

Western Kentucky University

From the Selected Works of Bruce Kessler

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My Trig Book

Bruce Kessler, *Western Kentucky University*



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MY TRIG BOOK

BY

BRUCE KESSLER

2ND EDITION



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Chapter 1: Finding Difficult Distances

MATH 117: Trigonometry

1.1 Introduction

In many instances, we can find distances just by using the Pythagorean theorem, which says that in a right triangle, the sum of the squares of the lengths of the legs is equal to the sum of the square of the length of the hypotenuse.

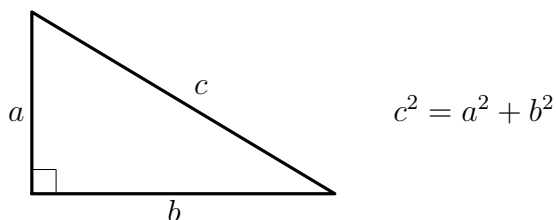


Figure 1

Example: Ann drives 8 blocks east and 6 blocks north (see Figure 2).

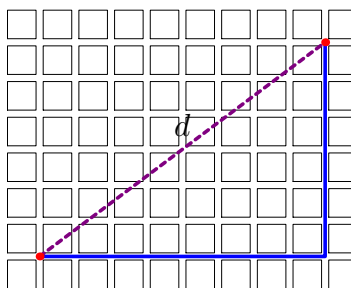


Figure 2

How far is she from her starting point, in miles? (1 block \approx 0.1 mi)

Since the city blocks are arranged at right angles, Ann's path forms the legs of a right triangle. We need to convert the distances to miles, using unit conversion (affectionately known as the ol' "multiply-by-one trick").

$$8 \text{ blocks} \cdot \frac{0.1 \text{ mi}}{1 \text{ block}} = 0.8 \text{ mi} \quad 6 \text{ blocks} \cdot \frac{0.1 \text{ mi}}{1 \text{ block}} = 0.6 \text{ mi}$$

Then, using the Pythagorean theorem,

$$d^2 = (0.8)^2 + (0.6)^2,$$

$$d^2 = 0.64 + 0.36 = 1, \text{ and}$$

$$d = 1 \text{ mile.}$$

(I should point out that this is not the only valid answer. This is the *Euclidean*, or “as-the-crow-flies,” distance between the two points. Realistically, Ann could not get to her destination by traveling 1 mile. She has basically taken the shortest path to her destination, traveling $8 + 6 = 14$ blocks, or 1.4 miles. This is called the “taxicab” distance between the two points.)

Now suppose that the right angles are removed from the problem.

Example: Ann walk 0.8 miles east, and then turns 75° to the north and walks another 0.6 miles (see Figure 3).

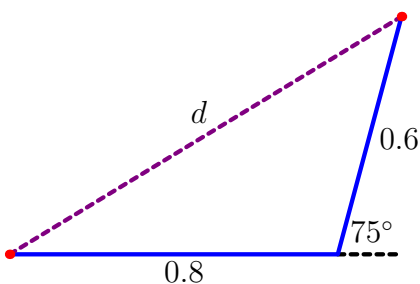


Figure 3

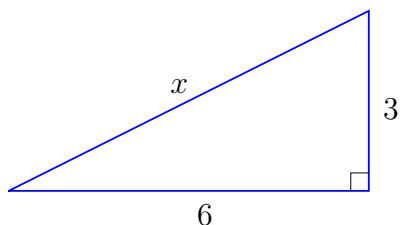
How far is she from her starting point, in miles?

Because the angle formed by the 0.8-mile and 0.6-mile sides is larger than a right angle, we can be sure that d is greater than it was in the last problem, but how much greater? Finding the answer to this question will require the development of several different identities based on the angles present in a triangle.

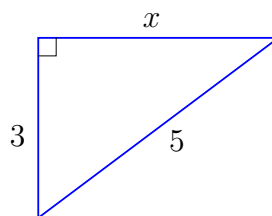
1.1 Exercises

Find the exact length of the missing sides of each diagram.

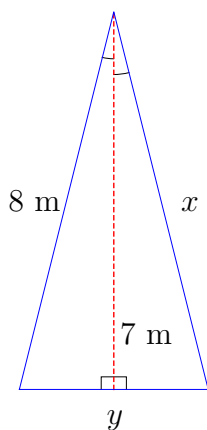
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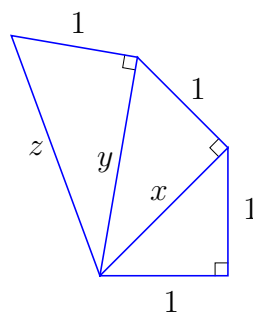
2)



3)



4)



5) Suppose that you are measuring a distance along the ground with a tape measure, but you are too lazy to bend over and hold the tape to the ground. Instead, you hold the end of the tape measure at your waist, 3 ft above the ground. (Figure 4) If the horizontal distance is actually 14 ft, at least how much is your measurement off?

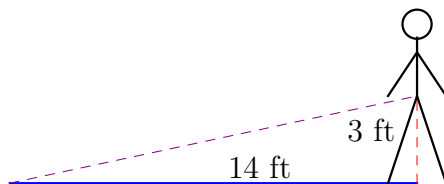


Figure 4

6) Suppose the same situation as in problem 5), except that the actual horizontal distance is 24 ft. At least how much is your measurement off now? Is the error more or less than in problem 5)? Is the percentage error more or less than in problem 5)? What other factors would affect the accuracy of your measure?

7) Suppose that you are trying to measure the distance between two points on the ground, but there is a curb in the way. (Figure 5) The curb is a hexagon, with dimensions shown at the bottom in Figure 5. If the actual measurements are as shown on the top in Figure 5, then at least how much would a measurement over the curb be off? Assuming that the tape is read correctly but not pulled tight at all, what would be the upper bound on the error?

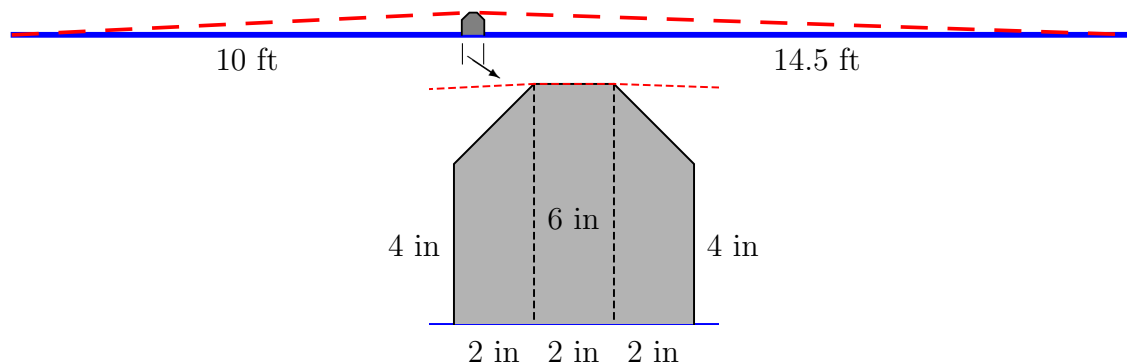


Figure 5

8) What is the Euclidean distance from point A to point B in Figure 6? What is the taxicab distance from point A to point B ? Does the taxicab distance depend on the path taken from A to B ? Speculate as to whether one of these types of distance measures is greater than or equal to the other for the same two points located at random intersections.

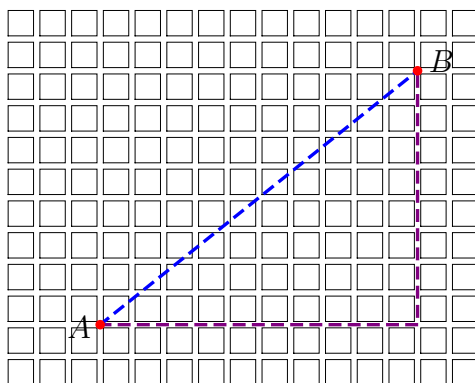


Figure 6

9) In isosceles right triangles where each leg has length x , find a formula for the length of the hypotenuse.

10) In right triangles where one leg has length x and the hypotenuse is twice that length, find a formula for the length of the other leg.

1.2 Finding Distances Based on Angles

The Sine of an Angle θ

The sine of an angle $0 < \theta < 90^\circ$, denoted $\sin \theta$, can be defined as the ratio of the length of the leg of a right triangle opposite the angle to the length of the hypotenuse of the right triangle (see Figure 7 below). Loosely, we say that

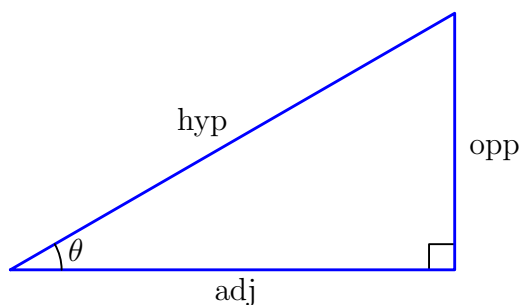


Figure 7

$$\sin \theta = \frac{\text{opp}}{\text{hyp}} \text{ for } 0 < \theta < 90^\circ.$$

Most of the time, we will be dependent upon our calculators to give us the sine of an angle, but for certain angles, we can calculate their sine explicitly. Since the sum of the angles in a triangle is 180° , a right triangle with a 45° angle will actually have two 45° angles, and will be an isosceles triangle. If the hypotenuse has length 2 and the legs are of equal length, then we can solve for their length using the Pythagorean theorem.

$$2^2 = a^2 + a^2$$

$$2a^2 = 4$$

$$a^2 = 2$$

$$a = \sqrt{2}$$

Then $\sin 45^\circ = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{2}}{2}$ (see Figure 8). Also, if we cut an equilateral triangle of side length 2 in half along a line of symmetry, then we have a right triangle. The length of the hypotenuse is 2, and the length of one leg is 1, so we can solve for the length of the other leg.

$$2^2 = a^2 + 1^2$$

$$a^2 = 3$$

$$a = \sqrt{3}$$

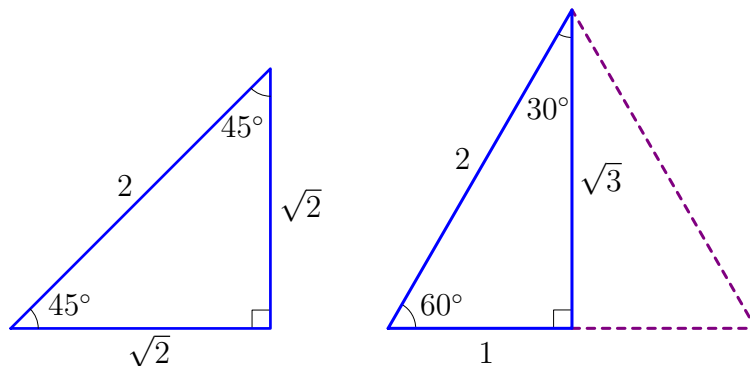


Figure 8 – Reference Triangles

Then $\sin 30^\circ = \frac{\text{opp}}{\text{hyp}} = \frac{1}{2}$ and $\sin 60^\circ = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{3}}{2}$ (see Figure 8).

We can extend this concept to any angle θ by drawing the angle on coordinate axes, with the angle starting on the positive x -axis, and ending at a ray drawn from the origin. If, as in Figure 9, we let (x, y) be a point on the ray but not the origin, and let $r = \sqrt{x^2 + y^2}$, then we can think about the sine of an angle θ as the ratio of y to the distance r from the origin; that is

$$\sin \theta = \frac{y}{r}.$$

This means that $\sin 0^\circ = \sin 180^\circ = 0$, $\sin 90^\circ = 1$, and $\sin 270^\circ = -1$. It also means that

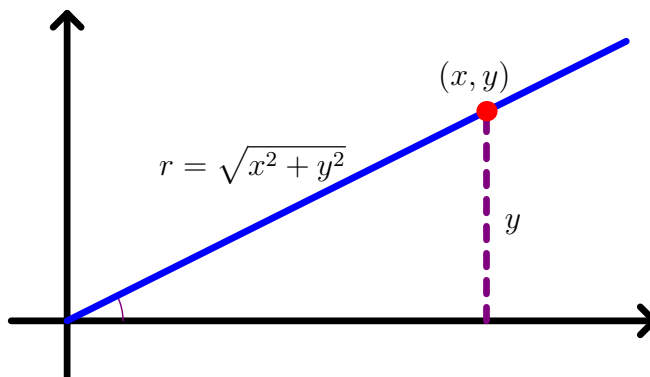


Figure 9

we can use our reference triangles to calculate certain angles. See Figure 10 for the diagram showing the calculation of $\sin 135^\circ$ and $\sin(-60^\circ)$.

Notice that, because of the definition, that $\sin(-\theta) = -\sin \theta$ for all angles θ .

The Cosine of an Angle θ

The cosine of an angle $0 < \theta < 90^\circ$, denoted $\cos \theta$, can be defined as the ratio of the length of the leg of a right triangle adjacent to the angle to the length of the hypotenuse of the

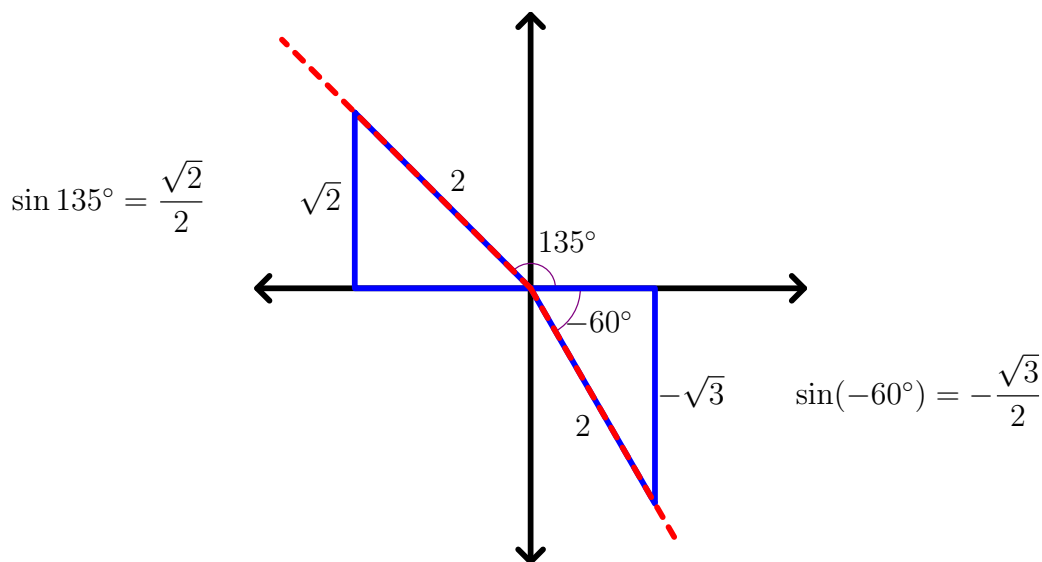


Figure 10

right triangle (see Figure 7 again). Loosely, we say that

$$\cos \theta = \frac{\text{adj}}{\text{hyp}} \text{ for } 0 < \theta < 90^\circ.$$

Then, using the reference triangles in Figure 8, we see that $\cos 45^\circ = \frac{\sqrt{2}}{2}$, $\cos 30^\circ = \frac{\sqrt{3}}{2}$, and $\cos 60^\circ = \frac{1}{2}$.

We can extend this concept to any angle θ in a similar fashion used for calculating the sine of any angle. If, as in Figure 11, we let (x, y) be a point on the ray but not the origin, and let $r = \sqrt{x^2 + y^2}$, then we can think about the cosine of an angle θ as the ratio of x to the distance r from the origin; that is

$$\cos \theta = \frac{x}{r}.$$

This means that $\cos 90^\circ = \cos 270^\circ = 0$, $\cos 0^\circ = 1$, and $\cos 180^\circ = -1$. As before, we can use our reference triangles to calculate the cosine of certain angles. See Figure 12 for the diagram showing the calculation of $\cos 135^\circ$ and $\cos(-60^\circ)$.

Notice that, because of the definition, that $\cos(-\theta) = \cos \theta$ for all θ .

How Are They Related?

Consider the point (x, y) on the terminal ray of the angle θ in Figure 9 and Figure 11, and let $r = \sqrt{x^2 + y^2}$. Then, from above,

$$\sin \theta = \frac{y}{r} \text{ and } \cos \theta = \frac{x}{r}.$$

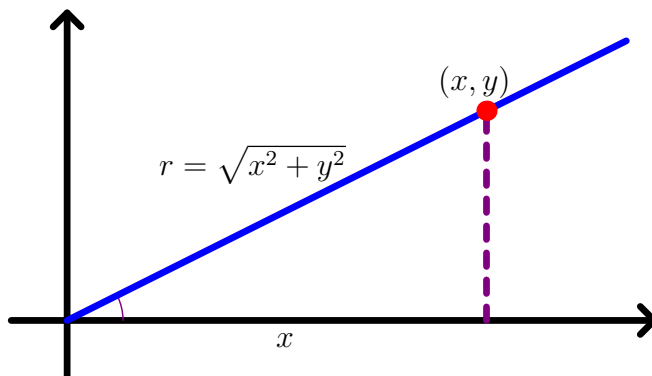


Figure 11

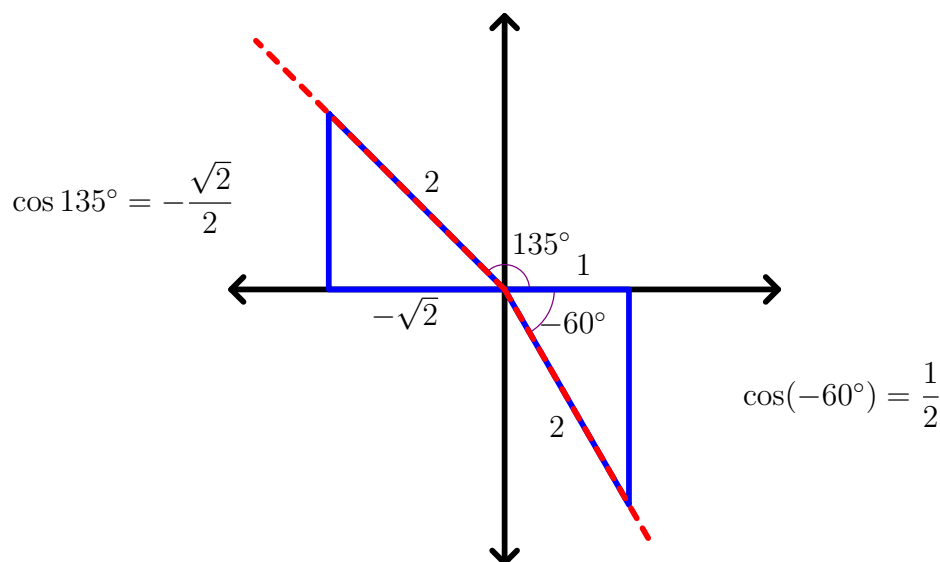


Figure 12

That means that

$$\sin^2 \theta + \cos^2 \theta = \frac{y^2}{r^2} + \frac{x^2}{r^2} = \frac{x^2 + y^2}{r^2} = \frac{r^2}{r^2} = 1 \quad (1)$$

for all values of θ . Since the above statement is basically a restatement of the Pythagorean theorem, we call the relationship in (1) a *Pythagorean Identity*. (It turns out that there are a couple more that we will find later.)

The right triangle diagram shown below in Figure 13 holds the key to another relationship between the sine and the cosine.

$$\sin \theta = \frac{\text{opp}}{\text{hyp}} = \cos(90^\circ - \theta) \quad \text{and} \quad \cos \theta = \frac{\text{adj}}{\text{hyp}} = \sin(90^\circ - \theta) \quad (2)$$

The sine and cosine are said to be *cofunctions* of each other, and while the above diagram only holds for $0^\circ < \theta < 90^\circ$, the relationships in (2) actually hold for all θ . To see this, consider the diagram in Figure 14. Let (a, b) be a point on the terminal ray of the angle θ .

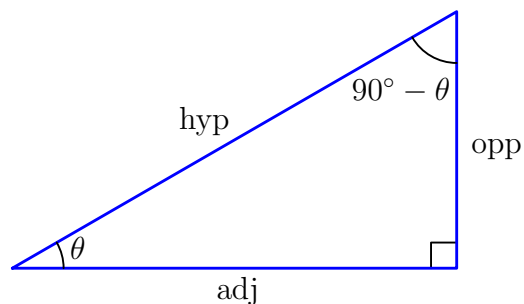


Figure 13

Then the terminal ray for the angle $90^\circ - \theta$ is the above ray reflected across the line $y = x$, and so the reflected point will have coordinants (b, a) . Again,

$$\sin \theta = \frac{b}{r} = \cos(90^\circ - \theta) \quad \text{and} \quad \cos \theta = \frac{a}{r} = \sin(90^\circ - \theta).$$

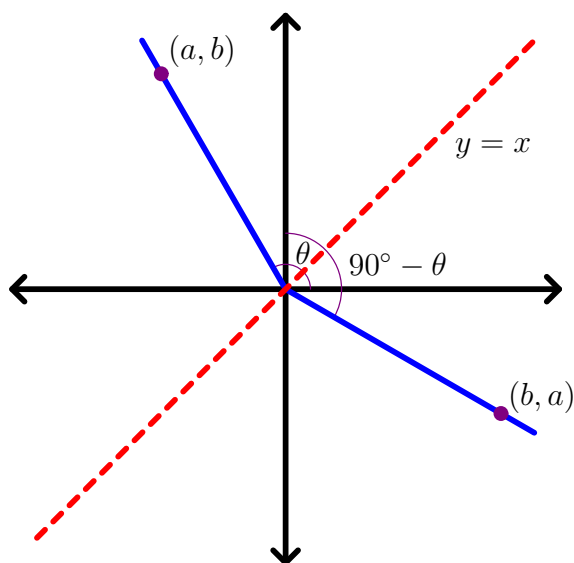


Figure 14

Finding Right-Angle Distances

We are now able to find the horizontal and vertical component of any distance based on that distance and its direction with respect to our chosen axes.

Example: *Ann travels 1 mile in a direction that is 37° north of east. How far north of her starting point is she? How far east?*

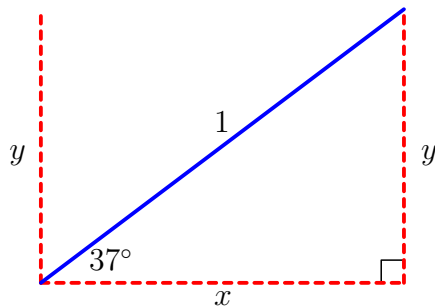


Figure 15

Let y denote the distance traveled north, and let x denote the distance traveled east.

$$\sin 37^\circ = \frac{y}{1} \qquad \cos 37^\circ = \frac{x}{1}$$

$$y = \sin 37^\circ \qquad x = \cos 37^\circ$$

$$y \approx 0.60 \text{ mi} \qquad x \approx 0.80 \text{ mi}$$

Ann has traveled roughly 0.6 miles north and 0.8 miles east.

Example: *The safety instructions on a 24-foot ladder say to never use the ladder at an angle steeper than 70° from horizontal (there is a gauge on the side of the ladder to indicate a “safe” angle). What is the maximum height that the ladder can reach when used safely?*

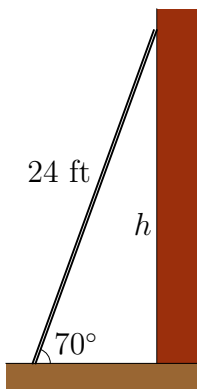


Figure 15

Let h denote the height of the top end of the ladder when placed against a wall at an angle of 70° above horizontal. Then

$$\sin 70^\circ = \frac{h}{24}, \text{ and so } h = 24 \sin 70^\circ \approx 22.55 \text{ ft.}$$

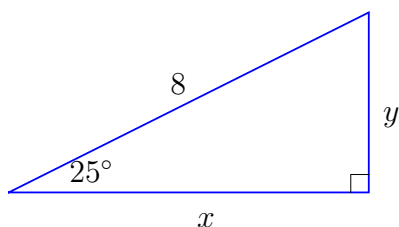
1.2 Exercises

Find the exact value of the sine and cosine of the following by drawing reference triangles on coordinate axes.

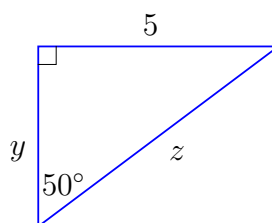
- | | |
|----------------|-----------------|
| 1) 300° | 2) -60° |
| 3) 150° | 4) -135° |
| 5) 120° | 6) -120° |
| 7) 225° | 8) -210° |

Find the length of the missing sides of each diagram, accurate to two decimal places.

