and Mahony (BBM) equation

$$u_t + u_x + uu_x - u_{xxt} = 0, \tag{1.3}$$

where the dispersion term u_{xxx} is replaced by $-u_{xxt}$ to reflect a bounded dispersion relation. It is contended that (1.3) is in important respects the preferable model over (1.1) which is not a suitable model for long waves.

The initial value problem for this equation has been investigated previously by Refs. 21–23, while the existence and stability of solitary waves have been investigated by Refs. 2,5,6,8, and 9. In Ref. 5, Eq. (1.3) is recast as an integral equation, and its solutions global in time are found for initial data in $W^{1,2}(\mathbb{R})$. Also, the authors showed that its solutions have better smoothness properties than those of (1.1). In Ref. 8, global solutions are found for initial data in $H^s(\mathbb{R})$, $s \geq 0$.

The linearized version of (1.3) has the dispersion relation $\omega(k) = \frac{k}{1+k^2}$, according to which both phase and group velocities $C_p = \frac{1}{1+k^2}$ and $C_g = \frac{1-k^2}{(1+k^2)^2}$ are bounded for all k. Moreover, both velocities tend to zero for large k, which implies that fine scale features of the solution tend not to propagate. The preference of (1.3) over (1.1) became clear in Ref. 5, when the authors attempted to formulate an existence theory for (1.1), respective to nonperiodic initial condition $u(x, 0)$ defined on $(-\infty, \infty)$.

A generalization to (1.3) to include a viscous term is provided by the equation

$$u_t + u_x + uu_x - u_{xxt} = \nu u_{xx}, \tag{1.4}$$

where $\nu > 0$ is transformed kinematic viscosity coefficient. The linearized version of (1.4) has a complex dispersion relation $w(k) = \frac{k - i\nu k^2}{1+k^2}$, so $u(x,t) = e^{ik(x - \frac{t}{1+k^2})} e^{-\frac{\nu k^2}{1+k^2} t}$. The real part of the frequency coincides with the bounded dispersion relation of the nondissipative case, while the imaginary part contains the viscosity coefficient ν which must be positive to have decaying waves. In this case the harmonic solutions exhibit both dispersive and dissipative effects.

II. EXISTENCE OF DISSIPATIVE SOLUTIONS

In this section we propose an easy theorem based on dynamical system theory,[4,12] in which we prove the existence of solutions to (1.4) for both negative and positive traveling wave velocities ($c < 0, c > 0$). Also, we will show that the solution tends to the fixed points of the dynamical system $c - 1 \pm k$, for any k and large traveling wave variable ζ. We show that on boundaries the solution does not necessarily decay to zero, but there is flow of energy through the boundaries to the solution and back to the boundaries.

Theorem 1 below is proven by showing that a certain dynamical system in the plane has a heteroclinic orbit connecting two critical points. The Lyapunov function $\mathcal{L}$ is used to show that a certain trajectory of the dynamical system is bounded. The Poincare-Bendixson Theorem[16] can be inferred to show that the trajectory approaches either a closed loop or a fixed point. The arguments in the proof of Theorem 1 show that in the case of a closed loop, $\mathcal{L}$ would have to be constant along the loop, which is impossible by (2.12). Therefore, the trajectory must approach a fixed point instead, which we show is a function of the viscosity ν and velocity c.

An alternative to the dynamical systems approach to finding heteroclinic solutions to ODEs similar to (2.3), perhaps with more complicated nonlinearities, are variational methods and critical

point theory. There is an extensive literature on applying variational methods to these types of equation without the dissipative term. See, for example, Ref. 7 and the references therein. The inclusion of the dissipative term makes this approach more difficult. It is necessary to use more complicated function spaces and more complicated arguments, although this approach was used successfully in Ref. 3.

Theorem 1: *Let* $c \neq 0$ *and* $k > 0$. (1.4) *has a traveling wave solution of the form* $u(x, t) = u(x - ct) = u(\zeta)$ *with* $u(\zeta) \to (c - 1) + k$ *as* $\zeta \to -\infty$ *and* $u(\zeta) \to (c - 1) - k$ *as* $\zeta \to \infty$. *If* $c < 0$, *then* $u(\zeta) < (c - 1) + k$ *for all real* ζ. *If* $c > 0$, *then* $u(\zeta) > (c - 1) - k$ *for all real* ζ.

Proof: The class of soliton solutions is found by employing the form of the traveling wave solutions of (1.4) which takes the form of the *ansatz*

$$u(x, t) = u(\zeta), \tag{2.1}$$

where $\zeta \equiv x - ct$ is the traveling wave variable, and c is a nonzero translational wave velocity. The substitution of (2.1) in (1.4) leads to

$$(1 - c)u' + uu' + cu''' = \nu u'', \tag{2.2}$$

where $' = \frac{d}{d\zeta}$. Integrating,

$$(1 - c)u + \frac{1}{2}u^2 + cu'' = \nu u' + A, \tag{2.3}$$

for some constant A. With $\phi = u$ and $\psi = u'$, (2.3) is equivalent to the dynamical system

$$\begin{aligned} \phi' &= \psi, \\ \psi' &= \frac{\nu}{c}\psi + \frac{c-1}{c}\phi - \frac{1}{2c}\phi^2 + \frac{A}{c}, \end{aligned} \tag{2.4}$$

or

$$\mathbf{u}' = \mathbf{F}(\mathbf{u}) \tag{2.5}$$

in vector notation. The fixed points of (2.4) are

$$\mathbf{z}^{\pm} = \begin{bmatrix} \phi^{\pm} \\ 0 \end{bmatrix} = \begin{bmatrix} c - 1 \pm \sqrt{(c-1)^2 + 2A} \\ 0 \end{bmatrix} = \begin{bmatrix} c - 1 \pm k \\ 0 \end{bmatrix}, \tag{2.6}$$

for $A = \frac{k^2 - (c-1)^2}{2}$. We will show that (2.4) has a trajectory from $\mathbf{z}^+$ to $\mathbf{z}^-$.

First, we will consider the $c < 0$ case. The linearization of (2.4) at $\mathbf{z}^+$ is

$$\mathbf{u}' = \begin{bmatrix} \phi' \\ \psi' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{\sqrt{(c-1)^2+2A}}{c} & \frac{\nu}{c} \end{bmatrix} \begin{bmatrix} \phi - \phi^+ \\ \psi \end{bmatrix} \equiv M \begin{bmatrix} \phi - \phi^+ \\ \psi \end{bmatrix}. \tag{2.7}$$

The determinant of M is negative, so $\mathbf{z}^+$ is a saddle point, and M has eigenvalues $\lambda^{\pm}$ with $\lambda^- < 0 < \lambda^+$.

The unit eigenvector corresponding to λ^+ and pointing to the left is

$$\mathbf{v} = \frac{\begin{bmatrix} -1 \\ -\lambda^+ \end{bmatrix}}{\sqrt{1 + (\lambda^+)^2}}. \tag{2.8}$$

Let $\mathbf{g}(\zeta) = \begin{bmatrix} g_1(\zeta) \\ g_2(\zeta) \end{bmatrix}$ be a solution of (2.4) with $\mathbf{g}(\zeta) \to \mathbf{z}^+$ and

$$\left| \frac{\mathbf{g}(\zeta) - \mathbf{z}^+}{|\mathbf{g}(\zeta) - \mathbf{z}^+|} - \mathbf{v} \right| \to 0 \tag{2.9}$$

as $\zeta \to -\infty$. We will show that $\mathbf{g}(\zeta) \to \mathbf{z}^-$ as $\zeta \to \infty$ and $g_1(\zeta) < \phi^+$ for all real ζ.

Define

$$P_3(\phi) = \frac{1}{6}\phi^3 + \frac{1}{2}(1-c)\phi^2 - A\phi. \tag{2.10}$$

P_3 is increasing on $(-\infty, \phi^-]$, decreasing on $[\phi^-, \phi^+]$, and increasing on $[\phi^+, \infty)$. Define

$$\begin{aligned} \mathcal{L}(\mathbf{u}) =& P_3(\phi) - P_3(\phi^+) + \frac{1}{2}c\psi^2 \\ =& \frac{1}{6}\phi^3 + \frac{1}{2}(1-c)\phi^2 - A\phi - P_3(\phi^+) + \frac{1}{2}c\psi^2. \end{aligned} \tag{2.11}$$

Then for all ζ,

$$\begin{aligned} \frac{d}{d\zeta}\mathcal{L}(\mathbf{g}(\zeta)) &= \Big(\frac{1}{2}g_1^2 + (1-c)g_1 - A\Big)g_1' + cg_2g_2' = \\ &= \Big(\frac{1}{2}g_1^2 + (1-c)g_1 - A\Big)g_2 + \\ &\quad cg_2\Big(\frac{c-1}{c}g_1 - \frac{1}{2c}g_1^2 + \frac{v}{c}g_2 + \frac{A}{c}\Big) = vg_2^2 \geq 0. \end{aligned} \tag{2.12}$$

Since $\mathcal{L}(\mathbf{z}^+) = 0$ then $\mathcal{L}(\mathbf{g}(\zeta)) > 0$ for all real ζ.

Let $\phi^{--} < \phi^-$ with

$$P_3(\phi^{--}) < P_3(\phi^+), \tag{2.13}$$

and let

$$B = P_3(\phi^-) - P_3(\phi^+) = \max\{P_3(\phi) \mid \phi^{--} \leq \phi \leq \phi^+\} - P_3(\phi^+). \tag{2.14}$$

Let $L > 0$ be large enough so that

$$\frac{1}{2}cL^2 + B < 0, \tag{2.15}$$

and define

$$R = (\phi^{--}, \phi^+) \times (-L, L). \tag{2.16}$$

We claim that $\mathbf{g}(\zeta) \in R$ for all real ζ. As $\zeta \to -\infty$, $\mathbf{g}(\zeta) \to \mathbf{z}^+$ along the direction $[1, \lambda^+]^T$. Therefore, $\mathcal{L}(\mathbf{g}(\zeta)) > \mathcal{L}(\mathbf{z}^+) = 0$ for all real ζ, and there exists $\zeta' \in \mathbb{R}$ such that $\mathbf{g}(\zeta) \in R$ for all $\zeta < \zeta'$.

So, it suffices to show that $\mathcal{L} \leq 0$ on ∂R. Let $\mathbf{u} \in \partial R$. If $\phi = \phi^+$, then $\mathcal{L}(\mathbf{u}) = \frac{1}{2}c\psi^2 \leq 0$. Similarly, if $\phi = \phi^{--}$, then $\mathcal{L}(\mathbf{u}) = P_3(\phi^{--}) - P_3(\phi^+) + \frac{1}{2}c\psi^2 < 0$. If $\phi^{--} \leq \phi \leq \phi^+$ and $\psi = L$, then $\mathcal{L}(\mathbf{u}) = P_3(\phi) - P_3(\phi^+) + \frac{1}{2}cL^2 \leq B + \frac{1}{2}cL^2 < 0$. Similarly, if $\phi^{--} \leq \phi \leq \phi^+$ and $\psi = -L$, $\mathcal{L}(\mathbf{u}) < 0$.

Define

$$L_\infty = \lim_{\zeta\to\infty} \mathcal{L}(\mathbf{g}(\zeta)). \tag{2.17}$$

L_∞ exists and is finite because $\mathcal{L}(\mathbf{g}(\zeta))$ is a non-decreasing function of ζ, $\mathcal{L}$ is bounded on R, and $\mathbf{g}(\zeta) \in R$ for all real ζ. Let C be the ω-limit set of $\mathbf{g}$, i.e.,

$$C = \{\mathbf{u} \mid \exists(\zeta_m) \subset \mathbb{R} \text{ with } \lim_{m\to\infty} \zeta_m = \infty,\ \mathbf{g}(\zeta_m) \to \mathbf{u}\}, \tag{2.18}$$

where C is nonempty because $\mathbf{g}(\zeta) \in R$ for all real ζ and R is bounded. C is compact and connected, and $\mathcal{L} = L_\infty$ on C. We must prove $C = \{\mathbf{z}^-\}$.