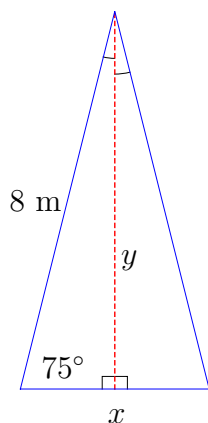
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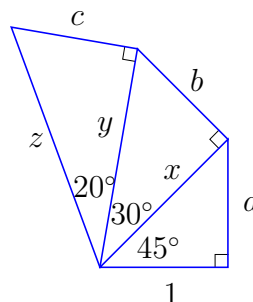
10)



11)



12)



Give the final location in terms of north/south, east/west of the starting point after traveling the following distances at the given angles.

- | | |
|--|--|
| 13) 15 steps at 30° north of due east | 14) 60 m at 45° north of due east, then 100 m at 30° west of due north |
| 15) 3 miles at 20° south of due east, then 1 mile due north | 16) 8 m at 40° north of due east, then 2 m due west, then 6 m at 40° north of due west |

17) While on a geological study abroad trip, you view a mountain range. (Figure 17) Using a range-finder, you determine that the top of the mountain is 4500 ft away. Using a protractor and a paper-clip on a string, you determine that the angle above horizontal from your position to the top of the mountain is 23° . You are 6 ft tall, and your eyes are 4 inches below the top of your head. How tall is the mountain in relation to ground level where you are standing?

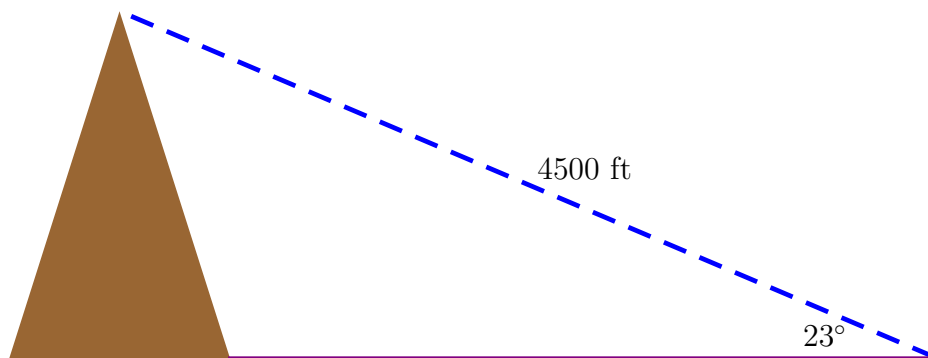


Figure 17

18) Suppose that you are a NASA scientist and you are calculating the angle of trajectory needed for a deep-space probe to reach a point in space 3.6 billion miles away, roughly the distance between Earth and the planet/asteroid (depends on who you talk to) Pluto. If the trajectory is off by 0.5° , how badly will the probe miss the planet?

1.3 Law of Cosines

This all started with a young lady named Ann traveling at non-right angles (how dare her!?!), so let's return to that problem (see page 5). Let's not just solve that problem, but find a general solution for the length of the third side of a triangle when we know the lengths of two sides and the angle between the sides of known length has measure θ .

The Theory

Let's suppose that we know the lengths of two sides, a and b , and the measure θ of the angle formed by the two sides. We want to find the length c of the third side. If $\theta = 90^\circ$, then we can just use the Pythagorean theorem. Let's break the problem up into two cases: where θ is acute, and where θ is obtuse.

Case 1: $0^\circ < \theta < 90^\circ$

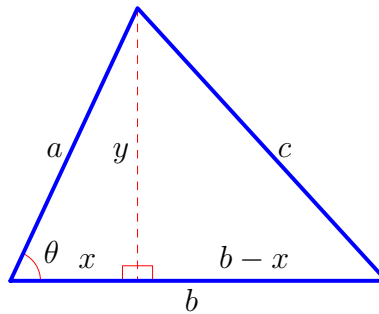


Figure 18

We are looking for the length c , but we will start with some other lengths. Drop an altitude from one end of the third side, as shown in Figure 18, and consider the lengths x and y . From our work above, we know that

$$x = a \cos \theta \text{ and } y = a \sin \theta.$$

Note that $x^2 + y^2 = a^2$ by the Pythagorean theorem. Also, by the Pythagorean theorem, we know that

$$c^2 = (b - x)^2 + y^2.$$

Then

$$c^2 = b^2 - 2bx + x^2 + y^2$$

$$c^2 = b^2 - 2b(a \cos \theta) + a^2$$

$$c^2 = a^2 + b^2 - 2ab \cos \theta.$$

Case 2: $90^\circ < \theta < 180^\circ$

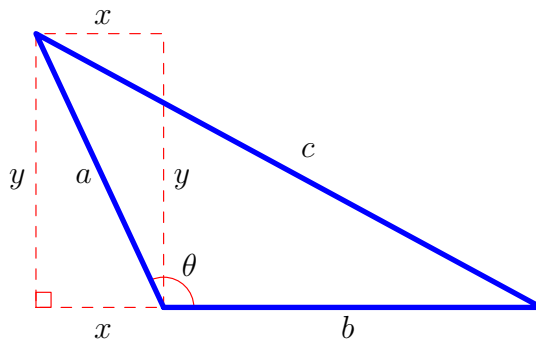


Figure 19

Again, consider the lengths x and y formed by dropping an altitude as in Figure 19. It is perhaps easier to calculate the lengths by looking at the top and right side of the dashed red rectangle, respectively.

$$x = a \sin(\theta - 90^\circ) = -a \sin(90^\circ - \theta) = -a \cos \theta$$

$$y = a \cos(\theta - 90^\circ) = a \cos(90^\circ - \theta) = a \sin \theta$$

Note that $x^2 + y^2 = a^2$ by the Pythagorean theorem, and also

$$c^2 = (b + x)^2 + y^2.$$

Then

$$c^2 = b^2 + 2bx + x^2 + y^2$$

$$c^2 = b^2 + 2b(-a \cos \theta) + a^2$$

$$c^2 = a^2 + b^2 - 2ab \cos \theta.$$

Therefore, ...

Notice that if $\theta = 90^\circ$, then $\cos \theta = 0$, and the formula simplifies to the Pythagorean theorem. Therefore, we can say without reservation that in a triangle with two sides of lengths a and b , the length of the third side c is given by the formula

$$c^2 = a^2 + b^2 - 2ab \cos \theta, \tag{3}$$

where θ is the angle between the two known sides. We call this formula in (3) the **Law of Cosines**. To solve for c , we take the positive square root of the righthand side.

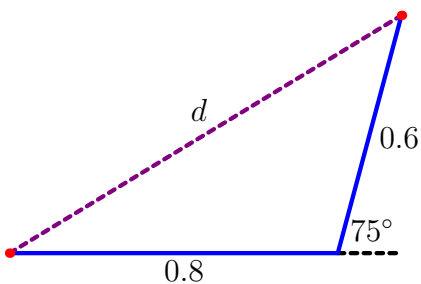


Figure 20

Finding Ann's Distance Traveled

Recall the problem (from page 5).

Example: Ann walks 0.8 miles east, and then turns 75° to the north and walks another 0.6 miles (see Figure 20). How far is she from her starting point, in miles?

The angle between the two sides of known length in this problem is $\theta = 180^\circ - 75^\circ = 105^\circ$. Thus,

$$d^2 = 0.8^2 + 0.6^2 - 2(0.8)(0.6) \cos 105^\circ$$

$$d^2 = 0.64 + 0.36 - 0.96 \cos 105^\circ$$

$$d^2 = 1 - 0.96 \cos 105^\circ$$

$$d = \sqrt{1 - 0.96 \cos 105^\circ}$$

$$d \approx \sqrt{1 - 0.96(-0.2588)} \approx 1.1 \text{ miles.}$$

Here we are depending on our calculator to tell us the cosine of 105° .

Solving for Other Unknowns in the Problem

Once we have this result, we can solve for other unknowns in the problem. For example, (see Figure 21) if Ann had wanted to walk directly to her current location, at what angle above due east would she have set out?

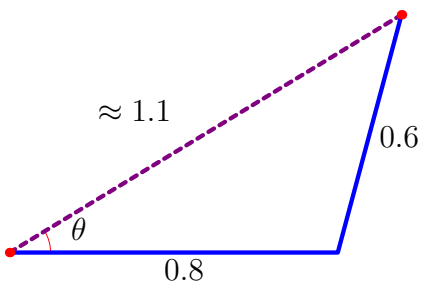


Figure 21

Letting $a = 0.8$, $b \approx 1.1$, and $c = 0.6$, we can solve (approximately) for θ .

$$0.6^2 \approx 0.8^2 + 1.1^2 - 2(0.8)(1.1) \cos \theta$$

$$1.76 \cos \theta \approx 0.64 + 1.21 - 0.36$$

$$1.76 \cos \theta \approx 1.49$$

$$\cos \theta \approx 0.8466$$

The inverse of the cosine function is found on our calculators as well, denoted either “ \cos^{-1} ” or “ \arccos .” Then

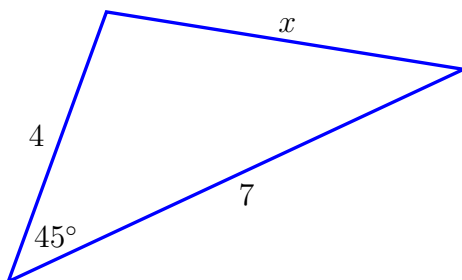
$$\theta \approx \cos^{-1} 0.8466 \approx 32.2^\circ.$$

I am a little concerned about using our approximation from the first problem to solve this one. It would be better (and possibly more accurate) to solve the problem directly.

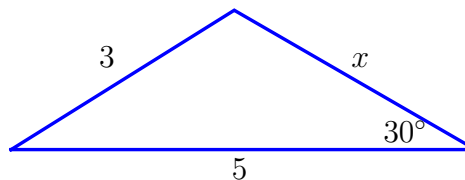
1.3 Exercises

Solve for the missing value in each diagram, exact answer first, then an approximation accurate to two decimal places.

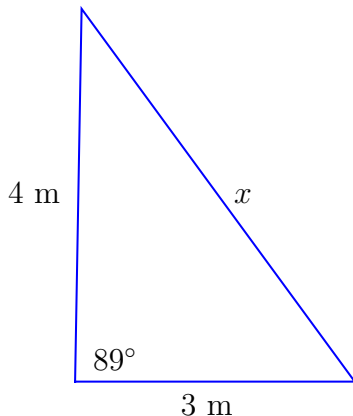
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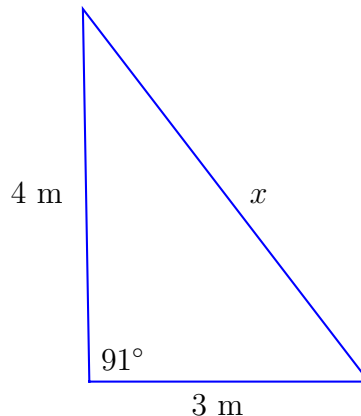
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Give the distance from the starting point after traveling the following distances at the given angles.

- | | |
|---|---|
| 5) 15 steps, turning 120° to the left, and then 15 steps | 6) 60 m, turning 45° to the right, and then 100 m |
| 7) 3 miles, turn 135° to the left, and then 1 mile | 8) 8 m, turn 110° to the left, then 2 m, turn 110° to the right, and then 8 m |

9) Suppose that you want to build a decorative wooden lamppost for your parent's backyard, based on the design shown below in Figure 22. Calculate the minimum lengths of the wooden beams needed to cut the two pieces (yellow and orange) put in at angles for support.

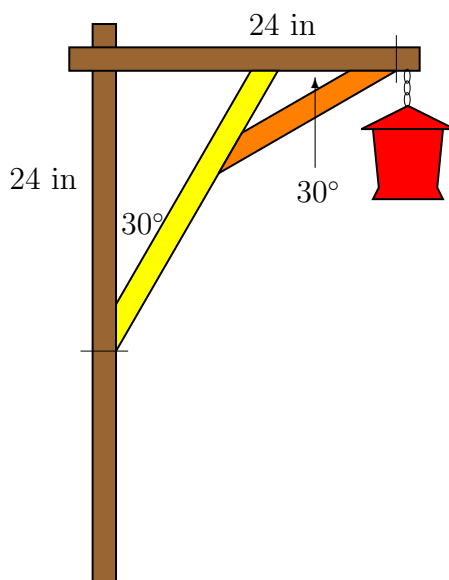


Figure 22

10) A sonar operator receives a “ping” from a “friendly” 700 yards away. After rotating 10° , they receive another “ping”, a suspected “hostile”, at a range of 1050 yards. How far apart are the “friendly” and the “hostile”?

11) A surveyor wants to measure the distance between points A and B , but there is a major geological structure between the two points. (Figure 23) The surveyor walks to a point C around the structure where both points A and B are in sight, and measures the range to each point, 559 yards and 774 yards, respectively. Given that the angle between the points at point C is 126° , what is the distance between A and B ?

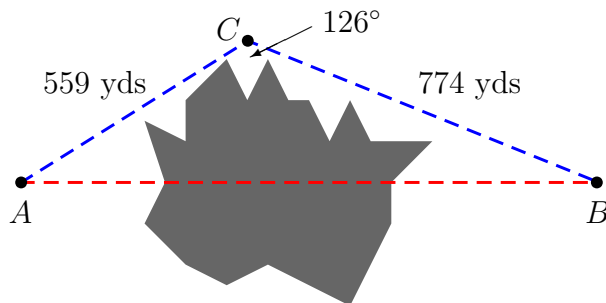


Figure 23

12) Find the measure of the least angle in a triangle of side lengths n , $n + 1$, and $n + 2$, for a general positive integer n . (Recall from your geometry course that the smallest angle in a triangle is opposite the smallest side.) Then, using your formula, find the measure for $n = 1$ and $n = 1,000,000$.

13) In a general triangle with side lengths n , $n + 1$, and $n + 2$, where n is an whole number, use the Law of Cosines to determine for which values of n the triangle is an obtuse triangle, and for what values of n is the triangle an acute triangle.

14) In a general isosceles triangle with side lengths n , n , and $n + 1$, where n is an whole number, use the Law of Cosines to determine for which values of n the triangle is an obtuse triangle, and for what values of n is the triangle an acute triangle?

1.4 Other Relationships with Sines and Cosines

This is just an observation. Suppose that two angles of triangle have measures A and B , and the sides opposite the two angles have lengths a and b , respectively (see Figure 24.) Let

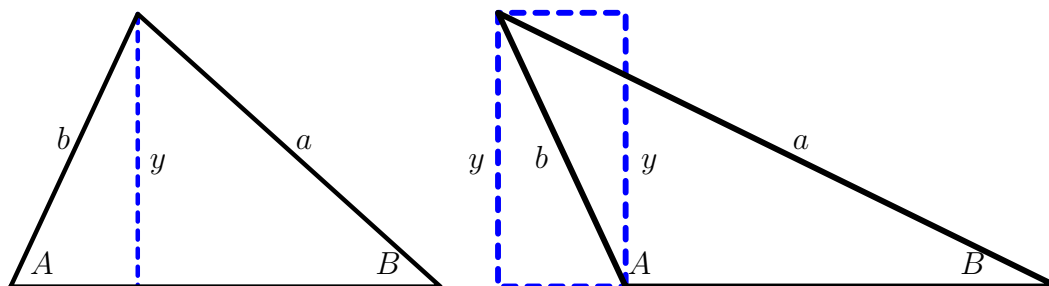


Figure 24

y be the length of the altitude drawn from the third angle. Then I have two different ways of calculating the length y :

$$y = b \sin A \quad \text{and} \quad y = a \sin B.$$

Then

$$b \sin A = a \sin B$$

and

$$\frac{\sin A}{a} = \frac{\sin B}{b}.$$

This relationship still works even if one of the angles is obtuse (see Figure 24 again):

$$y = b \cos(A - 90^\circ) = b \cos(90^\circ - A) = b \sin A \quad \text{and} \quad y = a \sin B,$$

so the relationship still holds. The formula

$$\frac{\sin A}{a} = \frac{\sin B}{b}. \tag{4}$$

where A and B are angles of a triangle and a and b are the lengths of their opposite sides, respectively, is called the *Law of Sines*.

Back to the Problem

How does that help us in finding the angle of her direct route? Recall all of the known information (shown in Figure 25.) Let θ be the measure of the angle that we wish to find, and since the sum of the measures of θ and the angle at the end of her route is 75° , then we let $75^\circ - \theta$ be the measure of that angle. Then, by the Law of Sines, we have

$$\frac{\sin \theta}{0.6} = \frac{\sin(75^\circ - \theta)}{0.8}$$

$$\frac{4}{3} \sin \theta = \sin(75^\circ - \theta).$$

The question is: what do we do with $\sin(75^\circ - \theta)$?

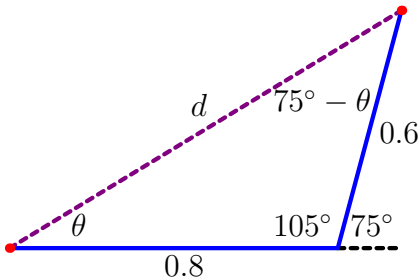


Figure 25

Sum and Difference Formulas

Let's approach this as generally as possible. Consider the measurements as shown in Figure 26. Then the following are certainly true for x and y :

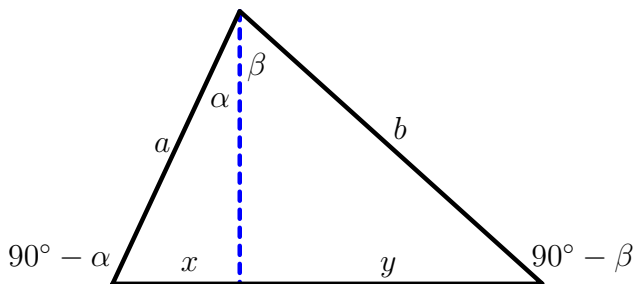


Figure 26

$$x = a \sin \alpha \quad \text{and} \quad y = b \sin \beta. \quad (5)$$

Also, by the Law of Sines, we have that

$$\frac{\sin(90^\circ - \alpha)}{b} = \frac{\sin(90^\circ - \beta)}{a}.$$

Then because of the cofunction properties of sines and cosines (equation (2)),

$$\frac{\cos \alpha}{b} = \frac{\cos \beta}{a}.$$

Solving for a , we get

$$a = \frac{b \cos \beta}{\cos \alpha}. \quad (6)$$

Now let's consider the sine of $\alpha + \beta$. Once again by the Law of Sines, we have

$$\frac{\sin(\alpha + \beta)}{x + y} = \frac{\sin(90^\circ - \alpha)}{b}.$$

Substituting x and y from equation (5) and solving for $\sin(\alpha + \beta)$, we get

$$\sin(\alpha + \beta) = \frac{\cos \alpha}{b} (a \sin \alpha + b \sin \beta).$$

Using the distributive law and substituting for a from equation (6), we get

$$\sin(\alpha + \beta) = \frac{\cos \alpha \sin \alpha}{b} \left(\frac{b \cos \beta}{\cos \alpha} \right) + \frac{\cos \alpha}{b} (b \sin \beta),$$

and finally

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta. \quad (7)$$

The difference is no problem, since

$$\sin(\alpha - \beta) = \sin(\alpha + (-\beta)).$$

Using equation (7) above, we get

$$\sin(\alpha - \beta) = \sin \alpha \cos(-\beta) + \cos \alpha \sin(-\beta).$$

Then, using the fact that $\sin(-\theta) = -\sin \theta$ and $\cos(-\theta) = \cos \theta$, we have the identity

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta. \quad (8)$$

While we are at it, we might as well go ahead and get the sum and difference formulas for cosine. Using the cofunction properties of the cosine, we know that

$$\cos(\alpha + \beta) = \sin(90^\circ - (\alpha + \beta)),$$

and so

$$\cos(\alpha + \beta) = \sin((90^\circ - \alpha) - \beta).$$

Then using the identity from equation (8) above, we have

$$\cos(\alpha + \beta) = \sin(90^\circ - \alpha) \cos \beta - \cos(90^\circ - \alpha) \sin \beta.$$

Again, using the cofunction properties, we have the identity

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta. \quad (9)$$

And since $\cos(\alpha - \beta) = \cos(\alpha + (-\beta))$, we can use equation (9) to get

$$\cos(\alpha - \beta) = \cos \alpha \cos(-\beta) - \sin \alpha \sin(-\beta)$$

and the identity

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta. \quad (10)$$

Back to our Problem

We had reduced to the problem to solving the trigonometric equation

$$\frac{4}{3} \sin \theta = \sin(75^\circ - \theta).$$

Using the difference formula for sines in equation (8), we have

$$\frac{4}{3} \sin \theta = \sin 75^\circ \cos \theta - \cos 75^\circ \sin \theta.$$

Then, by combining the $\sin \theta$ terms, we get

$$\left(\frac{4}{3} + \cos 75^\circ \right) \sin \theta = \sin 75^\circ \cos \theta$$

and

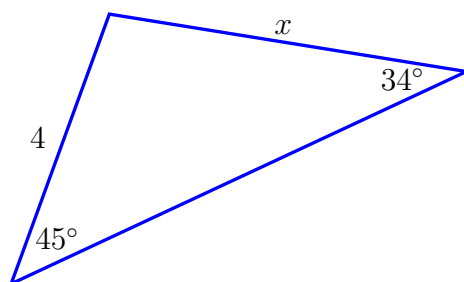
$$\frac{4 + 3 \cos 75^\circ}{3} \sin \theta = \sin 75^\circ \cos \theta.$$

The big question is: what do we do now to solve for θ ?

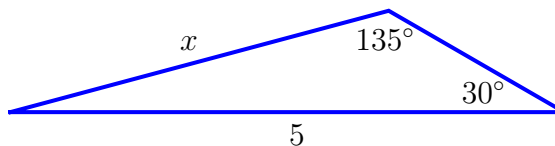
1.4 Exercises

Solve for the missing value in each diagram, exact answer first, then an approximation accurate to two decimal places.

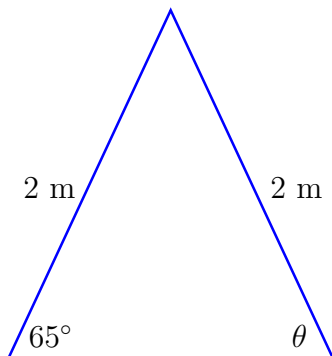
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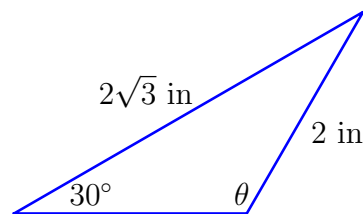
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4)



Express each of the following exactly in terms of $\sin \theta$ and $\cos \theta$, using the sum or difference formulas.

5) $\sin(\theta + 60^\circ)$

6) $\cos(\theta + 45^\circ)$

7) $\cos(\theta - 150^\circ)$

8) $\sin(\theta - 30^\circ)$

Find the exact values of the sine and cosine of the following angles, using the sum or difference formulas.

9) 15°

10) 75°

11) 105°

12) 165°

13) A 15-foot tall telephone pole is set on a hillside. The afternoon sun causes the pole to cast a shadow down the hill that is 18 feet long. We determine using a protractor and a paper clip on a string that the angle from horizontal from the end of the shadow up to the top of the telephone pole is 50° . (Figure 27) What is the angle of elevation of the hill?

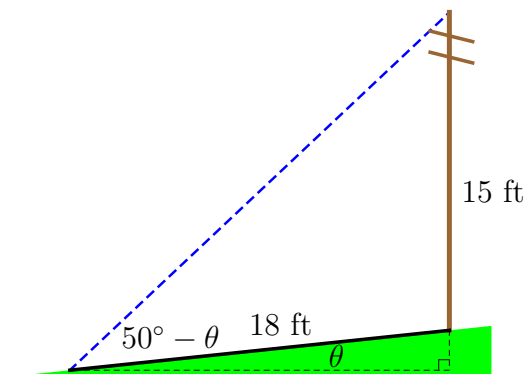


Figure 27

14) Suppose that a right triangle has legs of length a and b , and the acute angle θ is formed between the leg of length a and the hypotenuse. (Figure 28) Use the Law of Sines to develop a formula for the ratio $\frac{b}{a}$ in terms of the angle θ . Simplify as much as possible.