Suppose C contains a point $\mathbf{w}$ that is not on the ϕ-axis. Then $\mathbf{w}$ is not a stationary point of (2.4), and C contains another point $\mathbf{y}$. Let $\delta > 0$ with $\delta < \min(|\mathbf{w} - \mathbf{y}|, |w_2|)/2$. Then $\mathbf{g}$ crosses the annulus $B_\delta(\mathbf{w}) \setminus B_{\delta/2}(\mathbf{w})$ infinitely many times, and there exist sequences (ζ_m), $(\zeta\,'_m)$ with $\zeta_m \to \infty$ as $m \to \infty$, $\zeta_m < \zeta\,'_m < \zeta_{m+1}$ for all m, $|\mathbf{g}(\zeta_m) - \mathbf{w}| = \delta/2$, $|\mathbf{g}(\zeta\,'_m) - \mathbf{w}| = \delta$, and $\delta/2 < |\mathbf{g}(\zeta) - \mathbf{w}| < \delta$, for all m and all $\zeta \in (\zeta_m, \zeta\,'_m)$.

Let $K = \sup(|\mathbf{F}(\mathbf{u})| \mid \mathbf{u} \in B_\delta(\mathbf{w})) < \infty$, where $\mathbf{F}$ is from (2.5). Then $\zeta\,'_m - \zeta_m \geq \delta/(2K)$ for all m. For $\zeta \in (\zeta_m, \zeta\,'_m)$,

$$\frac{d}{d\zeta}\mathcal{L}(\mathbf{g}(\zeta)) \geq \nu g_2^2(\zeta) \geq \frac{\nu w_2^2}{4} > 0, \tag{2.19}$$

so

$$\begin{aligned}\mathcal{L}(\mathbf{g}(\zeta\,'_m)) - \mathcal{L}(\mathbf{g}(\zeta_m)) = \int_{\zeta_m}^{\zeta\,'_m} \frac{d}{d\zeta}\mathcal{L}(\mathbf{g}(\zeta))\,d\zeta \geq \\ \int_{\zeta_m}^{\zeta\,'_m} \frac{\nu w_2^2}{4}\,d\zeta = \frac{\nu(\zeta\,'_m - \zeta_m)w_2^2}{4} \geq \frac{\delta\nu w_2^2}{8K},\end{aligned} \tag{2.20}$$

for all m. Therefore,

$$\begin{aligned}L_\infty &= \mathcal{L}(\mathbf{g}(\zeta_1)) + (L_\infty - \mathcal{L}(\mathbf{g}(\zeta_1)) \geq \\ &\geq \mathcal{L}(\mathbf{g}(\zeta_1)) + \sum_{m=1}^{\infty} \mathcal{L}(\mathbf{g}(\zeta\,'_m)) - \mathcal{L}(\mathbf{g}(\zeta_m)) \geq \\ &\geq \mathcal{L}(\mathbf{g}(\zeta_1)) + \sum_{m=1}^{\infty} \frac{\delta\nu w_2^2}{8K} = \infty.\end{aligned} \tag{2.21}$$

But L_∞ is finite. This is a contradiction.

Therefore, C is a subset of the ϕ-axis. Since C is connected and compact, $C = [\phi_1, \phi_2] \times \{0\}$ for some $\phi^{--} < \phi_1 \leq \phi_2 < \phi^+$. Recall that $\mathcal{L} = L_\infty$ on C. $\mathcal{L}$ is not constant on any segment of the ϕ-axis of positive length. So C is a single point, which is a stationary point of (2.4) in R. There is only one such point, $\mathbf{z}^-$.

Now consider the case $c > 0$. Let $k > 0$. Since $-c < 0$, we have seen that there exists g solving (2.2) with c replaced by $-c$, that is,

$$(1 + c)g' + gg' - cg''' - \nu g'' = 0, \tag{2.22}$$

with $g(\zeta) \to (-c - 1) + k$ as $\zeta \to -\infty$, $g(\zeta) \to (-c - 1) - k$ as $\zeta \to \infty$, and $g(\zeta) < (-c - 1) + k$ for all real ζ.

Define $u : \mathbb{R} \to \mathbb{R}$ by $u(\zeta) = -g(-\zeta) - 2$. Then $g(\zeta) = -u(-\zeta) - 2$, $g'(\zeta) = u'(-\zeta)$, $g''(\zeta) = -u''(-\zeta)$, and $g'''(\zeta) = u'''(-\zeta)$. Substituting into (2.22) yields

$$\begin{aligned}0 &= (1 + c)u'(-\zeta) + (-u(-\zeta) - 2)u'(-\zeta) - cu'''(-\zeta) + \nu u''(-\zeta) \\ &= (c - 1)u'(-\zeta) - u(-\zeta)u'(-\zeta) - cu'''(-\zeta) + \nu u''(-\zeta) \\ &= -[(1 - c)u'(-\zeta) + u(-\zeta)u'(-\zeta) + cu'''(-\zeta) - \nu u''(-\zeta)].\end{aligned} \tag{2.23}$$

Therefore, u satisfies (2.2). By definition of u,

$$\begin{aligned}\lim_{\zeta\to-\infty} u(\zeta) &= \lim_{\zeta\to-\infty}(-g(-\zeta)-2) = \lim_{\zeta\to\infty}(-g(\zeta)-2)\\ &= -2-\lim_{\zeta\to\infty} g(\zeta) = -2-((-c-1)-k)\\ &= (c-1)+k\end{aligned} \tag{2.24}$$

and

$$\begin{aligned}\lim_{\zeta\to\infty} u(\zeta) &= \lim_{\zeta\to\infty}(-g(-\zeta)-2) = \lim_{\zeta\to-\infty}(-g(\zeta)-2)\\ &= -2-\lim_{\zeta\to-\infty} g(\zeta) = -2-((-c-1)+k)\\ &= (c-1)-k.\end{aligned} \tag{2.25}$$

For all real ζ, $g(\zeta) < (-c-1)+k$, so for all real ζ,

$$u(\zeta) = -g(-\zeta)-2 > -[(-c-1)+k]-2 = (c-1)-k. \tag{2.26}$$

Theorem 1 is proved.

III. EXACT SOLUTIONS VIA $\wp$ FUNCTIONS

A. Abel's equation

The importance of Abel's equation in its canonical forms stems from the fact that its integrability leads to closed form solutions to a general nonlinear ODE of the form

$$u_{\zeta\zeta} + q_0(u)u_\zeta + q_2(u) = 0. \tag{3.1}$$

This can be expressed by the following lemma.[18]

Lemma 1: Solutions to a general second order ODE of type (3.1) may be obtained via the solutions to Abel's equation (3.6), and vice versa using the following relationship

$$u_\zeta = \eta(u(\zeta)). \tag{3.2}$$

Proof: To show the equivalence, one just needs the simple chain rule

$$u_{\zeta\zeta} = \frac{d\eta}{du}u_\zeta = \eta\frac{d\eta}{du}, \tag{3.3}$$

which turns (3.1) into the second kind of Abel equation

$$\frac{d\eta}{du}\eta + q_0(u)\eta + q_2(u) = 0. \tag{3.4}$$

Moreover, via transformation of the dependent variable

$$\eta(u) = \frac{1}{y(u)}, \tag{3.5}$$

(3.4) becomes the Abel first kind without free or linear terms

$$\frac{dy}{du} = q_0(u)y^2 + q_2(u)y^3. \tag{3.6}$$

Comparing (3.1) with (2.3) we identify the constant, and quadratic polynomial coefficients

$$q_0(u) = -\frac{\nu}{c},$$
$$q_2(u) = \frac{u^2}{2c} + \frac{1-c}{c}u - \frac{A}{c}. \tag{3.7}$$

The need of dissipation is imperative since without the dissipation term $q_0 = 0$, the Abel's equation (3.6) becomes separable, as we will see next.

Progress of integration of (3.6) is based on the linear transformation $v = \int q_0(u)du = -\frac{\nu}{c}u$, which allows us to write Abel's equation (3.6) in canonical form

$$\frac{dy}{dv} = y^2 + g(v)y^3, \tag{3.8}$$

where $g(v)$ is the Appell's invariant and is the quadratic polynomial

$$g(v) = -\frac{c^2}{2\nu^3}v^2 + \frac{c(1-c)}{\nu^2}v + \frac{A}{\nu} = a_2v^2 + a_1v + a_0. \tag{3.9}$$

By Lemke's transformation[17]

$$y = -\frac{1}{z}\frac{dz}{dv}, \tag{3.10}$$

(3.8) can be written as a second-order differential equation with the *preserved Appel's invariant* $g(v)$,

$$z^2\frac{d^2v}{dz^2} + g(v) = 0. \tag{3.11}$$

B. No viscosity

If $\nu = 0$ (2.3) becomes

$$u_{\zeta\zeta} = -\frac{u^2}{2c} - \frac{1-c}{c}u + \frac{A}{c}. \tag{3.12}$$

The classical solutions obtained before by Ref. 6 with the assumption that $A = 0$ are

$$u(x,t) = 3(c-1)\mathrm{Sech}^2\Big[\frac{\sqrt{\frac{c-1}{c}}}{2}(x-ct)\Big] \text{ if } c > 1, \tag{3.13}$$
$$u(x,t) = -\frac{3c}{1+c}\mathrm{Sec}^2\Big[\frac{\sqrt{c}}{2}\Big(x - \frac{t}{1+c}\Big)\Big] \text{ if } 0 < c < 1,$$

see left and middle panels of Fig. 1. Solitary waves that translate with velocity $c > 1$ are called fast waves, while the periodic solutions that travel with velocity $0 < c < 1$ slow waves.

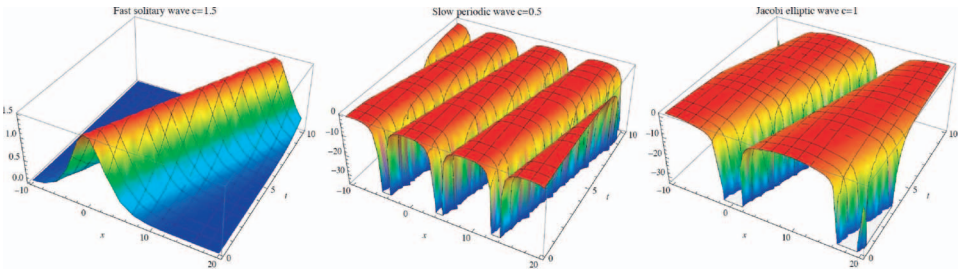

FIG. 1. Traveling waves $\nu = 0$, left $c = 1.5$ and middle $c = 0.5$, Eq. (3.13); right $c = 1$, Eq. (3.28).

In (3.6) if let $\nu = 0$, then the Abel's equation

$$\frac{dy}{du} = q_2(u)y^3 \tag{3.14}$$

is separable and leads to the *elliptic algebraic* equation

$$\eta^2 = -\frac{u^3}{3c} - \frac{1-c}{c}u^2 + \frac{2A}{c} - 2D. \tag{3.15}$$

Using again the substitution $u_\zeta = \eta$ (3.15) becomes the *elliptic differential* equation

$$u_\zeta^2 = -\frac{u^3}{3c} - \frac{1-c}{c}u^2 + \frac{2A}{c}u - 2D. \tag{3.16}$$

For verification, if one takes $\frac{d}{d\zeta}$ of the above, we recover (3.12).

Using a linear transformation, p. 311 of Ref. 11 of the dependent variable $u = -\sqrt[3]{12c}\hat{u} - (1-c)$, (3.16) can be written in standard form

$$\hat{u}_\zeta^2 = 4\hat{u}^3 - g_2\hat{u} - g_3, \tag{3.17}$$

with solution

$$\hat{u}(\zeta) = \wp(\zeta, g_2, g_3) \tag{3.18}$$

and invariants

$$g_2 = \sqrt[3]{\frac{12}{c^2}}(c-1)^2 > 0, \tag{3.19}$$

$$g_3 = \frac{2(1-c)^3}{3c}. \tag{3.20}$$

The limiting cases are obtained when $A = 0, D = 0$ to yield back to (3.13). We will not discuss here this reduction to hyperbolic or trigonometric functions, since this was done extensible before by many authors, but instead we refer the reader to Refs. 1 and 19 for the reduction and classification that depends on the sign of the invariants.

Instead, we will obtain a novel class of solutions that travel with a critical velocity $c = 1$. At the boundary between the fast and slow waves we will encounter periodic solutions in terms of rational combination of Jacobian elliptic functions.