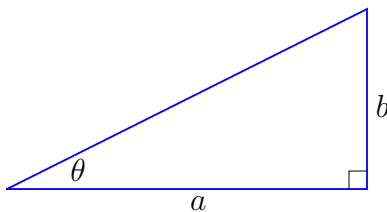


Figure 28

15) Suppose that we are trying to find the distance between two points A and B , as shown in Figure 29, but there is a small hill between the points. The peak of Mt. Kessler is visible from both point A and B , with angles of elevation of 38° and 58° , respectively. If we know that the height of Mt. Kessler is 3000 ft, then how far apart are points A and B ?

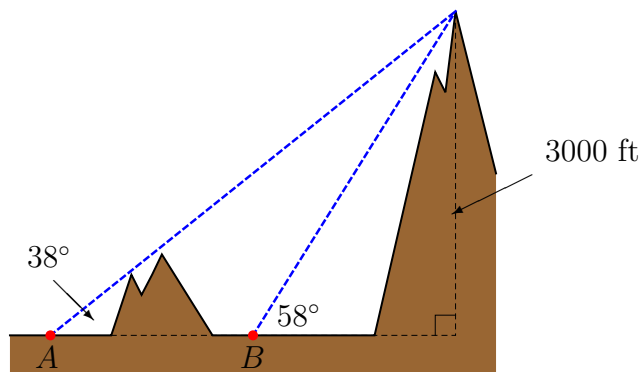


Figure 29

16) Suppose that a triangle has two sides of length a and b , and the angle between these two sides has measure θ , as in Figure 30. First assuming that θ is an acute angle as on the left, find a formula for the area of the triangle. Then show that the formula still holds if θ is an obtuse angle as on the right.

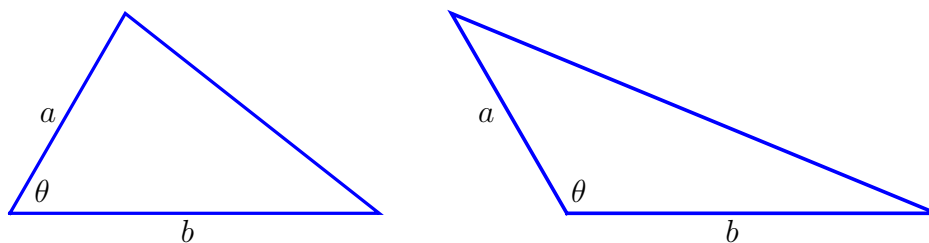


Figure 30

1.5 The Tangent of Angle θ

The tangent of an angle $0 < \theta < 90^\circ$, denoted $\tan \theta$, can be defined as the ratio of the length of the leg of a right triangle opposite the angle to the length of the leg of the right triangle adjacent to the angle (see Figure 7.) Loosely, we say that

$$\tan \theta = \frac{\text{opp}}{\text{adj}} \text{ for } 0 < \theta < 90^\circ.$$

Then, using the reference triangles in Figure 8, we see that $\tan 45^\circ = 1$, $\tan 30^\circ = \frac{1}{\sqrt{3}}$, and $\tan 60^\circ = \sqrt{3}$.

We can extend this concept to any angle θ in a similar fashion used for defining the sine and cosine of any angle. If, as in Figure 31, we let (x, y) be a point on the ray but not the origin, then we can think about the tangent of an angle θ as the ratio of y to x ; that is

$$\tan \theta = \frac{y}{x},$$

which is exactly the slope of the ray. This means that $\tan 0^\circ = \tan 180^\circ = 0$, and both $\tan 90^\circ$

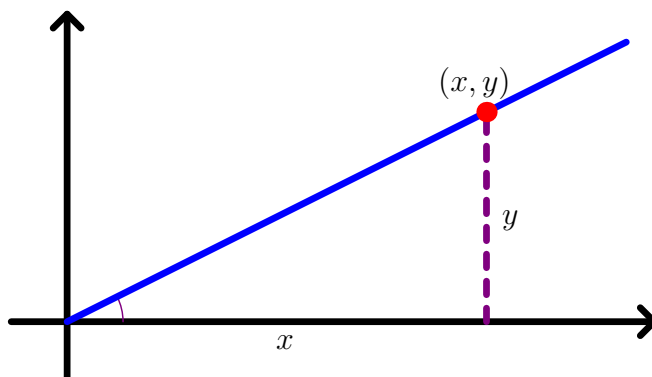


Figure 31

and $\tan 270^\circ$ are undefined. As before, we can use our reference triangles to calculate the cosine of certain angles. See Figure 32 for the diagram showing the calculation of $\tan 135^\circ$ and $\tan(-60^\circ)$.

Notice that, because of this definition of the tangent of θ , that $\tan(-\theta) = -\tan \theta$.

How is tangent related to sine and cosine?

Consider the point (x, y) on the terminal ray of the angle θ , and let $r = \sqrt{x^2 + y^2}$. Then, from above,

$$\sin \theta = \frac{y}{r} \text{ and } \cos \theta = \frac{x}{r}.$$

That means that

$$\frac{\sin \theta}{\cos \theta} = \frac{\frac{y}{r}}{\frac{x}{r}} = \frac{y}{r} \cdot \frac{r}{x} = \frac{y}{x} = \tan \theta$$

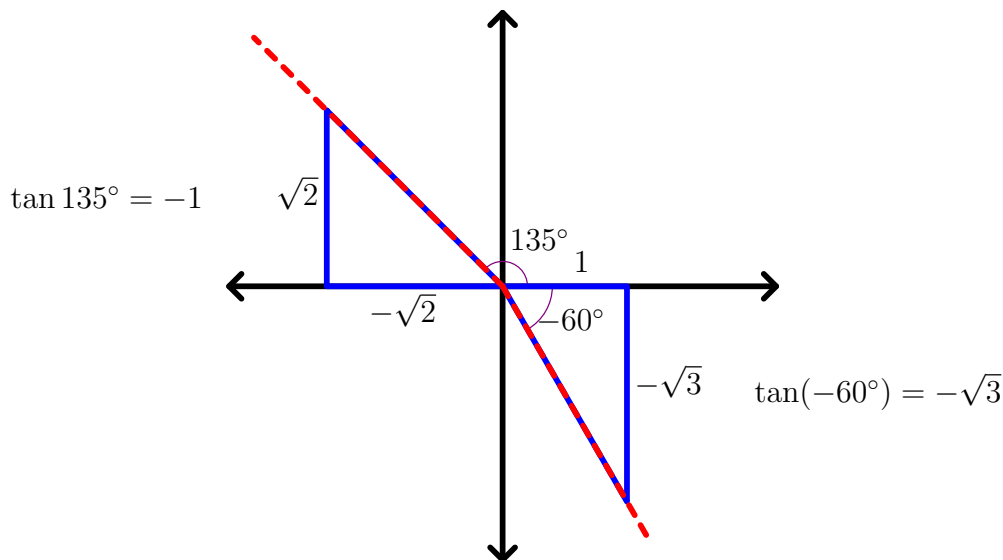


Figure 32

for all values of $\theta \neq 90^\circ + k \cdot 180^\circ$ for integer k (where x would equal 0).

The right triangle diagram shown below in Figure 33 holds the key to another relationship involving the tangent.

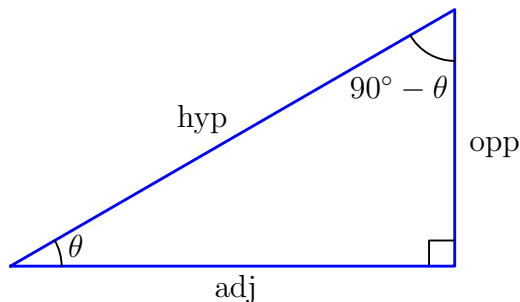


Figure 33

$$\tan(90^\circ - \theta) = \frac{\text{adj}}{\text{opp}} = \frac{1}{\frac{\text{opp}}{\text{adj}}} = \frac{1}{\tan \theta}$$

Since $\frac{1}{\tan \theta}$ is the cofunction of $\tan \theta$ (at least for $0 < \theta < 90^\circ$) and is a bit tedious to write as a fraction, we define a new trigonometric function, the *cotangent* of θ , as

$$\cot \theta = \frac{1}{\tan \theta} \text{ for } 0 < \theta < 90^\circ. \quad (11)$$

It is easy to see that, with this definition, the tangent will also be the cofunction of cotangent:

$$\cot(90^\circ - \theta) = \frac{1}{\tan(90^\circ - \theta)} = \frac{1}{\frac{\text{adj}}{\text{opp}}} = \frac{\text{opp}}{\text{adj}} = \tan \theta. \quad (12)$$

While the above diagram only holds for $0^\circ < \theta < 90^\circ$, the relationship in (11) motivates the definition of cotangent. Consider the diagram in Figure 34, and let (x, y) be a point on the terminal ray of the angle θ , with $y \neq 0$. Then we can think about the cotangent of an angle θ as the ratio of x to y ; that is

$$\cot \theta = \frac{x}{y}.$$

This definition will satisfy the cofunction identities wherever both tangent and cotangent are defined. The terminal ray for the angle $90^\circ - \theta$ is the above ray reflected across the line $y = x$, and so the reflected point will have coordinants (y, x) . If we assume that both $x \neq 0$ and $y \neq 0$, then

$$\cot \theta = \frac{1}{\tan \theta} = \frac{1}{\frac{b}{a}} = \frac{a}{b} = \tan(90^\circ - \theta)$$

and

$$\tan \theta = \frac{b}{a} = \frac{1}{\frac{a}{b}} = \frac{1}{\tan(90^\circ - \theta)} = \cot(90^\circ - \theta)$$

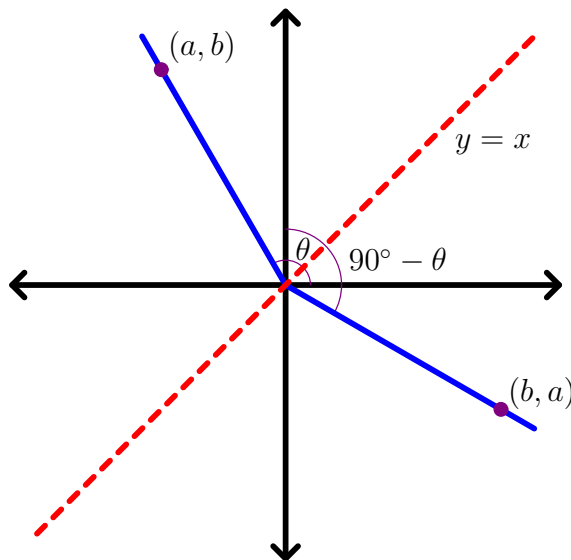


Figure 34

Back to our Problem

Recall the problem of finding the angle of her direct route, shown in Figure 35. We had reduced to the problem to solving the trigonometric equation

$$\frac{4 + 3 \cos 75^\circ}{3} \sin \theta = \sin 75^\circ \cos \theta.$$

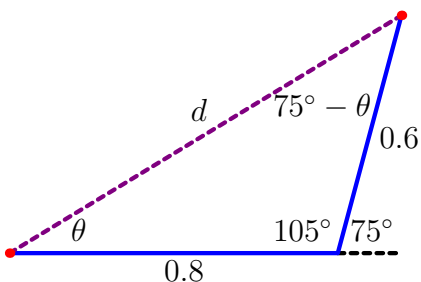


Figure 35

Get both the sine and cosine to the lefthand side of the equation:

$$(4 + 3 \cos 75^\circ) \sin \theta = 3 \sin 75^\circ \cos \theta,$$

$$\frac{\sin \theta}{\cos \theta} = \frac{3 \sin 75^\circ}{4 + 3 \cos 75^\circ}.$$

Then

$$\tan \theta = \frac{3 \sin 75^\circ}{4 + 3 \cos 75^\circ} \approx 0.606679,$$

and so, from the inverse tangent function “ \tan^{-1} ” or “arctan” on our calculator, we have that

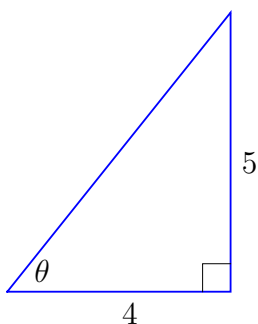
$$\theta \approx \tan^{-1} 0.606679 = 31.2443^\circ.$$

This means that our previous answer of 32.2° found by using an approximate distance from the Law of Cosines was off by roughly 1 degree.

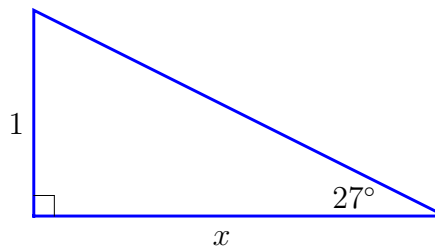
1.5 Exercises

Solve for the missing value(s) in each diagram, exact answer first, then an approximation accurate to two decimal places.

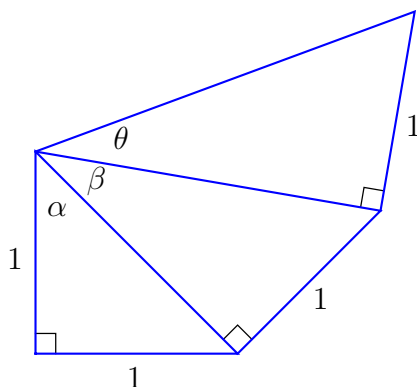
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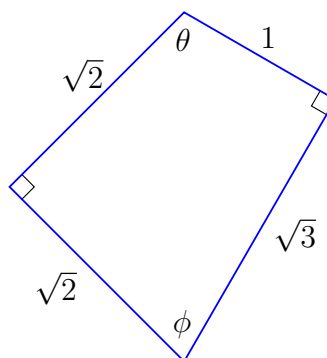
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4)



Solve the following trigonometric equations (exactly if possible) for solutions in the range $(-180^\circ, 180^\circ]$.

5) $\sin(\theta + 45^\circ) + \sin(\theta - 45^\circ) = 1$

6) $\cos(\theta + 60^\circ) - \cos(\theta - 60^\circ) = 1$

7) $\tan(\theta + 180^\circ) - 2\sin(\theta - 180^\circ) = 0$

8) $\sin(\theta + 90^\circ) - \cos(\theta + 270^\circ) = 0$

9) Suppose that you are one city-block (0.1 mile) away from a building with a billboard set up on top. (Figure 36) You determine with a protractor and paper clip on a string that the angle of inclination from your head to the top of the building is 12° , while the angle of inclination to the top of the billboard is 14° . If you are 5 feet, 6 inches tall, how tall is the billboard (in feet)?

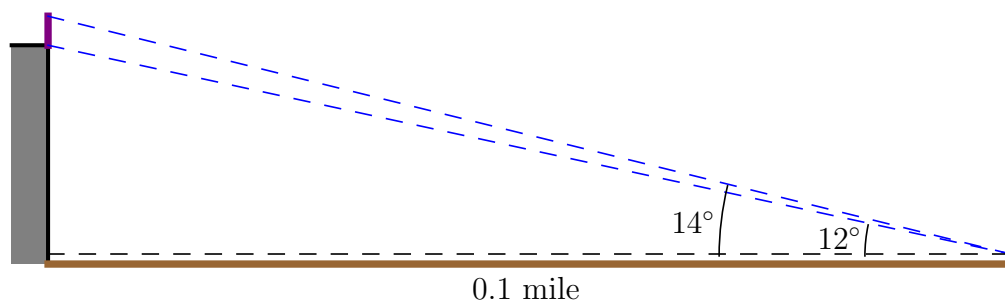


Figure 36

10) The **golden ratio** is the ratio ϕ of the long side of a rectangle to its short side when the rectangle is the union of a square and a similar rectangle with a different orientation, as shown in Figure 37. If a triangle is formed by cutting across the diagonal of this rectangle, what are the measures, accurate to two decimal places, of the two acute angles in the triangle?

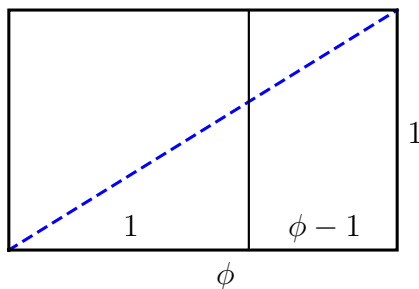


Figure 37

11) Develop a sum-and-difference formula for both $\tan(\alpha + \beta)$ and $\tan(\alpha - \beta)$ in terms of $\tan \alpha$ and $\tan \beta$.

12) Any expression of the form

$$a \sin \theta + b \cos \theta$$

can be rewritten in the form

$$A \sin(\theta + B).$$

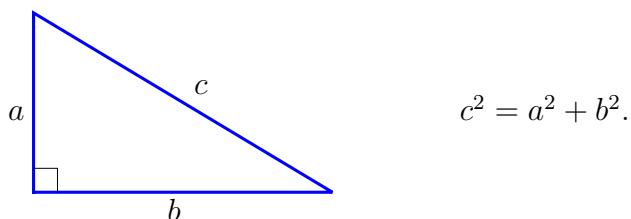
a) Use the sum formula for sine to express $A \sin(\theta + B)$ in terms of $\sin \theta$ and $\cos \theta$. Set this equal to $a \sin \theta + b \cos \theta$ and solve for A and B in terms of a and b .

b) Check your formula by finding the equivalent form for $\sqrt{2} \sin \theta + \sqrt{2} \cos \theta$, and substituting $\theta = 0^\circ$, 30° , and 45° into both expressions.

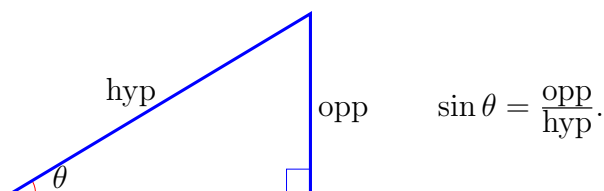
c) Recheck your formula by finding the equivalent form for $\sqrt{3} \sin \theta + \cos \theta$, and substituting $\theta = 0^\circ$, 30° , and 45° into both expressions.

1.6 Review: What have we learned (or relearned)?

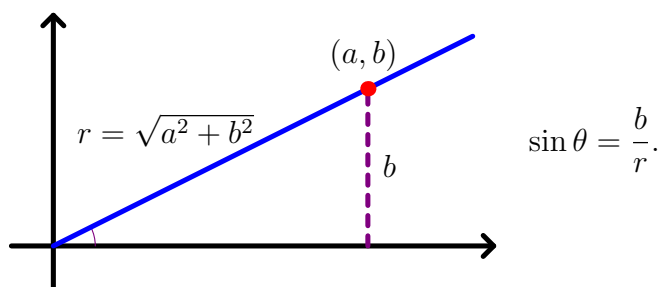
Theorem: Pythagorean Theorem *Let a and b be the lengths of two legs of a right triangle. Then the length of the hypotenuse c is given by*



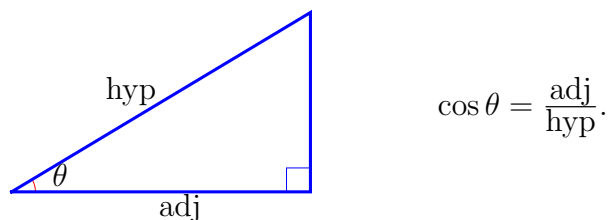
Definition: Sine of an Acute Angle *The sine of an acute angle θ in a right triangle is the ratio of the length of the side opposite θ to the length of the hypotenuse,*



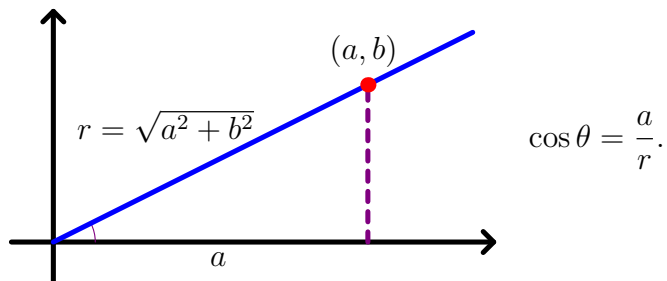
Definition: Sine of Any Angle *Let (a, b) be a point on a ray starting at $(0, 0)$ that forms the angle θ with the positive x -axis, and let $r = \sqrt{a^2 + b^2}$. The sine of an angle θ is the ratio*



Definition: Cosine of an Acute Angle *The cosine of an acute angle θ in a right triangle is the ratio of the length of the side adjacent to θ to the length of the hypotenuse,*



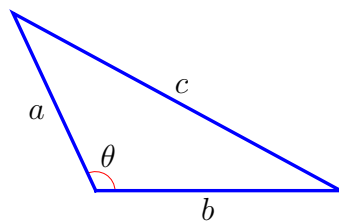
Definition: Cosine of Any Angle Let (a, b) be a point on a ray starting at $(0, 0)$ that forms the angle θ with the positive x -axis, and let $r = \sqrt{a^2 + b^2}$. The cosine of an angle θ is the ratio



Theorem: (A First) Pythagorean Identity

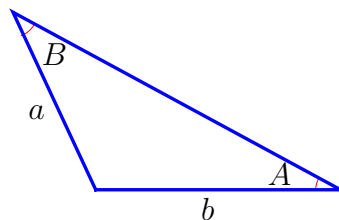
$$\sin^2 \theta + \cos^2 \theta = 1$$

Theorem: Law of Cosines Let a and b be the lengths of two sides of a triangle and let θ be the measure of the angle formed by those two sides. Then the length of the third side c is given by



$$c^2 = a^2 + b^2 - 2ab \cos \theta.$$

Theorem: Law of Sines Let a and b be the lengths of two sides of a triangle and let A and B be the measures of the angles opposite of those two sides, respectively. Then the length of the sides and the angle measures obey the proportion



$$\frac{\sin A}{a} = \frac{\sin B}{b}.$$

Theorem: Sum and Difference Identities

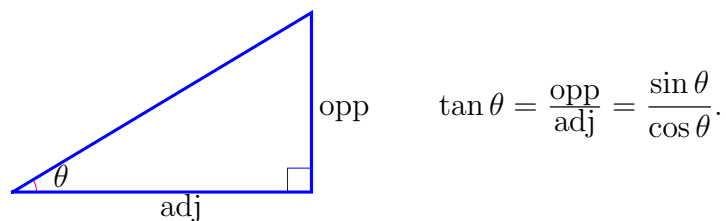
$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

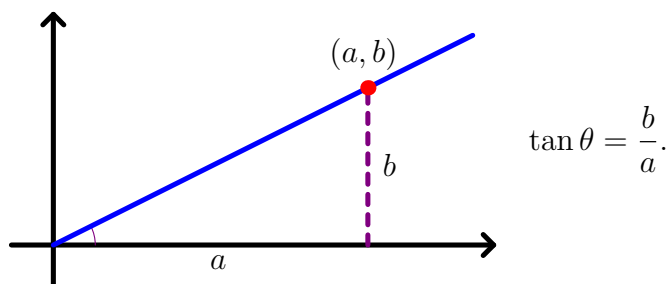
$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

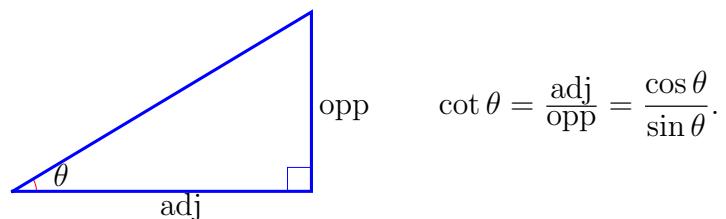
Definition: Tangent of an Acute Angle *The tangent of an acute angle θ in a right triangle is the ratio of the length of the side opposite θ to the length of the side adjacent to the angle,*



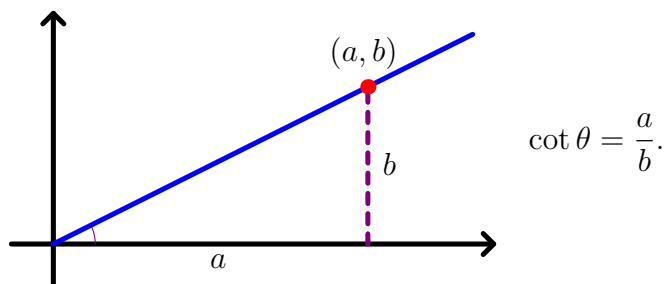
Definition: Tangent of $\theta \neq 90^\circ + 180^\circ k$, k integer *Let (a, b) , $a \neq 0$, be a point on a ray starting at $(0, 0)$ that forms the angle θ with the positive x -axis, and let $r = \sqrt{a^2 + b^2}$. The tangent of an angle θ is the ratio*



Definition: Cotangent of an Acute Angle *The cotangent of an acute angle θ in a right triangle is the ratio of the length of the side adjacent to θ to the length of the side opposite the angle,*



Definition: Cotangent of $\theta \neq 180^\circ k$, k integer Let (a, b) , $b \neq 0$, be a point on a ray starting at $(0, 0)$ that forms the angle θ with the positive x -axis, and let $r = \sqrt{a^2 + b^2}$. The cotangent of an angle θ is the ratio



Theorem: Negative Angle Identities

$$\sin(-\theta) = -\sin \theta \quad \cos(-\theta) = \cos \theta$$

$$\tan(-\theta) = -\tan \theta \quad \cot(-\theta) = -\cot \theta$$

Theorem: Cofunction Identities

$$\sin(90^\circ - \theta) = \cos \theta \quad \cos(90^\circ - \theta) = \sin \theta \quad (\text{for all } \theta)$$

$$\tan(90^\circ - \theta) = \cot \theta \quad \cot(90^\circ - \theta) = \tan \theta \quad (\text{when } \theta \neq 90^\circ k, k \text{ integer})$$

Theorem: Relational Identities

$$\tan \theta = \frac{\sin \theta}{\cos \theta} \quad (\text{when } \theta \neq 90^\circ + 180^\circ k, k \text{ integer})$$

$$\cot \theta = \frac{\cos \theta}{\sin \theta} \quad (\text{when } \theta \neq 180^\circ k, k \text{ integer})$$

$$\tan \theta = \frac{1}{\cot \theta} \quad \cot \theta = \frac{1}{\tan \theta} \quad (\text{when } \theta \neq 90^\circ k, k \text{ integer})$$

Review Exercises

- 1) Suppose that you are measuring a distance along the ground with a tape measure, but you are too lazy to bend over and hold the tape to the ground. Instead, you hold the end of the tape measure at your waist, 3 ft above the ground. (Figure 38) If your measure is 19 ft 3-1/4 inches, what would we expect the actual horizontal measure to be?