If $c = 1$ and $A = 0$, Eq. (3.12) becomes

$$\frac{u^2}{2} + u'' = 0. \tag{3.21}$$

By multiplying by u_ζ and integrating once we obtain

$$u_\zeta^2 + \frac{u^3}{3} = \frac{B^3}{3}, \tag{3.22}$$

where B is some nonzero constant of integration. In (3.15) if we let $D = -\frac{B^3}{6}$, we will also arrive to (3.22). Now let us use the substitution $\mu^2 = B - u$, where $\mu = \mu(\zeta)$. Hence (3.22) is

$$\mu_\zeta = \pm\frac{\sqrt{3}}{6}\sqrt{\mu^4 - 3B\mu^2 + 3B^2}. \tag{3.23}$$

Moreover, let us assume $\mu(\zeta) = \sqrt[4]{3}\sqrt{B}z(\zeta)$. Then (3.23) becomes

$$z_\zeta = \pm\frac{\sqrt{B}}{2\sqrt[4]{3}}\sqrt{z^4 - \sqrt{3}z^2 + 1}. \tag{3.24}$$

This ODE is solved using Jacobian elliptic functions.

Claim 1: If $Z = 1 + 2z^2\cos 2\alpha + z^4$, *then*

$$u(x,k) = \int_0^x \frac{dz}{\sqrt{Z}} = \frac{1}{2}\mathrm{sn}^{-1}\frac{2x}{1+x^2},$$

with $k = \sin\alpha$, *see p. 91 of Ref. 10.*

Proof: Putting $z = \tan\theta$, we find

$$u(x,k) = \int_0^{\tan^{-1}x} \frac{d\theta}{\sqrt{1-\sin^2\alpha\sin^2 2\theta}},$$

followed by $y = \sin 2\theta$, which will give us

$$u(x,k) = \frac{1}{2}\int_0^{\frac{2x}{1+x^2}} \frac{dy}{\sqrt{(1-y^2)(1-k^2y^2)}},$$

which is an elliptic integral of the first kind. Therefore, solving (3.24), we obtain the solution in z

$$\frac{2z}{1+z^2} = \pm\mathrm{sn}\Big(\frac{\sqrt{B}}{\sqrt[4]{3}}\zeta, k\Big), \tag{3.25}$$

where $k = \sin\frac{5\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}}$ is the modulus of the Jacobian elliptic function. It follows that for μ we will have

$$\frac{2a\mu}{a^2+\mu^2} = \pm\mathrm{sn}\Big(\frac{\sqrt{3}}{3}a\zeta, k\Big), \tag{3.26}$$

where $a = \sqrt{B}\sqrt[4]{3}$. By solving the quadratic yields

$$\mu = \pm a\frac{1 \pm \mathrm{cn}\big(\frac{\sqrt{3}}{3}a\zeta, k\big)}{\mathrm{sn}\big(\frac{\sqrt{3}}{3}a\zeta, k\big)}. \tag{3.27}$$

Therefore, the solution to the BBM equation without viscosity and velocity $c = 1$ is

$$\begin{aligned} u(x,t) =& B\Big[1 - \sqrt{3}\Big(\frac{1 \pm \mathrm{cn}\big(\frac{\sqrt{B}}{\sqrt[4]{3}}(x-t), k\big)}{\mathrm{sn}\big(\frac{\sqrt{B}}{\sqrt[4]{3}}(x-t), k\big)}\Big)^2\Big] \\ =& B\Big[1 - \sqrt{3}\frac{1 \mp \mathrm{cn}\big(\frac{\sqrt{B}}{\sqrt[4]{3}}(x-t), k\big)}{1 \pm \mathrm{cn}\big(\frac{\sqrt{B}}{\sqrt[4]{3}}(x-t), k\big)}\Big]. \end{aligned} \tag{3.28}$$

These critical solutions are $2K$ periodic with K obtained from the complete elliptic integral

$$\frac{K}{2} = \int_0^1 \frac{dz}{\sqrt{z^4 - \sqrt{3}z^2 + 1}} = 2.76806, \tag{3.29}$$

see right panel of Fig. 1. The same solution was previously obtained by Cornejo-Pérez and Rosu, using a factorization technique.[24]

The expressions (3.13) and (3.28) describe the whole class of solitary wave solutions with spectrum $c \in (0, \infty)$ which is in concordance with Ref. 6. They present periodic and solitary wave solutions to (1.3), with no restrictions on the constant A, which may be advantageous when adapting results to physical problems.

Remark 1: All the solutions obtained using simplifying assumptions $\nu = 0, A = 0$ are particular cases of general solutions of (2.3) in terms of Weierstrass elliptic functions $\wp$, as we will see next.

C. Viscosity present, $\nu > 0$

Consider (3.8) in the form of non-autonomous equation $F(y, y_v, v) = 0$. Painlevé proved in his doctoral thesis in 1887[20] that all integrals of non-autonomous equations do not have movable singular points, but only poles and fixed algebraic singularities. Poincare[25] proved in 1885 that any non-autonomous equation of genus $p = 0$ is reducible to Riccati equation, while if $p = 1$ is integrable via Weierstrass $\wp$ functions, after a linear fractional transformation. Since Riccati equation can be easily obtained from a linear ODE and the Appel invariant $g(v)$ has no singularities, one can conclude that the closed form solutions of (3.11) will not have movable singular points, but poles. Since $v(z)$ has only poles of order two, then the solution to (3.11), must be written in terms of Weierstrass $\wp$ functions via the transformation

$$v = Ez^p\omega(z) + F, \tag{3.30}$$

see Ince, p. 431 of Ref. 13. By substituting this ansatz in (3.11) with E, F constants, then for $p = \frac{2}{5}$, $\omega(z)$ will satisfy the elliptic equation

$$\omega'^2 = 4\omega^3 - g_2\omega - g_3 \tag{3.31}$$

with invariants g_2, g_3.

Moreover, z is an exponential function, because it satisfies

$$\frac{dv}{dz} = -\frac{1}{yz}, \tag{3.32}$$

which leads to

$$\frac{\nu}{c}d\zeta = \frac{dz}{z}. \tag{3.33}$$

The $\wp$ solutions can be combined into the general substitution

$$u(\zeta) = \sigma - g(\zeta), \tag{3.34}$$

where $g(\zeta) = e^{-n\zeta}\Omega(\zeta)$. σ and n are constants which are related to A and p.