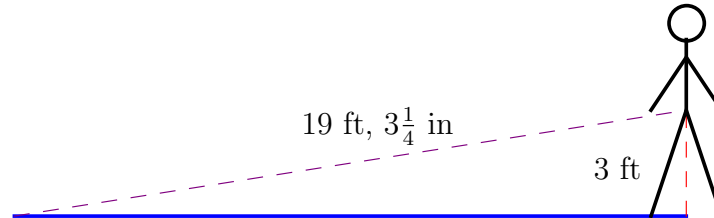


Figure 38

- 2) Suppose that you are trying to measure the distance between two points on the ground, but there is a curb in the way. (Figure 39) The curb is a hexagon, with dimensions shown at the bottom in Figure 39. If the measurements are as shown on the top in Figure 39 and we assume the tape is pulled as tight as possible, then what can we expect the actual distance to be?

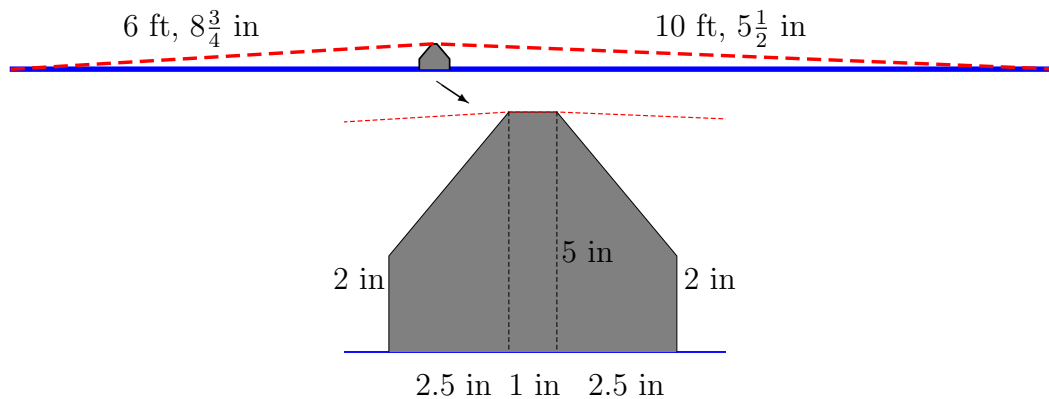


Figure 39

- 3) Give the final location in terms of north/south, east/west of the starting point after traveling 12 mi at 45° north of east, and then 10 mi at 30° north of east.

- 4) Suppose that you are 25 ft away from the base of a tree on level ground, and with a protractor and a paper clip on a string, you are able to determine that the angle from horizontal is 67° . (Figure 40) If you are 6 ft tall and we assume your eyes are 4 inches below the top of your head, then how tall is the tree?

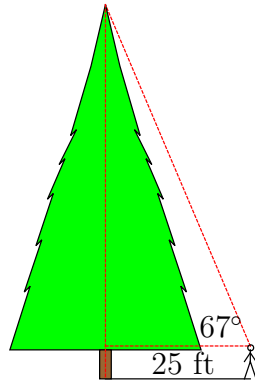


Figure 40

- 5) Give the distance from the starting point after traveling 12 mi, turning to the left 105° , and then traveling another 10 mi.
- 6) A submarine sonar operator picks up an enemy ship at 13° east of north at a range of 1200 m. They also pick up a known friendly ship at 2° west of north at a range of 1000 m. How far apart are the friendly and enemy ships?

7) Suppose that you are 27.5 ft away from the base of a tree on inclined ground, and with a protractor and a paper clip on a string, you are able to determine that the angle from horizontal to the top of the tree is 66° and the angle from horizontal to the base of the tree is -6° . (Figure 41) If you are 5 ft, 6 in tall and we assume your eyes are 4 inches below the top of your head, then

- what is the angle θ of incline (or decline) of the hill, and
- how tall is the tree?

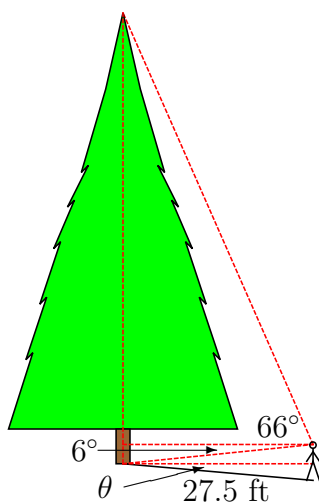
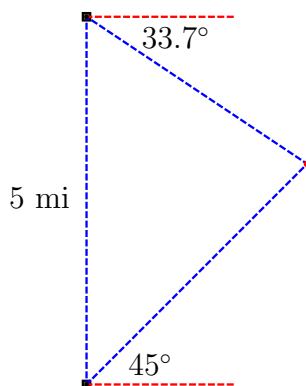


Figure 41

8) Two fire-spotting stations are located 5 miles north and south of each other. The south station spots a fire at 45° north of east. The north station spots the same fire at 33.7° south of east.



- Find the distance of the fire from each station.
- Find the position of the fire in miles north/south and miles east/west of the south station.

- 9) Suppose that you are visiting the observation deck (up at the top) of the St. Louis arch at a time of day when the shadow cast by the arch on the west lawn is exactly half as long as the arch is wide. (The arch has the same height and width.)



- a) Given that the sun rose that morning at 6:00 am and that the noon sun produces a shadow right under the arch, what time of day are you there?
- b) Given the same conditions, except that the shadow length is one-quarter the width of the arch, what time of day are you there?

- 10) Any expression of the form

$$a \sin \theta + b \cos \theta$$

can be rewritten in the form

$$A \cos(\theta - B).$$

- a) Use the difference formula for cosine to express $A \cos(\theta - B)$ in terms of $\sin \theta$ and $\cos \theta$. Set this equal to $a \sin \theta + b \cos \theta$ and solve for A and B in terms of a and b .
- b) Check your formula by finding the equivalent form for $\sqrt{2} \sin \theta + \sqrt{2} \cos \theta$, and substituting $\theta = 0^\circ$, 30° , and 45° into both expressions.
- c) Recheck your formula by finding the equivalent form for $\sqrt{3} \sin \theta + \cos \theta$, and substituting $\theta = 0^\circ$, 30° , and 45° into both expressions.

Chapter 2: Targeting a Projectile Launcher

MATH 117: Trigonometry

2.1 Introduction

Here is the set-up:

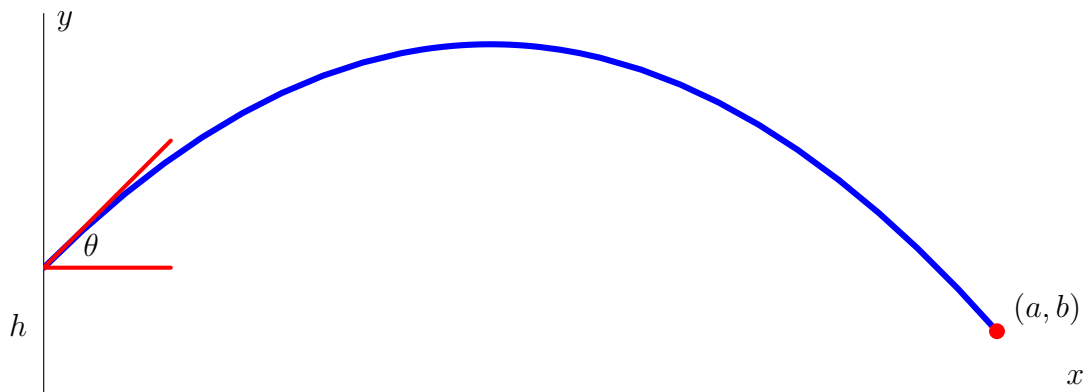


Figure 1

We would like to be able to set a projectile launcher to hit a target at (a, b) . We have the following parameters:

h – height of the launcher (given, in feet)

a – distance of the target from the launcher (given, in feet)

b – height of the target (given, in feet)

v – velocity of the projectile (constant, but not given,) and

θ – angle to the horizontal that the projectile is launched.

We will not be able to adjust the velocity v , but we will have to calculate it. We are going to assume that the projectile is very aerodynamic, so that we can ignore wind resistance. (You can factor in wind resistance if you like, but you will have to take **MATH 331: Differential Equations** before you can solve for the path.)

2.2 Describing the Path of the Projectile

We generally think about curves like this as a function (it looks remarkably parabolic,) with x as the independent variable and y as the dependent variable. However, in this case, both are changing with respect to a variable that we really don't seem to care much about – time. So, at least initially, I recommend that we think about the horizontal and vertical components as separate (but linked) dependent variables, both with the independent variable t , time in seconds after the launch of the projectile. So, ...

t	$x(t)$	$y(t)$
0	0	h
\vdots	\vdots	\vdots
?	a	b

In order to find formulas for both $x(t)$ and $y(t)$, we are going to have to determine how fast the projectile is going both horizontally and vertically. Let v_x be the horizontal velocity of the projectile, and let v_y be the vertical velocity of the dart.

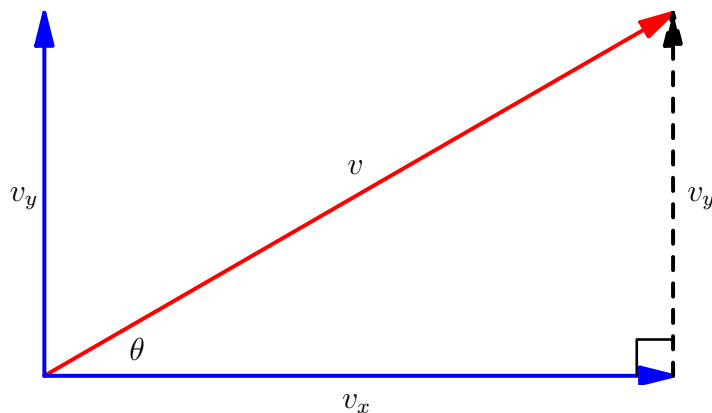


Figure 2

We use our knowledge of right triangle trigonometry to solve for v_x and v_y , we find that:

$$\begin{aligned} \cos \theta &= \frac{\text{adj}}{\text{hyp}} = \frac{v_x}{v} & \sin \theta &= \frac{\text{opp}}{\text{hyp}} = \frac{v_y}{v} \\ v_x &= v \cos \theta & v_y &= v \sin \theta. \end{aligned}$$

The Horizontal Component $x(t)$

For objects moving at a constant rate of speed, the relationship between the rate r (in a unit of length per unit of time,) the distance traveled d (in the same unit of length,) and the time t spent at that rate (in the same unit of time) is simple:

$$d = rt$$

– the dreaded “dirt” problem. In our case, the horizontal distance traveled is $x(t)$ feet, the horizontal rate is v_x feet per second, and the time is t seconds, so

$$x(t) = v_x t = vt \cos \theta. \quad (1)$$

The Vertical Component $y(t)$

The vertical component has the same basic properties, so

$$y(t) = v_y t + \dots = vt \sin \theta + \dots$$

However, there are two major differences. First, the projectile starts at height h feet. The formula for $y(t)$ is easily adjusted –

$$y(t) = vt \sin \theta + h + \dots$$

The second difference is pretty obvious, but not so easy to explain algebraically. Gravity, while it is not pulling the projectile left or right, is pulling it downward. The effect of gravity on the position is described by $-\frac{1}{2}gt^2$ (provable once you have had some integral calculus,) where g is the downward velocity per unit of time due to gravity, which, for feet and seconds, is $g = 32 \text{ ft/s}^2$. Therefore, the formula for the vertical component is

$$y(t) = -16t^2 + vt \sin \theta + h. \quad (2)$$

Can we get rid of the t ?

The presence of the variable t for time is problematic, especially since we are looking at the path of the projectile with x, y - axes. We are used to having y be a function of x . Is there any way to remove the t from our calculations?

We could evoke an old algebra trick: solve for t in one equation and substitute it into the other. Since the y -equation in (2) is quadratic in t and the x -equation in (1) is only linear, it makes sense to solve for t in the x -equation (1):

$$x = vt \cos \theta$$

$$t = \frac{x}{v \cos \theta}.$$

Substituting this into the y -equation (2) gives us

$$\begin{aligned} y &= -16 \left(\frac{x}{v \cos \theta} \right)^2 + v \left(\frac{x}{v \cos \theta} \right) \sin \theta + h \\ y &= \left(-\frac{16}{v^2 \cos^2 \theta} \right) x^2 + (\tan \theta)x + h. \end{aligned} \quad (3)$$

Notice that the equation is still quadratic and the coefficient of the squared term is negative, so the path of the projectile is a parabola that opens downward.

Whoa, Nelly!

We just divided by an unknown like it was no big deal! If $\theta = 90^\circ$, then $\cos 90^\circ = 0$, and so, $x = 0$ no matter what. In that case, we can not solve for t in the x equation and our equation for y in terms of x no longer makes sense. So, we should be more careful, and say that

$$y = \left(-\frac{16}{v^2 \cos^2 \theta} \right) x^2 + (\tan \theta)x + h \quad \text{for} \quad -90^\circ < \theta < 90^\circ.$$

Also, the expression $\frac{1}{\cos \theta}$ is kind of messy. We will clean this up in the next section by defining a new trigonometric function.

2.2 Exercises

Exercises 1) through 7) will require the use of either a graphing calculator in FUNCTION mode or the **Plot** command in *Mathematica*TM.

- 1) Find the equation for the trajectory of a projectile fired at velocity $v = \frac{16\sqrt{30}}{3}$ ft/s at angle $\theta = 30^\circ$ above horizontal from the ground ($h = 0$ ft). Determine how far the projectile will fly horizontally before hitting the ground. Check your answer by graphing your equation.
- 2) Find the equation for the trajectory of a projectile fired at velocity $v = \frac{16\sqrt{30}}{3}$ ft/s at angle $\theta = 60^\circ$ above horizontal from the ground ($h = 0$ ft). Determine how far the projectile will fly horizontally before hitting the ground. Check your answer by graphing your equation. Are there any similarities to your answer from problem 1)?
- 3) Find the equation for the trajectory of a projectile fired at velocity $v = 32$ ft/s at angle $\theta = 45^\circ$ above horizontal from the ground ($h = 0$ ft). Determine how far the projectile will fly horizontally before hitting the ground. Check your answer by graphing your equation.
- 4) Find the equation for the trajectory of a projectile fired at velocity $v = 40$ ft/s horizontally from a height of 4 ft off the ground. Determine how far the projectile will fly horizontally before hitting the ground. Check your answer by graphing your equation.
- 5) Graph the trajectory equation (3) with $\theta = 40^\circ$, $v = 40$ ft/s, and with $h = 0, 5, 10, \dots, 40$. As h increases, what is the effect on the horizontal distance the projectile travels before it hits the ground ($y = 0$)?
- 6) Graph the trajectory equation (3) with $\theta = 30^\circ$, $h = 1$ ft, and with $v = 5, 10, \dots, 50$ ft/s. As v increases, what is the effect on the horizontal distance the projectile travels before it hits the ground?
- 7) Graph the trajectory equation (3) with $h = 3$ ft, $v = 25$ ft/s, and with $\theta = 0^\circ, 5^\circ, \dots, 60^\circ$. As θ increases, what is the effect on the horizontal distance the projectile travels before it hits the ground?

8) Equation (3) is the result of removing the time variable t from the y -equation (2) using equation (1). If we want to consider the amount of time t that the projectile is in the air before it hits the ground, then we need to refer back to the original equation (2).

a) Solve equation (2) for t in terms of y , v , h , and θ .

b) Using the result of part a) above, calculate the amount of time t in seconds that the projectile is in the air when $h = 3$ ft, $v = 25$ ft/s, and with $\theta = 0^\circ, 5^\circ, \dots, 60^\circ$. As θ increases, what is the effect on the time in the air of the projectile before it hits the ground?

9) Suppose that you want to adjust the formula in (3) to allow for the variables x , y , and h to be measured in meters. If the acceleration due to gravity is $g = 9.8$ m/s², re-derive the formula accordingly.

10) Suppose that you wanted to find the initial velocity v using a law enforcement radar gun, in miles-per-hour. Re-derive the formula in (3) so that x , y , and h are measured in miles and t is measured in hours.

2.3 The Secant of Angle θ

The secant of an angle $0 < \theta < 90^\circ$, denoted $\sec \theta$, can be defined as the ratio of the length of the hypotenuse of a right triangle to the length of the leg of the right triangle adjacent to the angle (see Figure 3.) Loosely, we say that

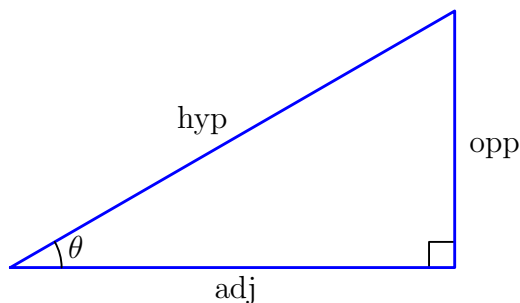


Figure 3

$$\sec \theta = \frac{\text{hyp}}{\text{adj}} \text{ for } 0 < \theta < 90^\circ.$$

Then, using our reference triangles, we see that $\sec 45^\circ = \sqrt{2}$, $\sec 30^\circ = \frac{2}{\sqrt{3}}$, and $\sec 60^\circ = 2$.

We can extend this concept to any angle θ in a similar fashion used for defining the sine and cosine of any angle. If, as in Figure 4, we let (x, y) , $x \neq 0$, be a point on the ray but not the origin, and we let $r = \sqrt{x^2 + y^2}$, then we can think about the secant of an angle θ as the ratio of r to x ; that is

$$\sec \theta = \frac{r}{x}.$$

This means that $\sec 0^\circ = \sec 180^\circ = 1$, and both $\sec 90^\circ$ and $\sec 270^\circ$ are undefined. As before,

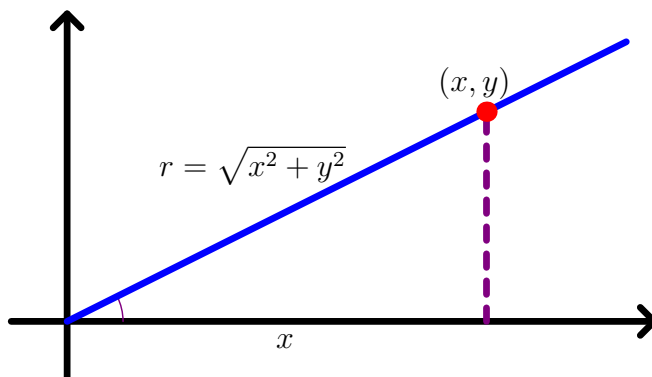


Figure 4

we can use our reference triangles to calculate the secant of certain angles. See Figure 5 for the diagram showing the calculation of $\sec 135^\circ$ and $\sec(-60^\circ)$.

Notice that, because of this definition of the secant of θ , that $\sec(-\theta) = \sec \theta$.

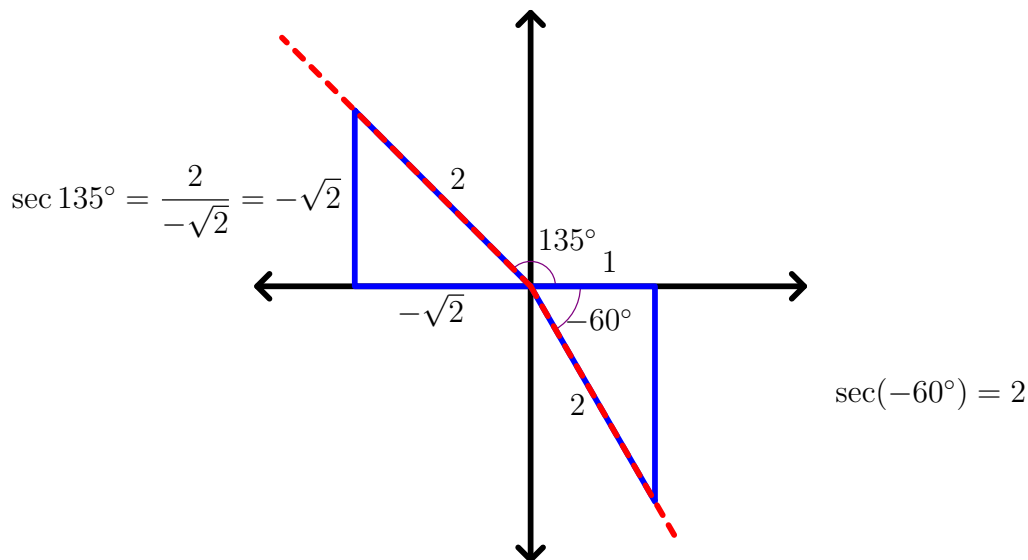


Figure 5

How is secant related to cosine?

Consider the point (x, y) on the terminal ray of the angle θ , and let $r = \sqrt{x^2 + y^2}$, as in Figure 4. Then, since $\cos \theta = \frac{x}{r}$, then

$$\frac{1}{\cos \theta} = \frac{1}{\frac{x}{r}} = \frac{r}{x} = \sec \theta$$

for all values of $\theta \neq 90^\circ + k \cdot 180^\circ$ for integer k (where x would equal 0).

The right triangle diagram shown below in Figure 6 holds the key to another relationship involving the secant.

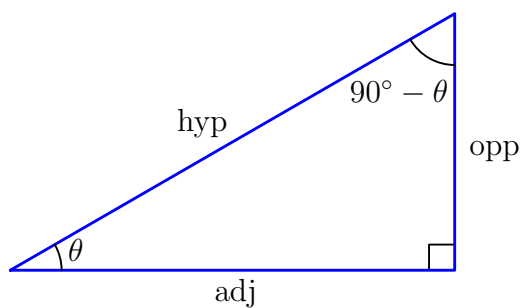


Figure 6

$$\sec(90^\circ - \theta) = \frac{\text{hyp}}{\text{opp}} = \frac{1}{\frac{\text{opp}}{\text{hyp}}} = \frac{1}{\sin \theta}$$

Since $\frac{1}{\sin \theta}$ is the cofunction of $\sec \theta$ (at least for $0 < \theta < 90^\circ$) and is a bit tedious to write as a

fraction, we define a new trigonometric function, the *cosecant* of θ , as

$$\csc \theta = \frac{1}{\sin \theta} \text{ for } 0 < \theta < 90^\circ. \quad (4)$$

It is easy to see that, with this definition, the secant will also be the cofunction of the cosecant:

$$\csc(90^\circ - \theta) = \frac{1}{\sin(90^\circ - \theta)} = \frac{1}{\frac{\text{adj}}{\text{hyp}}} = \frac{\text{hyp}}{\text{adj}} = \sec \theta. \quad (5)$$

While the above diagram only holds for $0^\circ < \theta < 90^\circ$, the relationship in (4) motivates the definition of cosecant. Consider the diagram in Figure 7, and let (a, b) be a point on the terminal ray of the angle θ , with $b \neq 0$, and let $r = \sqrt{a^2 + b^2}$. Then we can think about the cosecant of an angle θ as the ratio of r to b ; that is

$$\csc \theta = \frac{r}{b}.$$

This definition will satisfy the cofunction identities wherever both sine and cosine are defined. The terminal ray for the angle $90^\circ - \theta$ is the above ray reflected across the line $y = x$, and so the reflected point will have coordinants (b, a) . If we assume that both $b \neq 0$ and $a \neq 0$, then

$$\csc \theta = \frac{r}{b} = \sec(90^\circ - \theta)$$

and

$$\sec \theta = \frac{r}{a} = \csc(90^\circ - \theta)$$

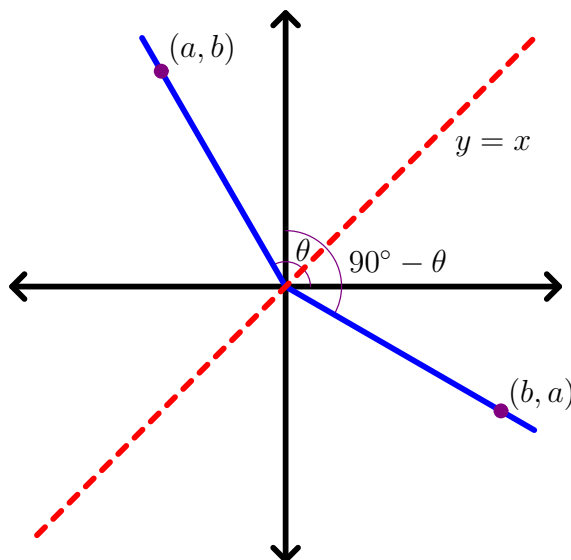


Figure 7

Back to our problem ...

When we last considered our projectile launcher problem in equation (3), we had removed the time variable t from the two equations to get the one position equation

$$y = \left(-\frac{16}{v^2 \cos^2 \theta} \right) x^2 + (\tan \theta)x + h \quad \text{for} \quad -90^\circ < \theta < 90^\circ.$$

Using our new trigonometric function, we can now state this cleaner:

$$y = \left(-\frac{16}{v^2} \sec^2 \theta \right) x^2 + (\tan \theta)x + h \quad \text{for} \quad -90^\circ < \theta < 90^\circ. \quad (6)$$

2.3 Exercises

For each of the following angles θ , draw the necessary reference triangles and find $\sin \theta$, $\cos \theta$, $\tan \theta$, $\cot \theta$, $\sec \theta$, and $\csc \theta$ exactly, if they exist.

- | | | | |
|-------------------------|--------------------------|--------------------------|---------------------------|
| 1) $\theta = 45^\circ$ | 2) $\theta = 60^\circ$ | 3) $\theta = -30^\circ$ | 4) $\theta = 150^\circ$ |
| 5) $\theta = 90^\circ$ | 6) $\theta = -90^\circ$ | 7) $\theta = 180^\circ$ | 8) $\theta = -45^\circ$ |
| 9) $\theta = 135^\circ$ | 10) $\theta = 225^\circ$ | 11) $\theta = 240^\circ$ | 12) $\theta = -120^\circ$ |

2.4 Finding the Velocity v of the Projectile Launcher

I'm assuming here that we do not have a radar gun (most classrooms do not.) That would make it too easy!

Do We Have To?

We could just fire our projectile launcher several times at different angles and fit the plotted data with a curve. Let's try this approach. With my projectile launcher, I fire the projectile at angles from horizontal ranging from 0° to 85° , increasing the angle in 5° increments. Each time, I keep the initial height of the launcher barrel constant, at 1 ft, and I measure the distance from the launcher ($x = 0$) to the place where the projectile lands on the ground. Here is some sample data (taken from the Virtual Launcher in **Virtual Launcher.nb**):

Angle ($^\circ$)	Distance (ft)	Angle ($^\circ$)	Distance (ft)	Angle ($^\circ$)	Distance (ft)
0	7.5	30	26.7	60	24.9
5	10.4	35	27.2	65	21.4
10	13.4	40	28.8	70	18.4
15	17.0	45	29.4	75	14.3
20	20.3	50	28.8	80	9.7
25	23.6	55	26.5	85	5.1

We could also include the point $(90, 0)$, since if the projectile is launched straight up, it would not travel horizontally. The data points are graphed below.

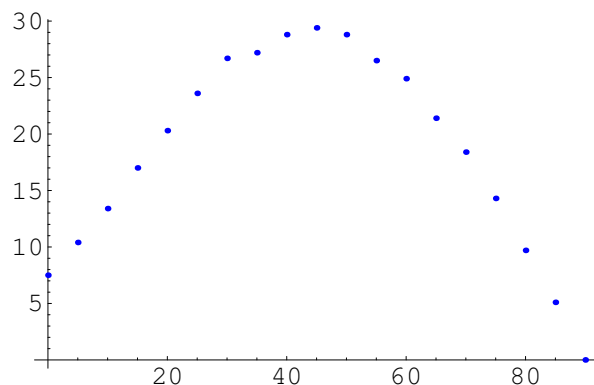


Figure 8

The question is: How do we model this data? Is it quadratic? It looks a little bit like a parabola, but if we use our calculators to get a least-squares quadratic fit to the data, it does not seem to match very well, particularly on the left part of the graph (see Figure 9.)

There are other problems, too. What if we want to change the initial firing height h ? What if we want to hit a target that is not on the ground? In those cases, this data is completely useless to us. To be completely general, we *are* going to have to solve for v , and that will require some algebra and trigonometry.

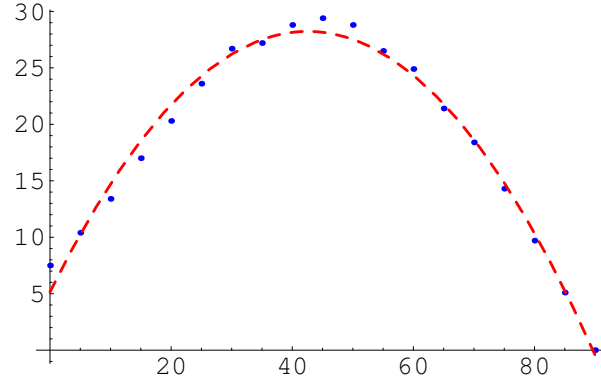


Figure 9

Solving Algebraically

Let's set the angle and take a few shots to determine the point that the launcher will hit at that angle. (In a perfect world, one shot would be enough, but since our launcher is not perfect, we will take a few shots and average the results.) Launching four projectiles (with the Virtual Launcher) with θ set to 30° and the firing height h set to 1, we get the distances 25.8 ft, 25.9 ft, 25.7 ft, and 25.8 ft (just a sample, your answers may vary.) The mean of these values is

$$\bar{x} = \frac{25.8 + 25.9 + 25.7 + 25.8}{4} = 25.8 \text{ ft,}$$

so we will assume that the trajectory has the point $(25.8, 0)$ on it.

Now, let's fill in the parts of our model from equation (6),

$$y = \left(-\frac{16}{v^2} \sec^2 \theta \right) x^2 + (\tan \theta)x + h.$$

$$\left. \begin{array}{l} h = 1 \\ \theta = 30^\circ \\ x = 25.8 \\ y = 0 \end{array} \right\} \implies 0 = \left(-\frac{16}{v^2} \sec^2 30^\circ \right) (25.8)^2 + (\tan 30^\circ)(25.8) + 1$$

Note that $\sin 30^\circ = \frac{1}{2}$ and $\cos 30^\circ = \frac{\sqrt{3}}{2}$, so $\sec 30^\circ = \frac{2}{\sqrt{3}}$ and $\tan 30^\circ = \frac{1}{\sqrt{3}}$. Then our equation becomes

$$0 = \left(-\frac{16}{v^2} \left(\frac{2}{\sqrt{3}} \right)^2 \right) (665.64) + \left(\frac{1}{\sqrt{3}} \right) (25.8) + 1, \text{ and}$$

$$\frac{42600.96}{3v^2} = \frac{25.8 + \sqrt{3}}{\sqrt{3}}.$$

Solving for v^2 first, we get

$$v^2 = \frac{42600.96}{25.8\sqrt{3} + 3}.$$

Then, taking the square root of both sides, we get

$$v = \sqrt{\frac{42600.96}{25.8 + \sqrt{3}}} \approx 29.9 \frac{\text{ft}}{\text{s}}.$$

We will use this value for v in all of our subsequent calculations with this example in the next section.

2.4 Exercises

All of the homework problems from this section are generated using the *Mathematica*TM file **Virtual Launcher.nb**. Enter the first non-text cell to initialize the value of the velocity variable. Enter the next non-text cell for a given angle of launch θ and a given height of launch h . Once you have finished your calculation of the velocity v , you may check your answer by entering the last non-text cell. There might be some slight variation in the two answers since the Virtual Launcher includes some built-in variation of the launch velocity. Set θ and h to the following values:

- | | |
|--------------------------------------|--|
| 1) $\theta = 30^\circ, h = 1$ foot | 2) $\theta = 60^\circ, h = 1$ foot |
| 3) $\theta = 45^\circ, h = 3$ feet | 4) $\theta = 0^\circ, h = 10$ feet |
| 5) $\theta = 20^\circ, h = 5$ feet | 6) $\theta = -5^\circ, h = 12$ feet |
| 7) $\theta = 10^\circ, h = 6.2$ feet | 8) $\theta = 25^\circ, h = 3.55$ feet |
| 9) $\theta = -30^\circ, h = 50$ feet | 10) $\theta = -45^\circ, h = 100$ feet |
| 11) $\theta = 85^\circ, h = 4$ feet | 12) $\theta = -89^\circ, h = 200$ feet |

2.5 Setting the Angle θ to Hit a Target

Let's continue the example that we started in the last section. We had found the velocity of the projectile launcher to be $v \approx 29.9\text{ft/sec}$. Substituting this into equation (6), we get

$$y = \left(-\frac{16}{894.01} \sec^2 \theta\right) x^2 + (\tan \theta)x + h \quad \text{for } 0^\circ \leq \theta < 90^\circ.$$

Suppose that we want to hit a target 16 ft away on the ground with the launcher 1 ft off of the ground. At what angle do we need to set the launcher?

We solve for θ by substituting our known values into our model in equation (6):

$$\left. \begin{array}{l} h = 1 \\ x = 16 \\ y = 0 \end{array} \right\} \implies 0 = \left(-\frac{16}{894.01} \sec^2 \theta\right) (16)^2 + (\tan \theta)(16) + 1$$

$$-\frac{4096}{894.01} \sec^2 \theta + 16 \tan \theta + 1 = 0.$$

Uh, oh. Now what?

Pythagorean Identities

Recall our definitions of sines and cosines using a right triangle and refer to Figure 10. We know

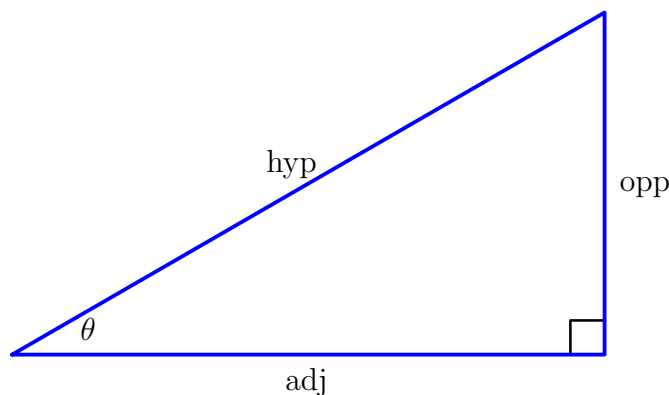


Figure 10

that

$$\begin{aligned} \sin \theta &= \frac{\text{opp}}{\text{hyp}}, & \cos \theta &= \frac{\text{adj}}{\text{hyp}}, \\ \tan \theta &= \frac{\sin \theta}{\cos \theta} = \frac{\text{opp}}{\text{adj}}, & \cot \theta &= \frac{\cos \theta}{\sin \theta} = \frac{\text{adj}}{\text{opp}}, \\ \sec \theta &= \frac{1}{\cos \theta} = \frac{\text{hyp}}{\text{adj}}, & \text{and } \csc \theta &= \frac{1}{\sin \theta} = \frac{\text{hyp}}{\text{opp}}. \end{aligned}$$

We also know from elementary geometry that since we are working with a right triangle,

$$\text{opp}^2 + \text{adj}^2 = \text{hyp}^2,$$

called the *Pythagorean theorem*.

If we divide each term by hyp^2 , then we have

$$\begin{aligned}\frac{\text{opp}^2}{\text{hyp}^2} + \frac{\text{adj}^2}{\text{hyp}^2} &= \frac{\text{hyp}^2}{\text{hyp}^2}, \\ \left(\frac{\text{opp}}{\text{hyp}}\right)^2 + \left(\frac{\text{adj}}{\text{hyp}}\right)^2 &= 1, \text{ and} \\ \sin^2 \theta + \cos^2 \theta &= 1.\end{aligned}$$

If we divide each term by adj^2 , then we have

$$\begin{aligned}\frac{\text{opp}^2}{\text{adj}^2} + \frac{\text{adj}^2}{\text{adj}^2} &= \frac{\text{hyp}^2}{\text{adj}^2}, \\ \left(\frac{\text{opp}}{\text{adj}}\right)^2 + 1 &= \left(\frac{\text{hyp}}{\text{adj}}\right)^2, \text{ and} \\ \tan^2 \theta + 1 &= \sec^2 \theta.\end{aligned}$$

If we divide each term by opp^2 , then we have

$$\begin{aligned}\frac{\text{opp}^2}{\text{opp}^2} + \frac{\text{adj}^2}{\text{opp}^2} &= \frac{\text{hyp}^2}{\text{opp}^2}, \\ 1 + \left(\frac{\text{adj}}{\text{opp}}\right)^2 &= \left(\frac{\text{hyp}}{\text{opp}}\right)^2, \text{ and} \\ 1 + \cot^2 \theta &= \csc^2 \theta.\end{aligned}$$

These three formulas hold for all values of θ , and are called the *Pythagorean Identities*. They may be of use to us in dealing with the $\sec^2 \theta$ that we encountered when solving for θ above.

And now, back to our equation ...

When we last saw our equation, we had it down to

$$-\frac{4096}{894.01} \sec^2 \theta + 16 \tan \theta + 1 = 0.$$

The difficulty, of course, is that we have apples and oranges in the equation, or, in this case, secants and tangents. However, using the Pythagorean identities, we can replace $\sec^2 \theta$ with $\tan^2 \theta + 1$:

$$\begin{aligned}-\frac{4096}{894.01} (\tan^2 \theta + 1) + 16 \tan \theta + 1 &= 0, \\ -\frac{4096}{894.01} \tan^2 \theta + 16 \tan \theta + \left(1 - \frac{4096}{894.01}\right) &= 0, \\ -\frac{4096}{894.01} \tan^2 \theta + 16 \tan \theta - \frac{3201.99}{894.01} &= 0, \text{ and} \\ -\frac{4096}{894.01} (\tan \theta)^2 + 16(\tan \theta) - \frac{3201.99}{894.01} &= 0\end{aligned}$$

This may not seem better than what we started with, but it is. Any equation of the form $ax^2 + bx + c = 0$ can be solved using the *quadratic formula*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

We can actually solve for $\tan \theta$ by using the quadratic formula:

$$\begin{aligned}\tan \theta &= \frac{-16 \pm \sqrt{(16)^2 - 4 \left(-\frac{4096}{894.01} \right) \left(-\frac{3201.99}{894.01} \right)}}{2 \left(-\frac{4096}{894.01} \right)}, \\ \tan \theta &= \frac{16 \mp \frac{\sqrt{(16)^2(894.01)^2 - 4(4096)(3201.99)}}{894.01}}{\frac{8192}{894.01}}, \\ \tan \theta &= \frac{16(894.01) \mp \sqrt{(16)^2(894.01)^2 - 4(4096)(3201.99)}}{8192}, \\ \tan \theta &\approx 0.2404, 3.2518.\end{aligned}$$

Then, using the inverse tangent, we have

$$\theta \approx 13.5^\circ, 72.9^\circ.$$

Both trajectories are shown in Figure 11. The solutions can also be tested using the Virtual Tar-getter in **Virtual_Tar-getter.nb**.

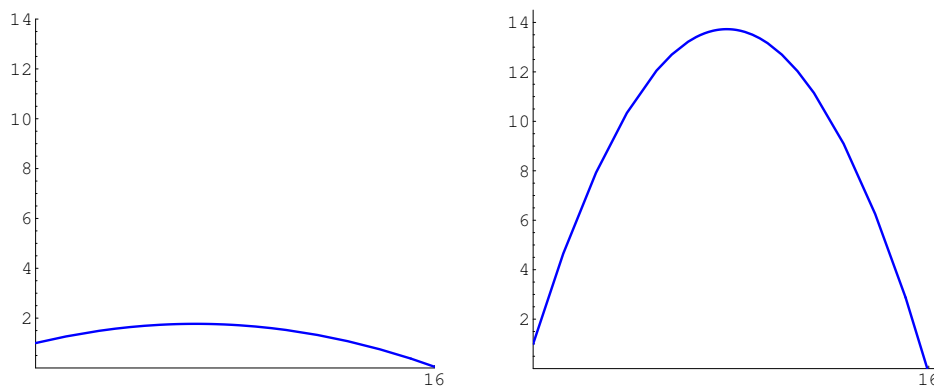


Figure 11

In practice, we would probably use the shallow trajectory, practically because our ceiling may not allow for the higher trajectory, but mostly because that the projectile is in the air longer with the higher trajectory, adding to the possible wind resistance, which we have ignored in our model. Note that both solutions are consistent with the data collected in Section 2.4 and shown in Figures 8 and 9.

In general, ...

I know what you are thinking: “By the time I do all of that paperwork, my target will be long gone.” True enough. That’s why we do all of the algebra and trigonometry ahead of time. In general, the (shallow) angle θ at which to set our launcher with velocity v from a firing height of h in order to hit a target a feet away at a height of b feet is

$$\theta = \tan^{-1} \left(\frac{v^2 - \sqrt{v^4 - 64(16a^2 + v^2(b - h))}}{32a} \right). \quad (7)$$

The derivation of this formula is left as an exercise.

Example: Find the angle of inclination needed to hit a target 15.5 ft away and 6.5 ft off of the ground using a launcher with velocity 25 ft/sec that is 3 ft off of the ground.

Using equation (7), we have

$$\theta = \tan^{-1} \left(\frac{(25)^2 - \sqrt{(25)^4 - 64(16(15.5)^2 + (25)^2(6.5 - 3))}}{32(15.5)} \right),$$

$$\theta = \tan^{-1} \left(\frac{625 - \sqrt{390625 - 64(3844 + 625(3.5))}}{496} \right),$$

$$\theta = \tan^{-1} \left(\frac{625 - \sqrt{4609}}{496} \right),$$

$$\theta \approx \tan^{-1}(1.1232) \approx 48.3^\circ.$$

We can use the Virtual Tar-Getter to check our answer. The graph of the trajectory is shown in Figure 12.

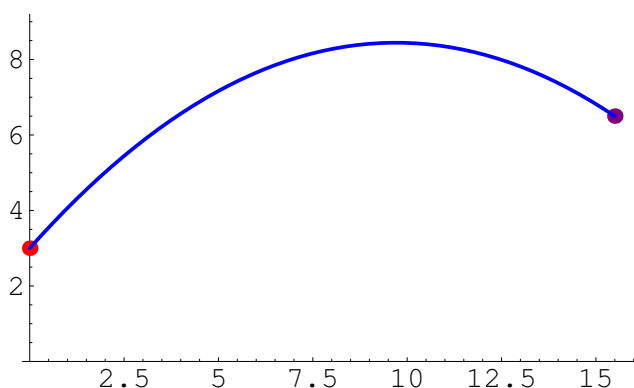


Figure 12

Equation (7) is provided only to show that this can be programmed into a calculator or computer and calculated instantly – please do not commit that horrible formula to memory! In the following homework exercises, we will find answers by solving equations for $\tan \theta$ and then θ , as we did earlier in this section.

2.5 Exercises

Verify the following identities by manipulating one side of the identity to look like the other side.

1) $(\sec \theta + \tan \theta)(\sec \theta - \tan \theta) = 1$

2) $(\csc \theta + \cot \theta)(\csc \theta - \cot \theta) = 1$

3) $\frac{\tan \theta + \cot \theta}{\tan \theta} = \csc^2 \theta$

4) $\frac{\tan \theta + \cot \theta}{\cot \theta} = \sec^2 \theta$

Solve for the firing angle(s) in the following situations with the given height h , velocity v , and position (a, b) in feet of the target, using (6). Verify your solution (if it exists) by using the Virtual Tar-Getter.

5) $h = 0$ ft, $v = 29.21$ ft/s, target at $(23.09, 0)$

6) $h = 0$ ft, $v = 32$ ft/s, target at $(32, 0)$

7) $h = 3$ ft, $v = 32$ ft/s, target at $(32, 3)$

8) $h = 0.14$ ft, $v = 30.91$ ft/s, target at $(21.86, 6)$

9) $h = 3$ ft, $v = 28.5$ ft/s, target at $(24, 4)$

10) $h = 16$ ft, $v = 30$ ft/s, target at $(30, 0)$

11) $h = 5$ ft, $v = 25$ ft/s, target at $(20.5, 4)$

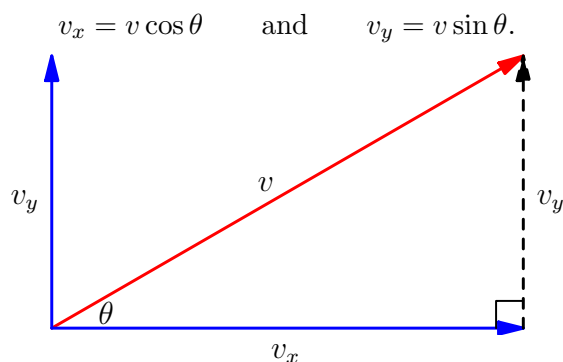
12) $h = 0.5$ ft, $v = 28$ ft/s, target at $(40, 0)$

13) Derive equation (7) by solving equation (6) for θ when $(x, y) = (a, b)$.

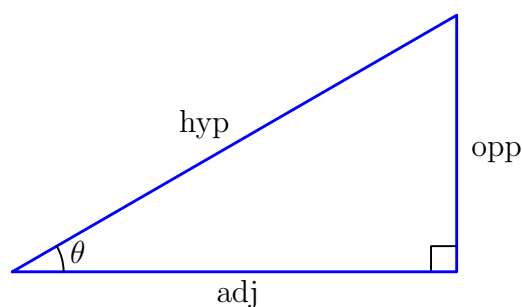
14) Adjust equation (7) to allow for units measured in meters (and meters per second) instead of feet. (See problem 9) in section 2.2.)