By substituting (3.34) into (2.3) we obtain the ODE

$$g'' - \frac{\nu}{c}g' - \frac{1}{2c}g^2 + \frac{1-c+\sigma}{c}g = 0. \tag{3.35}$$

The free term was eliminated by setting $A = \frac{\sigma^2}{2} + \sigma(1-c)$. Since $g(\zeta) = e^{-n\zeta}\Omega(\zeta)$, (3.35) becomes

$$\Omega'' - \left(2n + \frac{\nu}{c}\right)\Omega' + \left(n^2 + \frac{n\nu}{c} + \frac{1-c+\sigma}{c}\right)\Omega = \frac{1}{2c}e^{-n\zeta}\Omega^2. \tag{3.36}$$

Finally, let $\Omega(\zeta) = \omega(z(\zeta))$, and by chain rule with $' = \frac{d}{d\zeta}$ and $\dot{} = \frac{d}{dz}$, (3.36) is

$$(z')^2\ddot{\omega} + \left(z'' - \left(2n + \frac{\nu}{c}\right)z'\right)\dot{\omega} + \left(n^2 + \frac{1-c+\sigma+n\nu}{c}\right)\omega = \frac{1}{2c}e^{-n\zeta}\omega^2. \tag{3.37}$$

By letting

$$z'' - \left(2n + \frac{\nu}{c}\right)z' = 0, \tag{3.38}$$

we obtain

$$z'(\zeta) = c_1 e^{(2n+\frac{\nu}{c})\zeta}. \tag{3.39}$$

We also choose $\sigma = -(n^2c + n\nu + 1 - c)$ which cancels the linear term in (3.37).

We are left to solve

$$z'^2\ddot{\omega} - \frac{1}{2c}e^{-n\zeta}\omega^2 = 0 \tag{3.40}$$

subject to (3.39). Set $n = -p\frac{v}{c} = -\frac{2\nu}{5c}$, then

$$\sigma = \frac{14\nu^2}{25c} + c - 1,$$
$$A = -\frac{\sigma^2}{2} + \frac{14\nu^2\sigma}{25c}. \tag{3.41}$$

By substituting (3.39) into (3.40), we obtain

$$\ddot{\omega} = \frac{1}{2cc_1^2}\omega^2. \tag{3.42}$$

Letting $c_1 = \frac{1}{2\sqrt{3c}}$, we arrive at the elliptic equation

$$\ddot{\omega} = 6\omega^2, \tag{3.43}$$

which by multiplication by $\dot{\omega}$ and one quadrature becomes

$$\dot{\omega}^2 = 4\omega^3 - g_3. \tag{3.44}$$

Its solution is

$$\omega(z) = \wp(z + c_3, 0, g_3) \tag{3.45}$$

with invariants $g_2 = 0$, and g_3. If $g_3 = 1$, this is the equianharmonic case, see p. 652 of Ref. 1.

Moreover, $z(\zeta)$ is found by integrating (3.39) to get

$$z(\zeta) = c_2 + \frac{5\sqrt{3c}}{6\nu}e^{\frac{\nu\zeta}{5c}}, \tag{3.46}$$

which leads to

$$\Omega(\zeta) = \omega(z) = \wp(c_4 + \frac{5\sqrt{3c}}{6\nu}e^{\frac{\nu\zeta}{5c}}, 0, g_3), \tag{3.47}$$

where $c_4 = c_2 + c_3$, and g_3 are two integration constants that depend on the boundary conditions. Using all of the above, together with (3.41) we obtain

$$\sigma = \frac{14\nu^2}{25c} \pm \sqrt{\left(\frac{14\nu^2}{25c}\right)^2 - 2A}. \tag{3.48}$$

Then, the general solution to (3.31) is

$$u(\zeta) = \frac{14\nu^2}{25c} \pm \sqrt{\left(\frac{14\nu^2}{25c}\right)^2 - 2A(\nu, c)} - e^{\frac{2\nu\zeta}{5c}}\wp(c_4 + \frac{5\sqrt{3c}}{6\nu}e^{\frac{\nu\zeta}{5c}}, 0, g_3), \tag{3.49}$$

see Figs. 2 and 3. Once we fix the velocity c, and dissipation ν, then the constants σ and $A(\nu, c)$ are obtained using (3.41), which substituted back into (3.49), will give us the solution to (1.4). The two

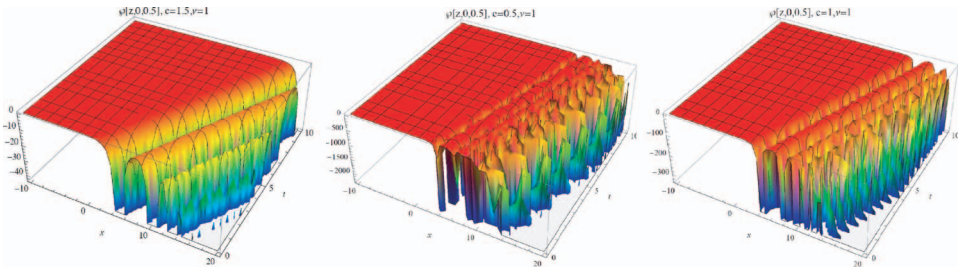

FIG. 2. Weierstrass solutions $\nu = 1$, Eq. (3.49), left $c = 1.5$; middle $c = 0.5$; right $c = 1$.

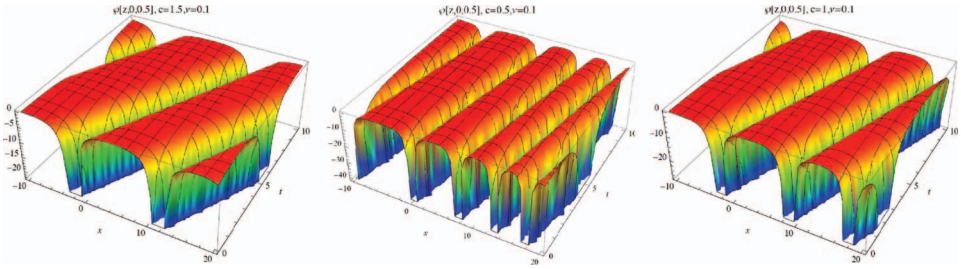

FIG. 3. Weierstrass solutions $\nu = 0.1$, Eq. (3.49), left $c = 1.5$; middle $c = 0.5$; right $c = 1$.

integration constants that depend on the boundary conditions are c_4 and g_3. It is crucial to notice that the fixed points of (2.4) depend on the velocity c and dissipation ν via the constant A, see (2.6).