2.6 Review: What have we learned (or relearned)?

Theorem: The horizontal and vertical components v_x and v_y of a vector v in the direction θ above horizontal is given by

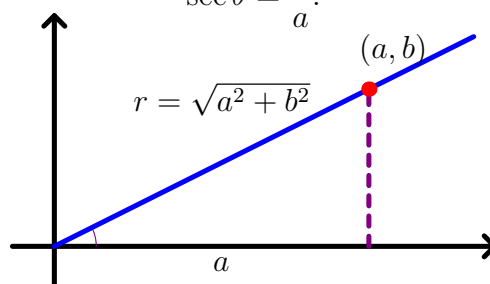


Definition: Secant of an Acute Angle The secant of an acute angle θ in a right triangle is the ratio of the length of the hypotenuse to the length of the side adjacent to θ ,

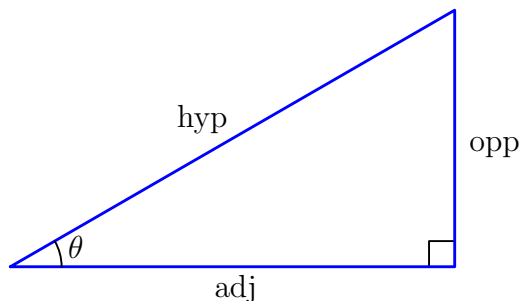


$$\sec \theta = \frac{\text{hyp}}{\text{adj}} \text{ for } 0 < \theta < 90^\circ.$$

Definition: Secant of Any Angle Let (a, b) be a point on a ray starting at $(0, 0)$ that forms the angle θ with the positive x -axis, and let $r = \sqrt{a^2 + b^2}$. The secant of an angle θ is the ratio

$$\sec \theta = \frac{r}{a}.$$


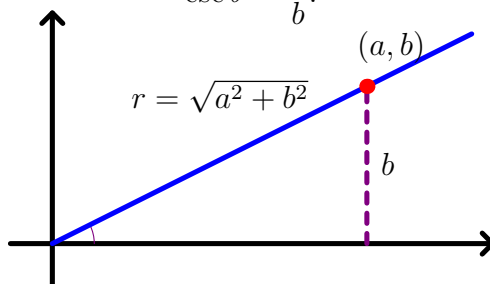
Definition: Cosecant of an Acute Angle *The cosecant of an acute angle θ in a right triangle is the ratio of the length of the hypotenuse to the length of the side opposite from θ ,*



$$\csc \theta = \frac{\text{hyp}}{\text{opp}} \text{ for } 0 < \theta < 90^\circ.$$

Definition: Cosecant of Any Angle *Let (a, b) be a point on a ray starting at $(0, 0)$ that forms the angle θ with the positive x -axis, and let $r = \sqrt{a^2 + b^2}$. The cosecant of an angle θ is the ratio*

$$\csc \theta = \frac{r}{b}.$$



Theorem: Cofunction Identities for Secant and Cosecant

$$\sec(90^\circ - \theta) = \csc \theta \quad \text{and} \quad \csc(90^\circ - \theta) = \sec \theta$$

Theorem: Relational Identities

$$\sec \theta = \frac{1}{\cos \theta} \quad (\text{when } \theta \neq 90^\circ + 180^\circ k, k \text{ integer})$$

$$\csc \theta = \frac{1}{\sin \theta} \quad (\text{when } \theta \neq 180^\circ k, k \text{ integer})$$

Theorem: Formula for the Trajectory of a Projectile *The height y of a projectile fired from initial height h at velocity v at an angle θ above horizontal is given by*

$$y = \left(-\frac{16}{v^2} \sec^2 \theta \right) x^2 + (\tan \theta)x + h \quad \text{for} \quad -90^\circ < \theta < 90^\circ.$$

Theorem: Pythagorean Identities

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\tan^2 \theta + 1 = \sec^2 \theta$$

$$1 + \cot^2 \theta = \csc^2 \theta$$

Review Exercises

The following projectile problems use the basic formula for the trajectory of the projectile given in equation (6) in section 2.3:

$$y = \left(-\frac{16}{v^2} \sec^2 \theta \right) x^2 + (\tan \theta)x + h \quad \text{for} \quad -90^\circ < \theta < 90^\circ.$$

- 1) Use the Virtual Launcher to fire six darts at a fixed angle of 45° from a height of 2 ft. Based on the average of the distances from the launcher of where the darts hit the ground, calculate the launcher's firing velocity. Check your answer by evaluating the variable "vreal".
- 2) Using the firing velocity found in problem 1) above, calculate the angle(s) θ needed to hit a target 18 ft away at a height 3 ft if the projectile is fired from a height of 3 ft. Check your answer using the Virtual Tar-getter.
- 3) Use the Virtual Launcher to fire six darts at a fixed angle of 30° from a height of 5 ft. Based on the average of the distances from the launcher of where the darts hit the ground, calculate the launcher's firing velocity. Check your answer by evaluating the variable "vreal".
- 4) Using the firing velocity found in problem 3) above, calculate the angle(s) θ needed to hit a target 6 ft away on the ground if the projectile is fired from a height of 1 ft. Check your answer using the Virtual Tar-getter.
- 5) Suppose that θ , $x = a$, $y = b$, and h are given amounts. Solve the trajectory formula for v .
- 6) Suppose that the projectile is fired straight up into the air ($\theta = 90^\circ$, see sections 2.3 and 2.4) from a height of 1.5 ft, and it takes 3 seconds to hit the ground.
 - a) Calculate the firing velocity of the launcher.
 - b) Calculate the angle(s) θ needed to hit a target 20 ft away on the ground, fired from a height of 1 ft. Check your answer using the Virtual Tar-getter.
 - c) Suppose that we miss the exact time of the dart in the air by ± 0.25 seconds. How far off could the measured firing velocity be? Using the shallow targeting angle, how far off could the landing spot of the dart be from the intended target?
 - d) Repeat the analysis from (c) for a time error of ± 0.1 seconds.

7) Verify the following identities.

a) $\frac{1}{1 - \sin \theta} + \frac{1}{1 + \sin \theta} = 2 \sec^2 \theta$

c) $\tan \theta + \cot \theta = \sec \theta \csc \theta$

e) $\cos^2 \theta - \sin^2 \theta = 2 \cos^2 \theta - 1$

b) $(\tan^2 \theta + 1)(1 - \cos^2 \theta) = \tan^2 \theta$

d) $\sec \theta + \tan \theta = \frac{\cos \theta}{1 - \sin \theta}$

f) $\cot^2 \theta (\sec^2 \theta - 1) = 1$

Chapter 3: Circular Arc Length and Angular Velocity

MATH 117: Trigonometry

3.1 Arc Length Using Degree Measure

You may recall from your high school geometry class that the ratio of the length around a circle, called the **circumference**, to the diameter of the circle is always the same constant, which we denote as π . If C denotes the circumference of a circle, and d denotes its diameter, then we say that

$$\pi = \frac{C}{d} \quad \text{or} \quad \pi = \frac{C}{2r}, \quad (1)$$

where r denotes the radius of the circle. We know that π is an irrational number (in spite of being defined as a fraction! What's up with that?!), roughly equal to

$$\pi \approx 3.14159265358979323846264338328 \dots$$

Given the definition of π in equation (1), it is easy to derive the formula for the circumference of a circle:

$$C = \pi d \quad \text{or} \quad C = 2\pi r.$$

The question is, what if we want to calculate the length of only part of the way around the circle?

The Measure of Arcs

In geometry, it is customary to denote an interval of a circle, called an **arc**, by either the endpoints of the arc (if they are on the same half of the circle) or by three points in order along the arc, topped by a small curve. See Figure 1 for an example of the notation. The **measure** of the arc, denoted $m(\cdot)$ is the angle measure of the central angle (vertex at the center of the circle) that circumscribes the arc. Again, see Figure 1 for an example of the notation.

Notice that the measure of the arc does not directly provide us with the length of the arc, although it does give us crucial information needed to find the length. If the radius of the circle is r and the measure of the arc is ϕ degrees, then since the circle is equidistant from the center, we have that the circular arc length s is given by

$$s = 2\pi r \left(\frac{\phi}{360^\circ} \right) = \frac{\pi}{180^\circ} \phi r. \quad (2)$$

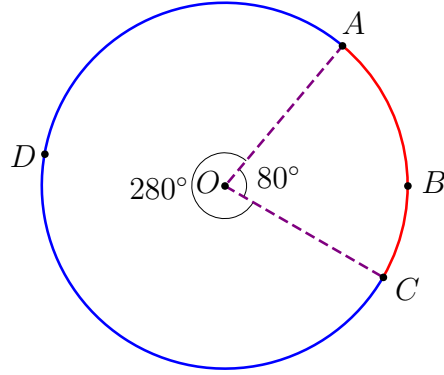


Figure 1: The red arc is denoted \widehat{AC} , \widehat{CA} , \widehat{ABC} , or \widehat{CBA} . The blue arc is denoted \widehat{ADC} or \widehat{CDA} . The measure of \widehat{AC} is $m(\widehat{AC}) = 80^\circ$ and the measure of \widehat{ADC} is $m(\widehat{ADC}) = 280^\circ$.

For example, if we assume that the radius of the circle in Figure 1 is 4 inches, then the length of arc \widehat{AC} is

$$s = \frac{\pi}{180^\circ}(80^\circ)(4) = \frac{16}{9}\pi \text{ inches,}$$

and the length of arc \widehat{ADC} is

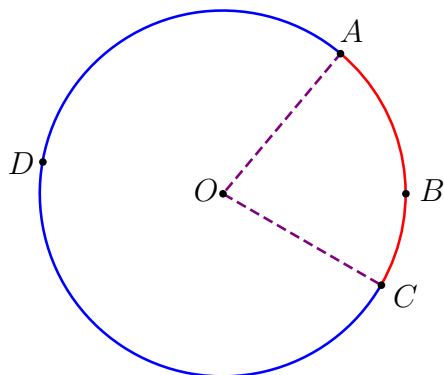
$$s = \frac{\pi}{180^\circ}(280^\circ)(4) = \frac{56}{9}\pi \text{ inches.}$$

Notice that the sum of the two lengths is 8π inches, the circumference of the circle.

While the formula for arc length in equation (2) is not that messy, we can certainly make it easier by changing our unit of measure for measuring angles.

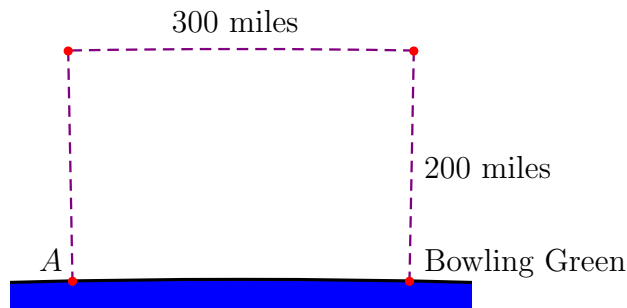
3.1 Exercises

Use the following diagram in problems 1) through 10).

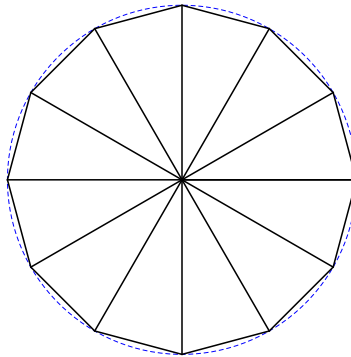


- 1) Given that $OA = 3$ in and $m(\widehat{ABC}) = 120^\circ$, find the arc length s of \widehat{ABC} .
- 2) Given that $OA = 12$ m and $m(\widehat{ABC}) = 72^\circ$, find the arc length s of \widehat{ABC} .
- 3) Given that $OA = 5$ m and $m(\widehat{ABC}) = 90^\circ$, find the arc length s of \widehat{ADC} .
- 4) Given that $OA = 8$ in and $m(\widehat{ABC}) = 50^\circ$, find the arc length s of \widehat{ADC} .
- 5) Given that $OA = 8$ ft and the arc length s of \widehat{ABC} is 8 ft, then find $m(\widehat{ABC})$.
- 6) Given that $OA = 10$ cm and the arc length s of \widehat{ABC} is 60 cm, then find $m(\widehat{ABC})$.
- 7) Given that $OA = 12$ cm and the arc length s of \widehat{ADC} is 20 cm, then find $m(\widehat{ABC})$.
- 8) Given that $OA = 2$ yd and the arc length s of \widehat{ADC} is 6 yd, then find $m(\widehat{ABC})$.
- 9) Given that $m(\widehat{ABC}) = 75^\circ$ and the arc length s is 4 in, then find the radius of the circle.
- 10) Given that $m(\widehat{ABC}) = 250^\circ$ and the arc length s is 10 m, then find the radius of the circle.
- 11) Suppose that a ribbon is wrapped around the world at the equator. If the diameter of the earth at the equator is 7,926 miles, and we decide to wrap a second ribbon around the world which will be 1 ft off the ground all the way around, how much longer would the second ribbon be than the first one?
- 12) Use the same strategy that we used to develop the formula for arc length in terms of the radius of the circle and the degree measure of the arc in (2) to find a formula for the area of a sector of a circle in terms of the radius and degree arc measure.

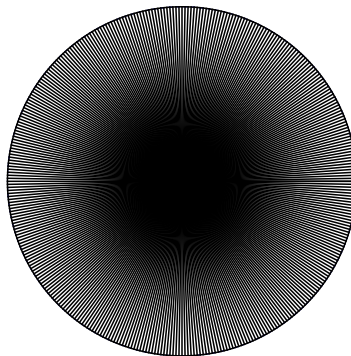
- 13) Suppose that the space shuttle, in geosynchronous orbit 200 miles above Bowling Green, leaves that position and travels 300 miles, still in its 200-mile-high orbit. If the shuttle is now directly above point A on the surface of the earth, how far along the earth's surface is point A from Bowling Green? (The average diameter of the earth is 7,918 miles.)



- 14) Suppose that you have 12 isosceles triangles, each with two sides of length 1 and an included angle of 30° , placed so that they all share a common vertex (shown below). Use the Law of Cosines to calculate the length of the third side of each triangle, and multiply by 12. Compare this result with the circumference of the superscribed circle of radius 1.



- 15) Suppose that you have 360 isosceles triangles, each with two sides of length 1 and an included angle of 1° , placed so that they all share a common vertex (shown below). Use the Law of Cosines to calculate the length of the third side of each triangle, and multiply by 360. Compare this result with the circumference of the superscribed circle of radius 1.



3.2 Radian Measure

Let's restate equation (2) for the arc length of a circular arc with radius r and degree measure ϕ :

$$s = \frac{\pi}{180^\circ} \phi r = \left(\frac{\pi}{180^\circ} \phi \right) r.$$

If we replace the expression $\frac{\pi}{180^\circ} \phi$ with a single value θ , then we have several nice consequences. First, our formula for the circular arc length s of an arc with radius r and measure θ in this new unit of measure is given will be given by

$$s = \theta r. \quad (3)$$

Second, if we consider the unit circle where $r = 1$ unit, then the measure of the angle and the circular arc length are the same ($s = \theta$), except, of course, for the unit of measure. (There are more nice consequences that we will merely foreshadow in this course, but will become immediately evident when you take calculus.)

Therefore, we define the *radian* measure θ of an angle to be

$$\theta = \frac{\pi}{180^\circ} \phi, \quad (4)$$

where ϕ is the degree measure of the angle. The circular arc length of an arc with radian measure θ and radius r is given in equation (3). The radian measure of many angles that we know the trigonometric values for are shown in the table below.

Angle Measure	Radian Measure	Angle Measure	Radian Measure
0°	$\frac{\pi}{180^\circ}(0^\circ) = 0 \text{ rad}$	90°	$\frac{\pi}{180^\circ}(90^\circ) = \frac{\pi}{2} \text{ rad}$
30°	$\frac{\pi}{180^\circ}(30^\circ) = \frac{\pi}{6} \text{ rad}$	180°	$\frac{\pi}{180^\circ}(180^\circ) = \pi \text{ rad}$
45°	$\frac{\pi}{180^\circ}(45^\circ) = \frac{\pi}{4} \text{ rad}$	270°	$\frac{\pi}{180^\circ}(270^\circ) = \frac{3\pi}{2} \text{ rad}$
60°	$\frac{\pi}{180^\circ}(60^\circ) = \frac{\pi}{3} \text{ rad}$	360°	$\frac{\pi}{180^\circ}(360^\circ) = 2\pi \text{ rad}$

Since $180^\circ = \pi \text{ rad}$, we can think of the conversion from degrees to radians, and likewise the conversion from radians to degrees, as a unit cancellation problem using the ole' "multiply-by-one" trick, where we multiply by a fraction equal to 1 arranged so that the appropriate units cancel.

Example: Convert 58° to radians.

$$58^\circ = 58^\circ \cdot \frac{\pi \text{ rad}}{180^\circ} = \frac{29\pi}{90} \text{ rad} \approx 1.0123 \text{ rad}$$

Example: Convert 1 radian to degrees.

$$1 \text{ rad} = 1 \text{ rad} \cdot \frac{180^\circ}{\pi \text{ rad}} = \frac{180^\circ}{\pi} \approx 57.2958^\circ$$

This example and Figure 2 illustrate the size of an angle with measure 1 radian.

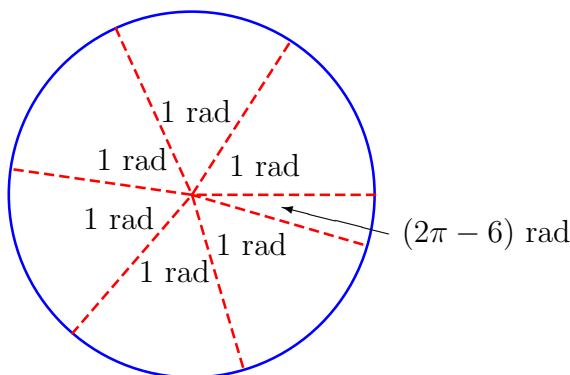


Figure 2

In spite of the fact that we just used radians as a unit in our unit cancellation trick, it is important to note that, because of the way we defined the radian, it is really more of a “non-unit”. Note that in the formula for arc length using degrees,

$$s = \frac{\pi}{180^\circ} \phi r,$$

we end up dividing the “degrees” out of the problem, leaving only a unit of length. In our formula for arc length using radians, $s = \theta r$, we never divide out the unit “radians”, leading one to think that the arc length should be given in “radians \times (unit of length)”. In practice, the radian is just an amount, although it is customary to put the notation “rad” or “radians” if the context makes it clear that we are talking about the measure of an angle.

If we use radians as a measure of the arc, then the length of the arc is much more straight-forward.

Example: Find the arc length of the arc with measure $\frac{4\pi}{9}$ in a circle of radius 4.

Using equation (3) with $\theta = \frac{4\pi}{9}$ and $r = 4$, we have

$$s = \left(\frac{4\pi}{9} \right) (4) = \frac{16}{9} \pi.$$

Note that $\frac{4\pi}{9} \text{ rad} = 80^\circ$, and that this solution is consistent with the example from the previous section.

3.2 Exercises

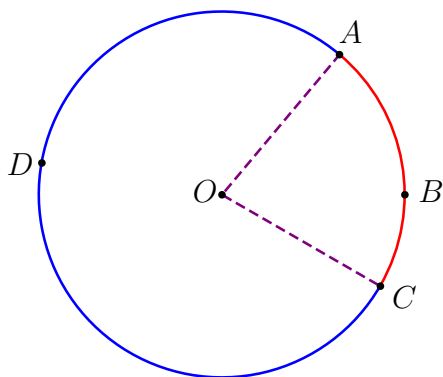
Convert the following degree measures to radians, exactly if possible, and then accurate to two decimal places.

- | | | | |
|---------------|----------------|----------------|----------------|
| 1) 35° | 2) 150° | 3) 240° | 4) 200° |
| 5) 60° | 6) 100° | 7) 144° | 8) 192° |

Convert the following radian measures to degrees, exactly if possible, and then accurate to two decimal places.

- | | | | |
|-------------------------|--------------------------|---------------------------|--------------------------|
| 9) $\frac{5\pi}{4}$ rad | 10) 2 rad | 11) $\frac{7\pi}{12}$ rad | 12) 1.3 rad |
| 13) $\frac{\pi}{3}$ rad | 14) $\frac{5\pi}{6}$ rad | 15) $\frac{\pi}{12}$ rad | 16) $\frac{\pi}{15}$ rad |

Use the following diagram in problems 17) through 26).



- 17) Given that $OA = 3$ in and $m(\widehat{ABC}) = \frac{2\pi}{3}$, find the arc length s of \widehat{ABC} .
- 18) Given that $OA = 12$ m and $m(\widehat{ABC}) = \frac{2\pi}{5}$, find the arc length s of \widehat{ABC} .
- 19) Given that $OA = 5$ m and $m(\widehat{ABC}) = \frac{\pi}{2}$, find the arc length s of \widehat{ADC} .
- 20) Given that $OA = 8$ in and $m(\widehat{ABC}) = \frac{5\pi}{18}$, find the arc length s of \widehat{ADC} .
- 21) Given that $OA = 8$ ft and the arc length s of \widehat{ABC} is 8 ft, then find $m(\widehat{ABC})$ in radians.
- 22) Given that $OA = 10$ cm and the arc length s of \widehat{ABC} is 60 cm, then find $m(\widehat{ABC})$ in radians.
- 23) Given that $OA = 12$ cm and the arc length s of \widehat{ADC} is 20 cm, then find $m(\widehat{ABC})$ in radians.

24) Given that $OA = 2$ yd and the arc length s of \widehat{ADC} is 6 yd, then find $m(\widehat{ABC})$ in radians.

25) Given that $m(\widehat{ABC}) = \frac{5\pi}{12}$ and the arc length s is 4 in, then find the radius of the circle.

26) Given that $m(\widehat{ABC}) = \frac{25\pi}{18}$ and the arc length s is 10 m, then find the radius of the circle.

3.3 Building a Spirograph™

You may have had a Spirograph™ when you were a child. The idea is that you have one large circle with gear cogs that you lay over a piece of paper, and smaller circular gears that roll around against the large circle. The smaller gears have holes that you insert a pencil or pen through, resulting in a very nice design that would be nearly impossible to draw otherwise. See Figure 3 for an illustration of the gears and some of the designs that can be drawn using the Spirograph™.

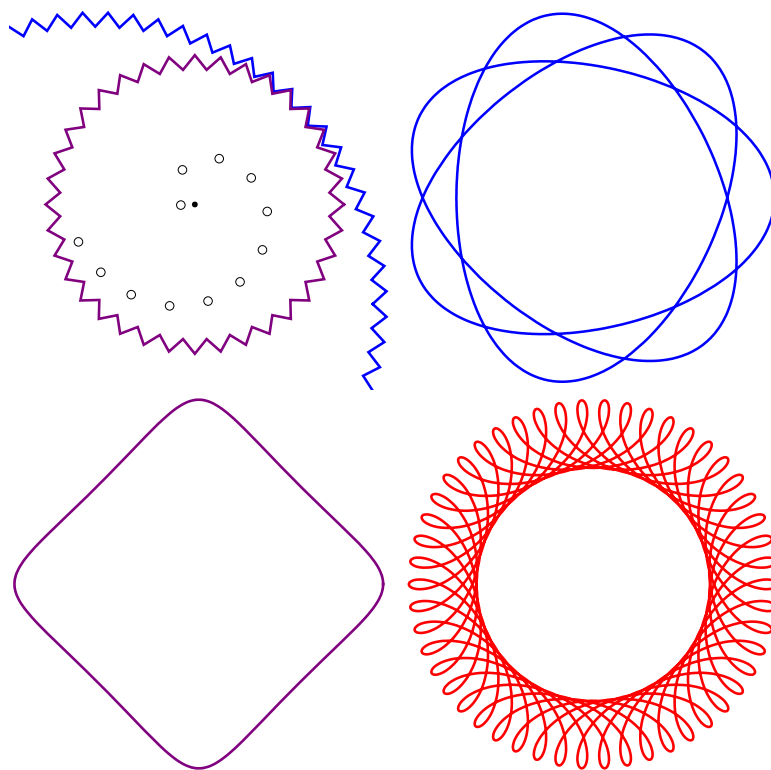


Figure 3: Top left, the gear design of the Spirograph™; all others, designs that could be drawn using the Spirograph™.

The question we would like to look at is, how can we describe and draw curves like this mathematically instead of mechanically? The curves are definitely not functions in the usual sense of the word (that is, they do not pass the vertical line test). How can our knowledge of circular arc length help with this problem?

The Theory

Let's model the situation as accurately, but as simply, as possible. The Spirograph™ gears are necessary to keep the inside gear from slipping as the design is being drawn, but in an abstract setting, we could certainly use circles, as illustrated in Figure 4, and simply not allow the circles to slip. Also, the Spirograph™ gears only have a finite number of holes

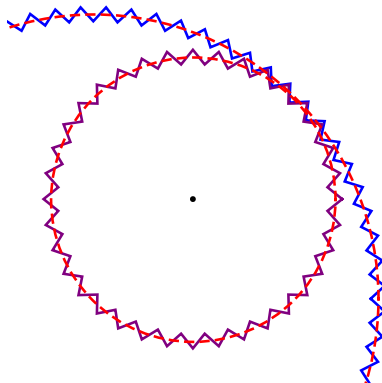


Figure 4: Replacing the gears with circles.

to put your pen into to construct a curve. We can allow for the pen to be inserted at any distance from the center of the small, inner circle.

Without loss of generality, let the outer circle have radius 1 unit, and let the inner circle have radius r , with $0 \leq r < 1$. Let k denote the distance from the center of the small circle that we insert our pen, so that $0 \leq k < r$. The one guiding principle in our model is that,

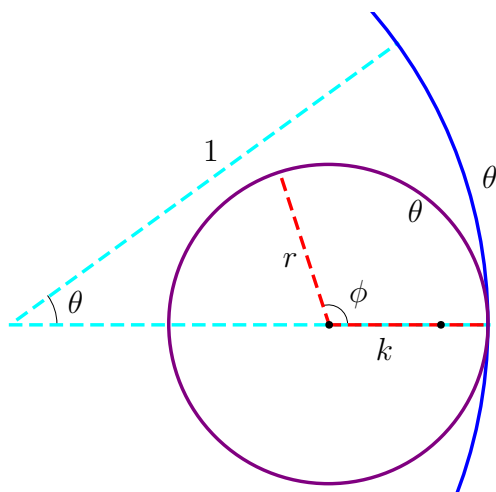


Figure 5

as we rotate the inner circle along the outer circle, the arc length covered by both must be equal. Let θ be the radian measure of the arc on the outer circle from the starting point to some arbitrary point where we have rotated the inner circle, and let ϕ be the radian measure of the arc on the inner circle. Then the arc length on the outer circle, $s = \theta(1) = \theta$, will equal the arc length on the inner circle, so that

$$s = \phi r = \theta.$$

Thus, we know immediately that

$$\phi = \frac{\theta}{r}.$$

Let's roll the small circle in Figure 5 up to the other end of the arc on the large circle, as shown in Figure 6. For any chosen values for r and k , we want to be able to describe

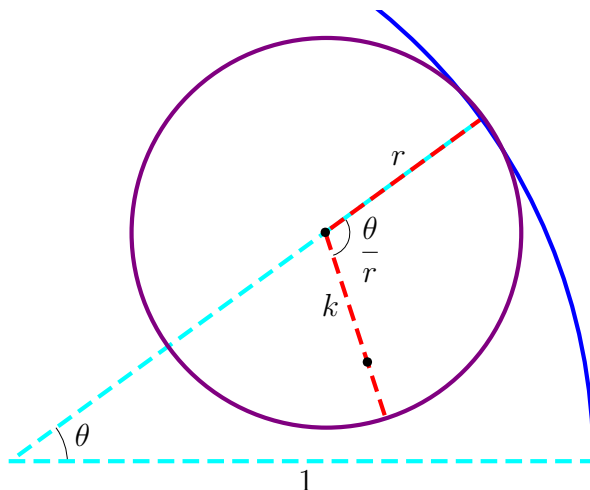


Figure 6

the location of the “pen point” in terms of the angle θ . The easiest way to do this is by developing formulas for both the x and y components of the position, called **parametric** equations, similar to the way that we developed the formula for a projectile in motion in Chapter 2. And since there are two circles to consider in the problem, we will consider the x and y contributions from each of them separately.

Let's consider the position of the center of the small circle, which is $1 - r$ units away from the center of the big circle, as shown in Figure 7. Let x_1 and y_1 be the horizontal

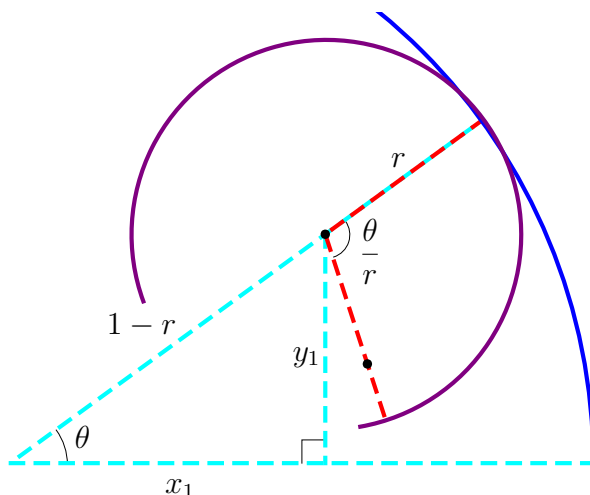


Figure 7

and vertical components of the center of the small Spirograph™. Then, based on the right triangle definitions of sine and cosine, we know that the center of the small circle is located

at (x_1, y_1) , where

$$x_1(t) = (1 - r) \cos \theta \quad \text{and} \quad y_1(t) = (1 - r) \sin \theta.$$

Notice that the small circle does not get rotated the full $\phi = \frac{\theta}{r}$ radians, and that it is actually spun in a clockwise direction, which we would consider a negative angle. Thus, the

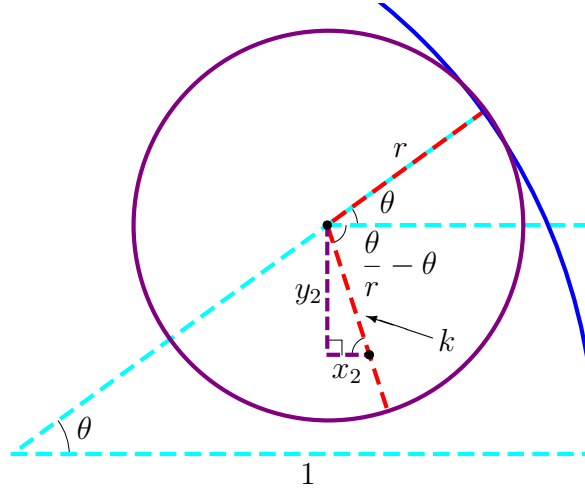


Figure 8

horizontal and vertical components x_2 and y_2 , respectively, from the center of the small circle to the pen point are

$$x_2(t) = k \cos \left(- \left(\frac{\theta}{r} - \theta \right) \right) = k \cos \left(\frac{1-r}{r} \theta \right)$$

and

$$y_2(t) = k \sin \left(- \left(\frac{\theta}{r} - \theta \right) \right) = -k \sin \left(\frac{1-r}{r} \theta \right),$$

respectively.

So adding both the horizontal components x_1 and x_2 and then the vertical components y_1 and y_2 gives us the parametric equations for $(x(t), y(t))$, the position of the pen point,

$$x(t) = (1 - r) \cos \theta + k \cos \left(\frac{1-r}{r} \theta \right) \quad \text{and} \quad y(t) = (1 - r) \sin \theta - k \sin \left(\frac{1-r}{r} \theta \right). \quad (5)$$

The following examples show the result of using the parametric equations in (5) to design SpirographTM curves.

Example: Draw the SpirographTM curve where the radius of the small gear is one-third that of the big circle and the pen point is $\frac{7}{30}$ the big radius from the center of the small gear.

In this case, we let $r = \frac{1}{3}$ and $k = \frac{7}{30}$. Using the equations in (5), the parametric equations for the curve will be

$$x(t) = \left(1 - \frac{1}{3} \right) \cos \theta + \frac{7}{30} \cos \left(\frac{1 - \frac{1}{3}}{\frac{1}{3}} \theta \right) = \frac{2}{3} \cos \theta + \frac{7}{30} \cos(2\theta)$$

and

$$y(t) = \left(1 - \frac{1}{3}\right) \sin \theta - \frac{7}{30} \sin \left(\frac{1 - \frac{1}{3}}{\frac{1}{3}} \theta\right) = \frac{2}{3} \sin \theta - \frac{7}{30} \sin(2\theta).$$

Letting t run from 0 to 2π , we get the curve shown in Figure 9.

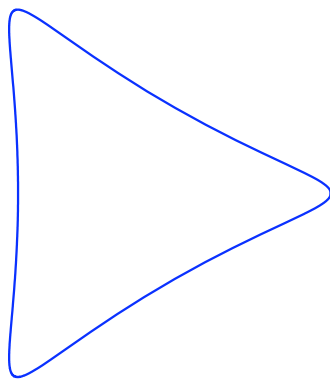


Figure 9

We may graph the curve on a graphing calculator or by using symbolic manipulation software. On a TI graphing calculator, we will need to go into **PAR** mode (for “parametric”) and **RADIAN** mode. We may have to set the minimum and maximum values of both x and y in the viewing window, as well as the minimum and maximum values of t used in generating the graph. The *Mathematica*TM file **spirograph.nb** on the course webpage contains code to generate the curve, as well as an animation of the curve being drawn with a SpirographTM. (The movie file **spirograph.mov** on the course webpage, which shows the curve at the bottom-right of Figure 3 being drawn, was made using this code.)

Example: Draw the SpirographTM curve where the radius of the small gear is $\frac{6}{11}$ that of the big circle and the pen point is $\frac{4}{11}$ the big radius from the center of the small gear.

In this case, we let $r = \frac{6}{11}$ and $k = \frac{4}{11}$. Using the equations in (5), the parametric equations for the curve will be

$$x(t) = \left(1 - \frac{6}{11}\right) \cos \theta + \frac{4}{11} \cos \left(\frac{1 - \frac{6}{11}}{\frac{6}{11}} \theta\right) = \frac{5}{11} \cos \theta + \frac{4}{11} \cos \left(\frac{5}{6} \theta\right)$$

and

$$y(t) = \left(1 - \frac{6}{11}\right) \sin \theta - \frac{4}{11} \sin \left(\frac{1 - \frac{6}{11}}{\frac{6}{11}} \theta\right) = \frac{5}{11} \sin \theta - \frac{4}{11} \sin \left(\frac{5}{6} \theta\right).$$

Letting t run from 0 to 2π , we get the curve shown on the left in Figure 10. To complete the figure, we need to spin it around 5 more times, so we let t run from 0 to 12π . The completed curve is shown on the right in Figure 10. Can you develop a general rule for how many times around the circle we have to spin the small gear in order to complete the curve? (See problem 7) in the homework.)

It is important to point out that if we wanted to scale the whole process to a larger circle, say of radius 2 instead of 1, then we only need to multiply $x(t)$ and $y(t)$ in equation (5) by the radius

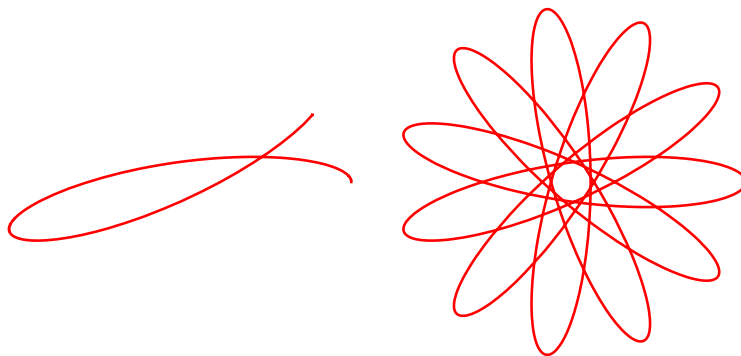


Figure 10

of the new circle. The values r and k would become the ratios of the gear ratio and distance to the pen-point, respectively, to the new outer gear radius.

3.3 Exercises

In exercises 1) through 6), give the parametric equations that would draw SpirographTM curves inside a circle of radius 1 with a smaller, inner gear of radius r using the pin-point at distance k from the center of the gear. Then graph each curve using a graphing calculator, your own *Mathematica*TM code, or the code in the file **spirograph.nb**. Determine the value p needed so that $0 \leq \theta \leq p$ gives the complete curve.

1) $r = \frac{1}{2}, k = \frac{1}{4}$

2) $r = \frac{3}{7}, k = \frac{1}{5}$

3) $r = \frac{5}{7}, k = \frac{3}{7}$

4) $r = \frac{2}{3}, k = \frac{1}{3}$

5) $r = \frac{13}{23}, k = \frac{1}{2}$

6) $r = \frac{25}{47}, k = \frac{1}{4}$

7) Graph the curves as above with $r = \frac{n}{5}$, where n assumes each of the values $n = 1, 2, 3, 4$, with $k = \frac{1}{8}$ in each case.

a) Speculate as to a formula for the smallest value p such that θ in the range $0 \leq \theta \leq p$ gives the complete graph.

b) Repeat the experiment with $r = \frac{n}{6}$, where n assumes each of the values $n = 1, 2, 3, 4, 5$, with k chosen arbitrarily, except that $0 \leq k < r$. Revise your formula for p if necessary.

8) The requirement that k is strictly less than r comes from the fact that, in a real SpirographTM, the pen-point cannot be on the outer edge of the inner gear. However, in our model, we can set $k = r$.

a) Graph the SpirographTM (sort of) curves with $r = \frac{1}{n}$, where n assumes each of the values $n = 2, 3, 4, 5$, and let $k = r$ in each case. Describe the types of curves created.

b) Graph the SpirographTM (sort of) curves with $r = \frac{1}{n}$ and then $r = \frac{n-1}{n}$, each time with $k = r$, for $n = 3, 4, 5, 6$. For each value of n , describe the two curves and compare them to each other.

9) Graph SpirographTM curves with rational r , $\frac{1}{2} < r < 1$ (you pick the values), and with $k = 1 - r$. Describe the types of curves created. How are the graphs dependent upon your choice of r ?

10) Graph SpirographTM curves with irrational r , $0 < r < 1$ (you pick the values), and with any value k , $0 \leq k \leq r$. What is the value p (if it exists) such that $0 \leq \theta \leq p$ renders the entire graph?

11) If we use a small gear of radius r in our model of the SpirographTM, and we place the pen point at the center of the small gear, it makes sense that the resulting curve would be a circle of radius $1 - r$. Derive the parametric curves for x and y in this situation, and then find the x, y -equation of the resulting curve, to prove algebraically that the resulting curve is indeed a circle of radius $1 - r$.

12) Consider the parametric equations

$$\begin{cases} x(\theta) = a \cos \theta \\ y(\theta) = b \sin \theta, \end{cases} \quad \text{for } a, b > 0.$$

Find the x, y -equation of the resulting curve, and determine what kind of geometric object the resulting curve is. Graph the parametric equations for various values of a and b to verify your findings.

3.4 Generalizing Spirograph™ Curves

Let's revisit the construction of the parametric equations that describe Spirograph™ curves, and see if we can develop a more general method for this type of motion. Recall from (5) in Section 3.3 that the position of the pen point on a Spirograph™ curve can be described with the parametric equations

$$\begin{cases} x(t) = (1-r)\cos\theta + k\cos\left(\frac{1-r}{r}\theta\right) \\ y(t) = (1-r)\sin\theta - k\sin\left(\frac{1-r}{r}\theta\right), \end{cases} \text{ for } 0 \leq \theta \leq p, \quad (6)$$

where the radius of the outer gear is 1 unit, the radius of the smaller gear is r , $0 < r < 1$, and the distance from the center of the small gear to the pen point is k , $0 \leq k < r$. The value of p is $n \cdot 2\pi$, where n is the number of times we have to go around the outer gear in order to draw the complete curve, and as you have probably learned through experimentation while doing the homework from Section 3.3, n is the numerator of the simplified rational number r . (More on that later in this section.)

A second look at the equations . . .

From problem 11) in Section 3.3, we know that the parametric equations

$$\begin{cases} x(\theta) = a\cos\theta \\ y(\theta) = a\sin\theta \end{cases}$$

give us a circle of radius a . Since the difference between the size of the gears is $1-r$, then we have the first terms of the parametric equations for the Spirograph™ curve in (6) (that get us out to the center of the small gear). Note that, in a more general modeling situation where we had circular motion, we could set the radius to any needed value.

Now consider the rotation of the smaller gear. The circumference of the outer gear is 2π , and the circumference of the small gear is $2\pi r$. If we think about rolling the small gear along a line of length 2π as illustrated in Figure 11, then we can roll it (clockwise)

$$\frac{\text{length of the line}}{\text{circumference}} = \frac{2\pi}{2\pi r} = \frac{1}{r}$$

times. However, when we wrap the line back into a circle, we give the small circle one additional counterclockwise spin, for a total of $1 - \frac{1}{r}$ counterclockwise rotations, or

$$1 - \frac{1}{r} = \frac{r-1}{r} = -\frac{1-r}{r}$$

rotations, which corresponds to the angle we calculated in Section 3.3. Thus, the coefficient of the variable θ in (6) tells us how many revolutions the small gear will turn as we move the small gear one revolution around the outer gear. With a pen point of radius k from the center of the small gear, we have the second terms of the parametric equations for the Spirograph™ curve in (6) (that get us out to the pen point). Note that, in a more general modeling situation where we had circular motion on top of circular motion, we could set the coefficient of θ to any needed value.

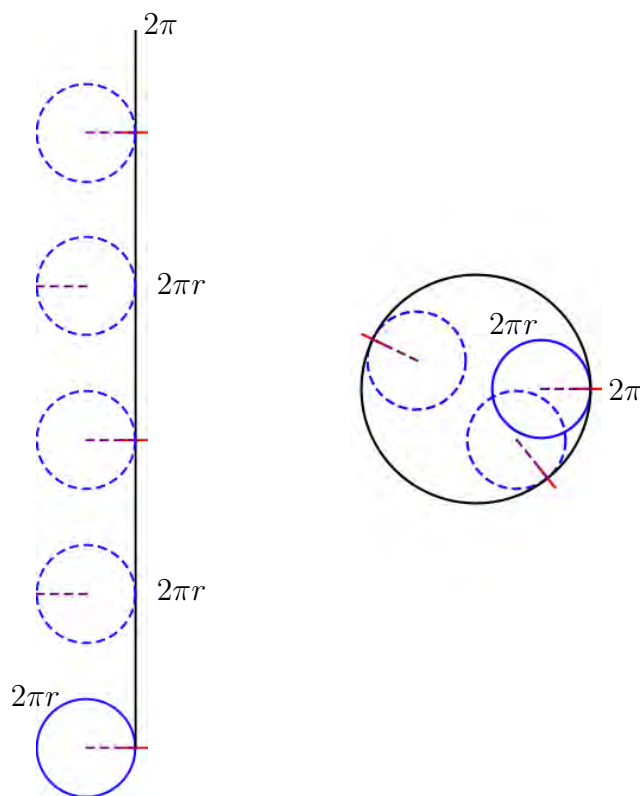


Figure 11

Note that this also explains why the number of times we have to take the small gear around the outside gear is the numerator of the value r times 2π . Suppose that $r = \frac{n}{q}$, where n, q are positive integers with $n < q$ (since $r < 1$), and where r is in simplest form. Then

$$1 - \frac{1}{r} = 1 - \frac{1}{\frac{n}{q}} = 1 - \frac{q}{n} = \frac{n - q}{n}.$$

Since n and q have no common factors, then the first time we get an integer number of revolutions at the place that we started is for the number of revolutions be n , at which time the small gear will have rotated $n - q$ times, or $q - n$ times clockwise.

Let's go for the moon!

So how can we adapt this construction to model circular-on-circular motion? Perhaps the oldest and most consistent example (whether we realize it or not) of this type of motion is the Earth's moon, orbiting our planet while the planet itself is orbiting the sun. So what does the Moon's path around the Sun look like? Is it a "loop-the-loop" path as shown at left in Figure 12, or a smooth, wavy path as shown at middle, or something even weirder like at right?

Looking down at the solar system (meaning that we see the northern hemisphere of the Earth and the "top" of the Sun and Moon), the Earth orbits counterclockwise around the Sun, and the Moon orbits counterclockwise around the Earth. (Not that it matters for this model, but the Earth also spins counterclockwise about its own axis.) Let the distance from the Sun to the Earth be 1

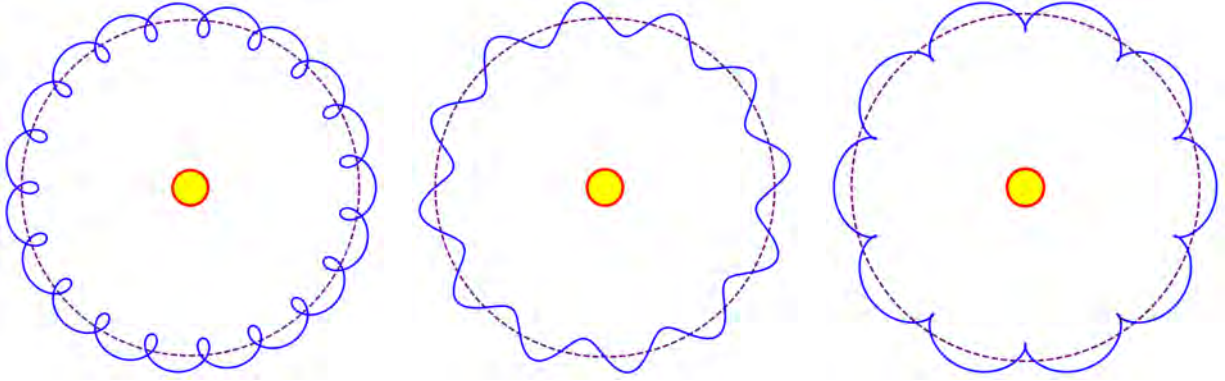


Figure 12

“unit”. Then the path of the Earth can be described with the parametric equations

$$\begin{cases} x_1(\theta) = \cos \theta \\ y_1(\theta) = \sin \theta. \end{cases}$$

Let the distance from the Earth to the Moon be 0.1 of these “units” (not drawn to scale) so that we can clearly see the path that the Moon takes. The Moon goes around a fixed-position Earth once every 27.3 days, so the Moon will make

$$\frac{365.25 \text{ days}}{27.3 \text{ days}} + 1 = \frac{392.55}{27.3} \text{ revolutions}$$

around the earth in one year, or 365.25 days. Remember that the “+1” comes from the counter-clockwise circular path of the center of Moon’s orbit. Then the path of the Moon relative to the Earth can be described with the parametric equations

$$\begin{cases} x_2(\theta) = 0.1 \cos \left(\frac{392.55}{27.3} \theta \right) \\ y_2(\theta) = 0.1 \sin \left(\frac{392.55}{27.3} \theta \right). \end{cases}$$

Then the parametric equations of the Moon relative to the Sun are found by combining the x -equations and the y -equations to get

$$\begin{cases} x(\theta) = \cos \theta + 0.1 \cos \left(\frac{392.55}{27.3} \theta \right) \\ y(\theta) = \sin \theta + 0.1 \sin \left(\frac{392.55}{27.3} \theta \right), \end{cases} \quad \text{for } 0 \leq \theta \leq 2\pi. \quad (7)$$

The path is illustrated in Figure 13. You can also view the file **orbitalmotion.nb** to view Earth and the Moon in motion.

Since in a model like this, we would typically think about days passing instead of radians, we may want to adjust the equations to have an independent variable t for time in days. To accomplish this, we basically do an in-line conversion inside of the trigonometric functions from time to radians. In this example, 365.25 days is equivalent to 2π radians, so if the input is in days, we want to multiply by one in such a way that the units “days” cancel, and we are left with radians in the numerator:

$$\theta = \frac{2\pi \text{ radians}}{365.25 \text{ days}} \cdot t \text{ days} = \frac{2\pi}{365.25} t.$$

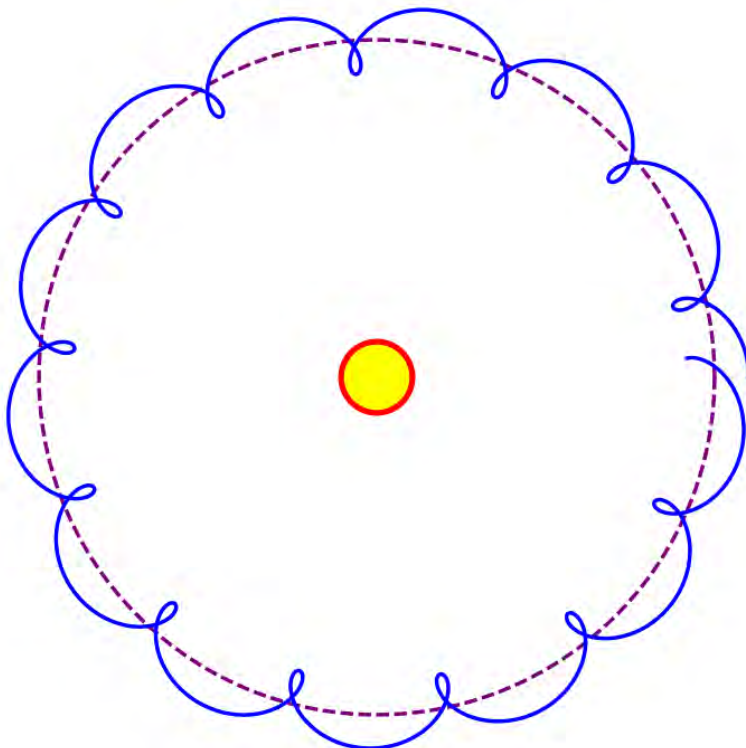


Figure 13

Then our adjusted parametric equations in (7) become

$$\begin{cases} x(t) = \cos\left(\frac{2\pi}{365.25}t\right) + 0.1 \cos\left(\frac{392.55}{27.3} \cdot \frac{2\pi}{365.25}t\right) \\ y(t) = \sin\left(\frac{2\pi}{365.25}t\right) + 0.1 \sin\left(\frac{392.55}{27.3} \cdot \frac{2\pi}{365.25}t\right), \end{cases} \quad \text{for } 0 \leq t \leq 365.25,$$

or, after simplifying,

$$\begin{cases} x(t) = \cos\left(\frac{8\pi}{1461}t\right) + 0.1 \cos\left(\frac{10468\pi}{132951}t\right) \\ y(t) = \sin\left(\frac{8\pi}{1461}t\right) + 0.1 \sin\left(\frac{10468\pi}{132951}t\right), \end{cases} \quad \text{for } 0 \leq t \leq 365.25. \quad (8)$$

The graph of these parametric equations will be identical to the graph shown in Figure 13.