Finally, it is worth mentioning that the solution in terms of $\wp$ functions is not unique. One can see that there is at least one other similar form previously obtained by Porubov, via the Carnevalle method and B*ä*cklund transformation.[15,26]

Remark 2: Letting $\nu = 0 \to n = 0$, $c_1 = 1$, and $c = 1 \to \sigma = 0$ in (3.42), and since $\omega = g$, but $f = -g$, then $\omega = -f$ so (3.44) becomes (3.21), with solution given by (3.28), see right panel of Fig. 1.

Remark 3: Letting $\nu = 0 \to n = 0$ and $\sigma = c - 1 \neq 0$, then according to (3.39), $z' = c_1$. Since $\omega = g$, then (3.37) is actually (3.12), and hence we recover the slow and fast waves solutions (3.13), see left and middle panels of Fig. 1.

Remark 4: If the second invariant $g_3 = 0$ in (3.44), then by integrating (3.44)

$$\omega(z) = \frac{1}{(c_5 \pm z)^2}. \tag{3.50}$$

Using (3.46) in (3.50) we obtain the kink solution

$$u(\zeta) = \frac{14\nu^2}{25c} \pm \sqrt{\left(\frac{14\nu^2}{25c}\right)^2 - 2A} - \frac{e^{\frac{2\nu\zeta}{5c}}}{\left(c_6 \pm \frac{5\sqrt{3c}}{6\nu} e^{\frac{\nu\zeta}{5c}}\right)^2}, \tag{3.51}$$

see Fig. 4.

An analysis when a $\wp$ function tends to a smooth kink can be found in Ref. 27.

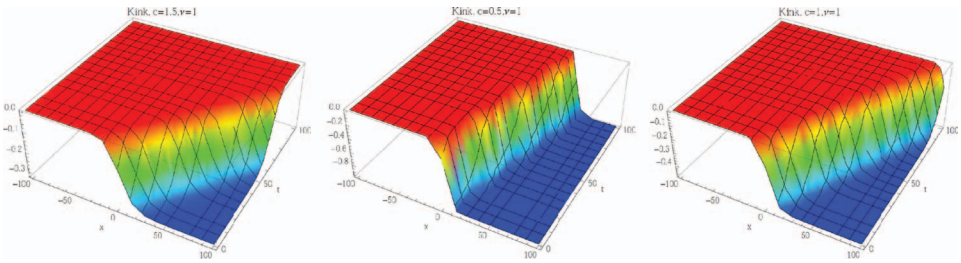

FIG. 4. Kink solutions $\nu = 1$, Eq. (3.51), left $c = 1.5$; middle $c = 0.5$; right $c = 1$.

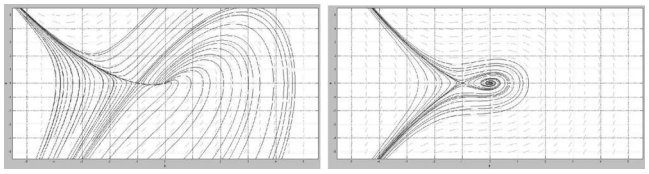

FIG. 5. Phase-plane for $c = 0.5$, left $\nu = 1$, (0.0) node, $(-1, 0)$ saddle; right $\nu = 0.1$, (0.0) spiral, $(-1, 0)$ saddle.

Remark 5: If $A = 0 \to c - 1 = \pm k = \pm\frac{14\nu^2}{25c}$, and one selects the lower branch of the radical, we obtain

$$\begin{aligned} u(\zeta) &= -e^{\frac{2\nu\zeta}{5c}}\wp(c_4 + \frac{5}{\nu}\sqrt{\frac{c}{3a}}e^{\frac{\nu\zeta}{5c}}, 0, g_3), \text{ if } g_3 \neq 0, \\ u(\zeta) &= -\frac{e^{\frac{2\nu\zeta}{5c}}}{\left(c_6 \pm \frac{5\sqrt{3c}}{6\nu}e^{\frac{\nu\zeta}{5c}}\right)^2}, \text{ if } g_3 = 0. \end{aligned} \tag{3.52}$$

IV. STABILITY OF THE VISCOUS WAVES

In this section we consider the two-mode dynamical system of (2.4) with equilibrium points in the phase plane (ϕ, ψ) at $(0, 0)$ and $(c - 1 \pm \sqrt{(c-1)^2 + 2A}, 0)$, where $A = A(\nu, c)$ is obtained from (3.41). All the equilibrium points lie only in the plane $\psi = 0$, in the (ϕ, ψ, c) space. The bifurcation curves are given by $\phi\left(\phi - (c-1) \mp \sqrt{(c-1)^2 + 2A}\right) = 0$, which are two constant lines. Near the origin $(\phi^-, 0) = (0, 0)$

$$\begin{bmatrix} \phi' \\ \psi' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1-c}{c} & \frac{\nu}{c} \end{bmatrix} \begin{bmatrix} \phi \\ \psi \end{bmatrix}. \tag{4.1}$$

Following standard methods of phase-plane analysis the characteristic polynomial of the Jacobian matrix of (4.1) evaluated at the fixed point $(\phi^-, 0)$ is

$$g_0(\lambda) = \lambda^2 - p_0\lambda + q_0 = 0, \tag{4.2}$$

where $p_0 = \frac{\nu}{c}$ and $q_0 = \frac{1-c}{c}$.

Since $\nu > 0$, $c > 0$, then $p_0 > 0$, hence the origin is *unstable*. Also, putting $\Delta = p_0^2 - 4q_0 = \frac{\nu^2 - 4c(1-c)}{c^2}$, we have the following cases:

(i) $0 < c < 1$, gives $q_0 > 0$. If $\nu > 2\sqrt{c(1-c)} > 0$, the origin is unstable *node*; if $0 < \nu < 2\sqrt{c(1-c)}$, the origin is unstable *spiral*, see Fig. 5 and

(ii) $c > 1$, gives $q_0 < 0 \Rightarrow \Delta > 0$, hence the origin is a *saddle* point, see Fig. 6.

Near the secondary fixed point $(\phi^+, 0) = (c - 1 \pm \sqrt{(c-1)^2 + 2A}, 0)$

$$\begin{bmatrix} \phi' \\ \psi' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{1-c}{c} & \frac{\nu}{c} \end{bmatrix} \begin{bmatrix} \phi \\ \psi \end{bmatrix}. \tag{4.3}$$

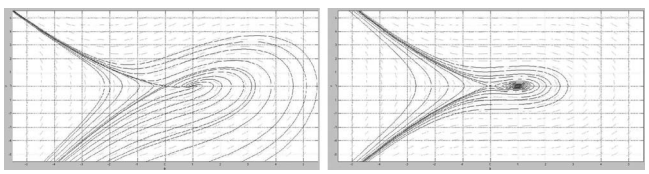

FIG. 6. Phase-plane for $c = 1.5$, left $\nu = 1$, (0.0) saddle, (1, 0) node; right $\nu = 0.1$, (0.0) saddle, (1, 0) spiral.

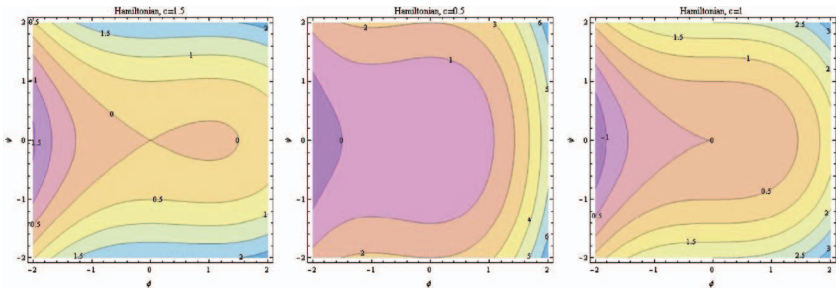

FIG. 7. Contour plot of Hamiltonian $\mathcal{H} = const$, see (4.5), left $c = 1.5$; middle $c = 0.5$; right $c = 1$.

The characteristic polynomial of the Jacobian matrix of (4.3) evaluated at the fixed point $(\phi^+, 0)$ is

$$g_1(\lambda) = \lambda^2 - p_1\lambda + q_1 = 0, \tag{4.4}$$

where $p_1 = p_0 = \frac{\nu}{c}$ and $q_1 = -q_0 = -\frac{1-c}{c}$.

Since $p_1 = p_0$, the second fixed point is also *unstable*, and moreover we have the following cases:

(i) $0 < c < 1$, gives $q_1 < 0 \Rightarrow \Delta > 0$, hence the fixed point is a *saddle* point, see Fig. 5 and
(ii) $c > 1$, gives $q_1 > 0$. if $\nu > 2\sqrt{c(c-1)}$, we have an unstable *node*; if $0 < \nu < 2\sqrt{c(c-1)}$, the fixed point is unstable *spiral*, see Fig. 6.

All the other remaining cases, i.e., both fixed points collide, are degenerate, since if $c = 1$, $g(\lambda) = \lambda^2 - \nu\lambda$.

Note that when there is no viscosity, the system has Hamiltonian, which is directly proportional with the Lyapunov function via

$$\mathcal{H}(\mathbf{u}) = \frac{1}{2}\psi^2 + \frac{1-c}{2c}\phi^2 + \frac{1}{6c}\phi^3 = \frac{1}{c}\mathcal{L}(\mathbf{u}), \tag{4.5}$$

see (2.11). Therefore, along any phase path $\mathcal{H}(\mathbf{u}) = constant$ is a conserved quantity, see Fig. 7. In this case, $p_0 = p_1 = 0$, and hence

(i) $(\phi^-, 0)$ is a *saddle* and $(\phi^+, 0)$ is a *center* when $q_1 > 0$, see left panel of Fig. 8,
(ii) $(\phi^-, 0)$ is a *center* and $(\phi^+, 0)$ is a *saddle* when $q_1 < 0$, see middle panel of Fig. 8, and
(iii) $c = 1$ is degenerate since both critical points collide at origin $\phi^- = \phi^+$, see right panel of Fig. 8.

Therefore, in this case the unstable spirals (which correspond to the case with small viscosity $\nu = 0.1$) become centers when there is no viscosity $\nu = 0$.

This is an example of a transcritical bifurcation where, at the intersection of the two bifurcation curves $\phi = 0$ and $\phi = 2(c - 1)$, the equilibrium changes from one curve to the other at the bifurcation point. As c increases through one, the saddle point collides with the unstable node, and then remains there while the unstable node or spiral moves away from $(\phi^-, 0)$.

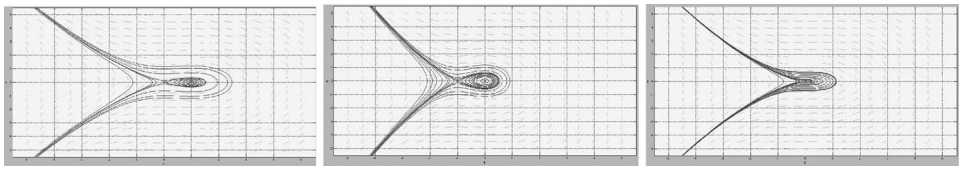

FIG. 8. Phase-plane for $\nu = 0$, left $c = 1.5$, (0.0) saddle, (1, 0) center; middle $c = 0.5$, (0.0) center, $(-1, 0)$ saddle; right $c = 1$ (0.0) degenerate.

V. SUMMARY

In this paper a basic theory for the BBM equation (1.3), and its extension (1.4) to include the dissipation term was shown. When the viscous term is not present, (1.3) has traveling wave solutions that depend critically on the traveling wave velocity. When the velocity $c > 1$ (1.4) has solitary waves solutions, while if $c < 1$, the solutions become periodic. At the interface between the two cases, when $c = 1$, the solutions are rational functions of cnoidal waves. Based on Ince's transformation, a theory including the dissipative term was presented, in which we have found more general solutions in terms of Weierstrass elliptic $\wp$ functions. Using dynamical systems theory we have shown that the solutions of (1.4) experience a transcritical bifurcation when $c = 1$, where at the intersection of the two bifurcation curves, stable equilibrium changes from one curve to the other at the bifurcation point. As the velocity changes, the saddle point collides with the node at the origin, and then remains there, while the stable node moves away from the origin. Also, the bifurcation curves and the fixed points of the dynamical system are functions of the traveling wave velocity c and dissipation constant ν.

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