Note to astronomy students: I am well aware that the above model is fraught with inaccuracies, and is fairly simplistic. First of all, as mentioned, it is not drawn to scale. Also, a circular orbit for the Earth around the Sun is assumed, although the Earth takes a slightly elliptical path around the Sun. To be completely correct, the Earth and the Moon each wobble about their common center of gravity, and this point has an elliptical orbit about the Sun. However, when drawn completely correct, one would be hard-pressed to spot the difference between that graph and the above graph when actually drawn to scale.

3.4 Exercises

In problems 1 – 4, suppose that a rotating arm, turning counterclockwise around a center axis, has another rotating arm positioned with its center 4 ft from the central axis. The second rotating arm has length 1 ft from the its axis.

1) Suppose that the smaller arm is aligned with the first arm, and that the smaller arm is stationary with respect to its axis.

a) Find the parametric equations that give the position of the end of the smaller arm as the larger arm turns about its axis, and graph its path.

b) Suppose that the larger arm turns at one revolution every 10 seconds. Adjust your answers to 1a) to give the parametric equations for the position of the end of the smaller arm in terms of seconds.

2) Suppose that the smaller arm rotates once clockwise with respect to its stationary axis, in the same time it takes the large arm to rotate once.

a) Find the parametric equations that give the position of the end of the smaller arm as the larger arm turns about its axis, and graph its path.

b) Suppose that the larger arm turns at one revolution every 5 seconds. Adjust your answers to 2a) to give the parametric equations for the position of the end of the smaller arm in terms of seconds.

3) Suppose that the smaller arm rotates once counterclockwise with respect to its stationary axis, in the same time it takes the large arm to rotate once.

a) Find the parametric equations that give the position of the end of the smaller arm as the larger arm turns about its axis, and graph its path.

b) Suppose that the larger arm turns at one revolution every 1 second. Adjust your answers to 3a) to give the parametric equations for the position of the end of the smaller arm in terms of seconds.

4) Suppose that the smaller arm rotates four times counterclockwise with respect to its stationary axis, in the same time it takes the large arm to rotate once.

a) Find the parametric equations that give the position of the end of the smaller arm as the larger arm turns about its axis, and graph its path.

b) Suppose that the larger arm turns at one revolution every 30 seconds. Adjust your answers to 4a) to give the parametric equations for the position of the end of the smaller arm in terms of seconds.

5) Reconsider the model of the Moon orbiting the Earth, which is orbiting the Sun, given in equation (8). Correct the model by using more accurate distances from the Sun to the Earth, and from the Earth to the Moon, and graph the path of the Moon about the Sun.

6) Reconsider the model of the Moon orbiting the Earth, which is orbiting the Sun, given in equation (8). How would the model (using the old distances) change if the Moon orbited clockwise about the Earth instead of counterclockwise? (You may need to revisit equation (7) first.) Graph this hypothetical path of the Moon about the Sun.

7) Reconsider the model of the Moon orbiting the Earth, which is orbiting the Sun, given in equation (8). Suppose that we replace the Moon with a man-made satellite, in geosynchronous orbit (stays fixed above a certain position on the Earth's surface). Develop parametric equations to describe the path of the satellite around the sun, using the same inaccurate distances as in equation (8) to help see the pattern.

3.5 Angular Velocity

Now that we have connected the ideas of angle/arc measure and the length of the circular arc, it only makes sense to connect the ideas of speed of angle change and the speed of the arc length change. *Linear velocity* is defined as

$$\text{linear velocity} = \frac{\text{change in distance or length}}{\text{change in time}}.$$

Note that we do not necessarily have to be moving in a line to have a change in distance or length, so linear velocity could be velocity along a curve or arc. An analogous definition of *angular velocity* would be

$$\text{angular velocity} = \frac{\text{change in measure of an angle}}{\text{change in time}}.$$

In circles, each point is equidistant from the center, so the radius r will not change. However, a change in the angle will cause a change in the arc length; that is, using the radian formula for arc measure in (3), if

$$s_1 = \theta_1 r \quad \text{and} \quad s_2 = \theta_2 r,$$

then

$$s_2 - s_1 = \theta_2 r - \theta_1 r = (\theta_2 - \theta_1)r, \quad \text{or} \quad \Delta s = (\Delta \theta)r.$$

Then

$$\text{linear velocity} = \frac{\Delta s}{\Delta t} = \frac{(\Delta \theta)r}{\Delta t} = \frac{\Delta \theta}{\Delta t} r = \text{angular velocity} \times r,$$

or, using v for linear velocity and ω for angular velocity,

$$v = \omega r.$$

Note the similarity with our arc length formula $s = \theta r$.

Although our formula is dependent upon measuring the angle in radians, we are typically less interested in measuring radians per unit of time or even degrees per unit of time than in measuring revolutions per unit of time. This will usually cause us to have to perform a unit conversion somewhere in our work, with $1 \text{ rev} = 2\pi \text{ rad}$.

Example: A mechanically-inclined person decides to build a go-cart, using an old lawnmower engine for the go-cart engine. The engine is designed to turn its shaft at a maximum of 600 rpm (revolutions per minute). Assuming that the crankshaft to axle ratio is 1:1 (geared to turn at the same speed) and that the tires powered by the engine are 9 inches in diameter (and that the engine has enough power to turn the wheels – another topic altogether), what will be the top speed of the go-cart in mph?

Let's start by converting 600 rpm to radians-per-minute:

$$600 \text{ rpm} = \frac{600 \text{ rev}}{\text{min}} \cdot \frac{2\pi \text{ rad}}{1 \text{ rev}} = 1200\pi \frac{\text{rad}}{\text{min}}.$$

This is the angular velocity of the crankshaft, which will equal the angular velocity of the axle. The radius of the tires attached to that axle is $\frac{9}{2} = 4.5$ inches, so the linear velocity of the tire is

$$\frac{1200\pi}{\text{min}} (4.5 \text{ in}) = 5400\pi \frac{\text{in}}{\text{min}}.$$

Lastly, we convert this into miles-per-hour:

$$5400\pi \frac{\text{in}}{\text{min}} = \frac{5400\pi \text{ in}}{\text{min}} \cdot \frac{60 \text{ min}}{1 \text{ hr}} \cdot \frac{1 \text{ mile}}{5280 \text{ ft}} \cdot \frac{1 \text{ ft}}{12 \text{ in}} = \frac{225\pi \text{ miles}}{44 \text{ hr}} \approx 16.06 \text{ mph}.$$

Example: The wheels on a small car are 15 inches in diameter. If the car is moving at 60 mph, what is the angular velocity of the wheels in revolutions-per-minute?

We have two units of length being used in the statement of the problem, so we need to convert our amounts to one of those two units of length. Since we are wanting to end up with revolutions per minute, it makes sense to convert miles-per-hour to inches-per-minute, to help avoid some very large numbers:

$$60 \text{ mph} = \frac{60 \text{ miles}}{\text{hr}} \cdot \frac{1 \text{ hr}}{60 \text{ min}} \cdot \frac{5280 \text{ ft}}{1 \text{ mile}} \cdot \frac{12 \text{ in}}{1 \text{ ft}} = 63360 \frac{\text{in}}{\text{min}}.$$

Also, since the diameter is 15 inches, then the radius $r = 7.5$ inches. Then we solve the following equation for the angular velocity, which I will denote with ω :

$$63360 \frac{\text{in}}{\text{min}} = \omega \cdot 7.5 \text{ in}$$

$$\omega = \frac{63360 \text{ in}}{\text{min}} \cdot \frac{1}{7.5 \text{ in}} = \frac{8448}{\text{min}}.$$

Lastly, we need to convert our radians-per-minute answer to a revolutions-per-minute answer:

$$\omega = \frac{8448}{\text{min}} \cdot \frac{1 \text{ rev}}{2\pi} = \frac{4224 \text{ rev}}{\pi \text{ min}} \approx 1344.54 \text{ rpm}.$$

3.5 Exercises

1) Suppose that a fly-wheel with radius r attached to a small motor is spinning at ω revolutions-per-minute (rpm). In each of the following cases, find the linear velocity v in the given units of a point on the outer edge of the fly-wheel.

- | | |
|---|---|
| a) $r = 4 \text{ cm}$, $\omega = 200 \text{ rpm}$; m/s | b) $r = 3.5 \text{ in}$, $\omega = 400 \text{ rpm}$; ft/s |
| c) $r = 2 \text{ in}$, $\omega = 360 \text{ rpm}$; ft/s | d) $r = 55 \text{ mm}$, $\omega = 600 \text{ rpm}$; m/s |

2) Suppose that a car or truck with a front-left tire of diameter d is traveling at v mph. In each of the following cases, find the angular velocity ω in rpm's of the front-left tire.

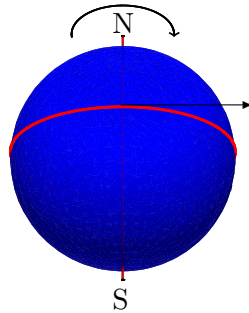
- | | |
|---|---|
| a) $d = 19 \text{ in}$, $v = 55 \text{ mph}$ | b) $d = 20 \text{ in}$, $v = 70 \text{ mph}$ |
| c) $d = 15 \text{ in}$, $v = 65 \text{ mph}$ | d) $d = 24 \text{ in}$, $v = 35 \text{ mph}$ |

3) The speedometer in a vehicle works by taking the rpm-reading from a tire of the vehicle and converting that into an mph-reading equivalent to the linear velocity of the outside of the tire. If one were to replace that tire on a vehicle with a new tire of a different radius without an adjustment to the speedometer, then the speedometer would give an incorrect reading of the vehicle's speed. In the following cases, tires of the correct radius for the speedometer have been replaced with tires of a different radius. Calculate a formula for getting the correct speed v from the incorrect speedometer reading \tilde{v} .

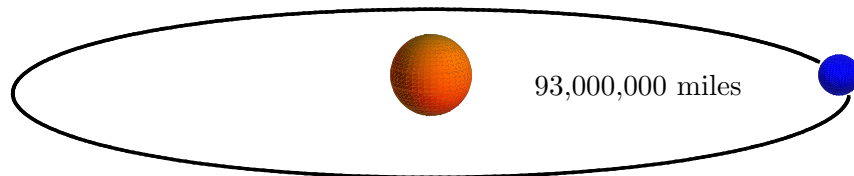
- | | |
|--|---|
| a) old $r = 10 \text{ in}$, new $r = 15 \text{ in}$ | b) old $r = 9.5 \text{ in}$, new $r = 12 \text{ in}$ |
|--|---|

4) Using the same set up as in 3) above, find a general formula to calculate the correct speed v from the incorrect speedometer reading \tilde{v} when the tires of correct radius $r = a$ are replaced with tires of radius $r = b$.

5) Have you ever wondered why space shuttle launches take place in the southern part of the United States, and why after an initial vertical launch, they turn and track to the east? Given that the earth has a diameter of 7,926 miles at the equator, find the linear velocity in mph of an object at the equator due to the rotation of the earth.



6) Let's assume that the earth is in a circular orbit around the sun with radius 93 million miles. (This is blatantly false – the earth is in an elliptical orbit around the sun – but the ellipse is nearly circular, with the radius at the major and minor axes only differing by 3% of the average radius.) Remember that one year is approximately 365.25 days, taking into account leap years. How fast is the earth moving in mph in its orbit around the sun? (The illustration below is not drawn to scale.)



3.6 Review: What have we learned (or relearned)?

Theorem: Circular Arc Length (Degrees) *The arc length s with measure ϕ degrees along a circle of radius r is given by*

$$s = \frac{\pi}{180^\circ} \phi r.$$

Definition: Radians *The radian measure θ of an angle with degree measure ϕ is given by*

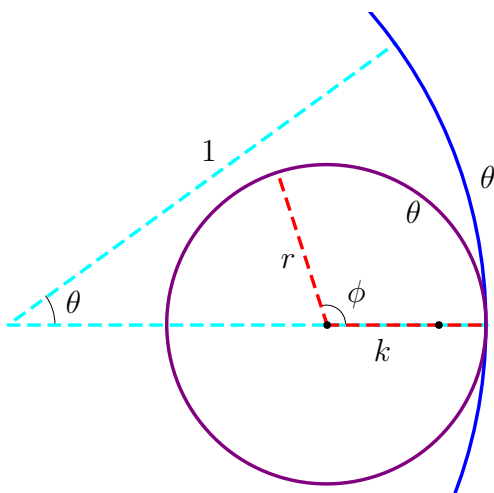
$$\theta = \frac{\pi}{180^\circ} \phi.$$

Theorem: Circular Arc Length (Radians) *The arc length s with measure θ radians along a circle of radius r is given by*

$$s = \theta r.$$

Theorem: SpirographTM Curves *The parametric equations for SpirographTM curves where the inner gear has radius r that of the outer gear, $0 < r < 1$, and the distance of the pen point from the center of the small gear is k the radius of the outer gear, $k \leq r$, are*

$$\begin{cases} x(t) = (1 - r) \cos \theta + k \cos \left(\frac{1-r}{r} \theta \right) \\ y(t) = (1 - r) \sin \theta - k \sin \left(\frac{1-r}{r} \theta \right), \end{cases} \text{ for } 0 \leq \theta \leq \text{numerator of } r \cdot 2\pi.$$



Theorem: Circular-Motion-on-Circular-Motion Curves *The parametric equations for curves where an axis rotating counterclockwise located r units from a center point, has an object spinning about it at radius k , rotating n times counterclockwise about its axis per one revolution about the first axis, are*

$$\begin{cases} x(t) = r \cos \theta + k \cos ((n + 1)\theta) \\ y(t) = r \sin \theta + k \sin ((n + 1)\theta), \end{cases} \text{ for } 0 \leq \theta \leq 2\pi.$$

Definition: Linear Velocity *Linear velocity v is the ratio of the change in distance Δd to the change in time Δt ,*

$$v = \frac{\Delta d}{\Delta t}.$$

Definition: Angular Velocity *Angular velocity ω is the ratio of the change in angle $\Delta\theta$ to the change in time Δt ,*

$$\omega = \frac{\Delta\theta}{\Delta t}.$$

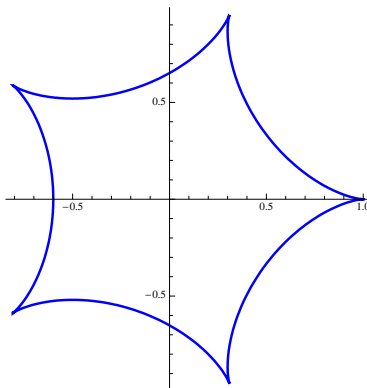
Theorem: Linear Velocity of a Spinning Wheel *The linear velocity v of a circle of radius r spinning at angular velocity ω radians per unit of time is given by*

$$v = \omega r.$$

Review Exercises

- 1) A communications satellite is in geosynchronous orbit 250 miles above the earth. If we assume that the satellite is 2500 miles from the earth's axis of rotation, how many miles will the satellite travel each day relative to the earth?
- 2) Recall that the earth is in a (roughly) circular orbit around the sun, with radius 93 million miles, and that a year is approximately 365.25 days. Light from the sun takes 8 minutes, 19 seconds to reach the earth.
 - a) Find both the degree and radian measure of the angle the earth moves in its orbit around the sun during the time that light travels from the sun to the earth.
 - b) How many miles does the earth travel around the sun during that time?
- 3) Recall that $1 \text{ radian} \approx 57.30^\circ$. If we start to travel around a circle in increments of 1 radian, will we ever land at our starting point? If so, give the number of radians needed. If not, explain why.
- 4) Given that the angle measure of an arc is k radians and the length of that arc is $2k$ feet, what is the radius of the circle that contains the arc?
- 5) Find the parametric equations that will draw a SpirographTM curve with an inner gear of radius two-thirds that of the outer gear, and pen-point radius on the small gear that is one-third that of the outer gear. Graph the curve using either your graphing calculator or *Mathematica*TM.
- 6) Find parametric equations that will draw a 12-petal rose where all of the petals meet in the center of the graph, along with the range of θ needed to draw the complete graph. Graph the curve using either your graphing calculator or *Mathematica*TM.

- 7) Suppose that you want to draw a five-pointed “throwing” star, with arcs connecting the points as shown below.



- a) Find the parametric equations and the appropriate range of θ that will draw the star in a counterclockwise direction.
- b) Find the parametric equations and the appropriate range of θ that will draw the same star in a clockwise direction.
- 8) Suppose that the planet Krypton orbits its sun 100 million miles from the sun's center (counterclockwise from observed from “above”) once each 500 days, and Krypton's moon orbits Krypton once each day, also counterclockwise when observed from above Krypton's north pole, at a distance of 150,000 miles from the center of Krypton. Find the parametric equations that show the path of Krypton's moon about Krypton's sun, in terms of days t .
- 9) Suppose that a robotic arm in a factory of length 1.5 feet spins counterclockwise at one revolution per second, and that at the end of that arm is another robotic arm of the same length that spins counterclockwise about its axis at 3 revolutions per second. Find the parameter equations that give the position of the end of the second arm in terms of seconds t .
- 10) If a tractor with 52-inch diameter back tires is moving at 25 mph, how many rpm's are the back tires turning?
- 11) The moon has a diameter of 2,159 miles, while the earth has a diameter of 7,918 miles. The average center-to-center distance between the earth and the moon is 238,863 miles. The moon makes a complete orbit around the earth every 27.3 days. How fast is the moon moving in its orbit around the earth (in mph)?
- 12) Suppose that you buy a “tricked out” truck whose tire radii have been changed without adjusting the speedometer, and you want to put the original size tires back on the truck. You take the truck out on the interstate, and drive it at a constant “60 mph” according to the speedometer, but it only takes 45 seconds to drive between two mile markers. If the truck has 32-inch diameter wheels on it now, what was the truck's original tire size to which the speedometer is still set?

Chapter 4: Trigonometric Functions as Functions

MATH 117: Trigonometry

4.1 Introduction

We have previously defined and discussed all of the trigonometric functions without making light of one simple fact: they are functions! That is, for every angle we plug in to one of these things, we get one value out of them. In algebra, when we were given a function $f(x)$, we were able to evaluate it at any value x that we wanted, and then graph it. We only know the values for the trigonometric functions at very select values: multiples of 30° (or $\frac{\pi}{6}$ radians), and at multiples of 45° (or $\frac{\pi}{4}$ radians). See the table below with the values of the trigonometric functions for the acute angles and $\frac{\pi}{2}$. (The other non-acute angles would be plus/minus versions of these values, correct?)

θ	$\sin \theta$	$\cos \theta$	$\tan \theta$	$\cot \theta$	$\sec \theta$	$\csc \theta$
$0 = 0^\circ$	0	1	0	undef.	1	undef.
$\frac{\pi}{6} = 30^\circ$	$\frac{1}{2} = 0.5$	$\frac{\sqrt{3}}{2} \approx 0.886$	$\frac{1}{\sqrt{3}} \approx 0.557$	$\sqrt{3} \approx 1.732$	$\frac{2}{\sqrt{3}} \approx 1.155$	2
$\frac{\pi}{4} = 45^\circ$	$\frac{1}{\sqrt{2}} \approx 0.707$	$\frac{1}{\sqrt{2}} \approx 0.707$	1	1	$\sqrt{2} \approx 1.414$	$\sqrt{2} \approx 1.414$
$\frac{\pi}{3} = 60^\circ$	$\frac{\sqrt{3}}{2} \approx 0.886$	$\frac{1}{2} = 0.5$	$\sqrt{3} \approx 1.732$	$\frac{1}{\sqrt{3}} \approx 0.557$	2	$\frac{2}{\sqrt{3}} \approx 1.155$
$\frac{\pi}{2} = 90^\circ$	1	0	undef.	0	undef.	1

We can improve this to multiples of $\frac{\pi}{12} = 15^\circ$ by using the sum and difference formulas from Chapter 1. Note that $\frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}$, and recall that

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta \quad \text{and} \quad \cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta.$$

Then

$$\begin{aligned} \sin \frac{\pi}{12} &= \sin \left(\frac{\pi}{4} - \frac{\pi}{6} \right) = \sin \frac{\pi}{4} \cos \frac{\pi}{6} - \cos \frac{\pi}{4} \sin \frac{\pi}{6} \\ \sin \frac{\pi}{12} &= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2} = \frac{\sqrt{3} - 1}{2\sqrt{2}} \\ \sin \frac{\pi}{12} &= \frac{\sqrt{3} - 1}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{6} - \sqrt{2}}{4} \approx 0.259, \end{aligned}$$

and also

$$\begin{aligned}\cos \frac{\pi}{12} &= \cos \left(\frac{\pi}{4} - \frac{\pi}{6} \right) = \cos \frac{\pi}{4} \cos \frac{\pi}{6} + \sin \frac{\pi}{4} \sin \frac{\pi}{6} \\ \cos \frac{\pi}{12} &= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2} = \frac{\sqrt{3} + 1}{2\sqrt{2}} \\ \cos \frac{\pi}{12} &= \frac{\sqrt{3} + 1}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{6} + \sqrt{2}}{4} \approx 0.966.\end{aligned}$$

Then using the cofunction identities

$$\sin \theta = \cos \left(\frac{\pi}{2} - \theta \right) \quad \text{and} \quad \cos \theta = \sin \left(\frac{\pi}{2} - \theta \right),$$

we have that

$$\sin \frac{5\pi}{12} = \sin \left(\frac{\pi}{2} - \frac{\pi}{12} \right) = \cos \frac{\pi}{12} = \frac{\sqrt{6} + \sqrt{2}}{4} \approx 0.966$$

and

$$\cos \frac{5\pi}{12} = \cos \left(\frac{\pi}{2} - \frac{\pi}{12} \right) = \sin \frac{\pi}{12} = \frac{\sqrt{6} - \sqrt{2}}{4} \approx 0.259.$$

The rest of the trigonometric function values for these angles can be deduced from the sine and cosine values (left as an exercise). These values are plotted in Figure 1. They give a

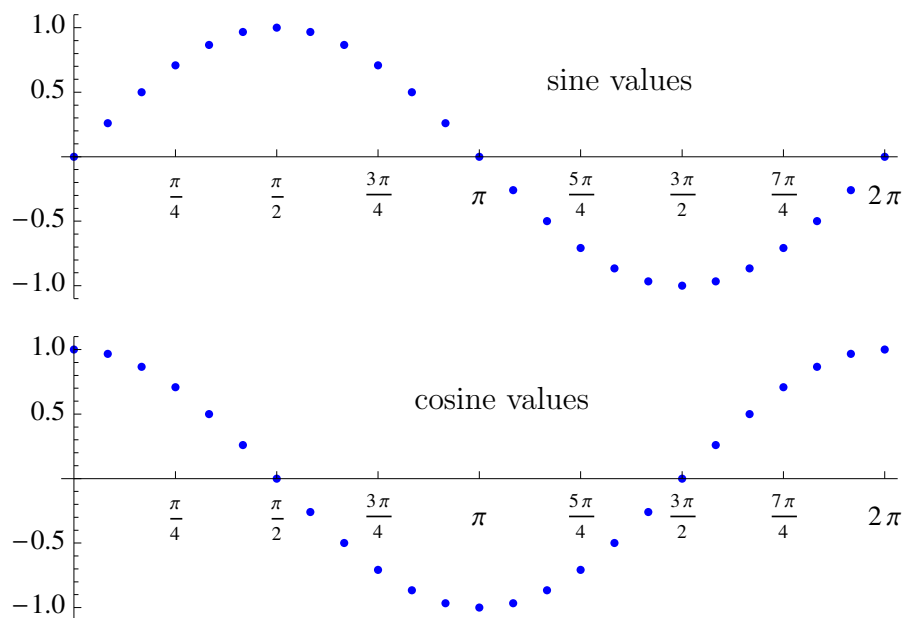


Figure 1

pretty good indication as to what the graphs of sine and cosine look like.

So we are left with a couple of questions. Can we calculate the values of things like $\sin 5^\circ$ exactly? And what do the graphs of all of the trigonometric values look like? We will try to tackle these questions in the following sections.

4.1 Exercises

1) Verify again that $\sin \frac{\pi}{12} = \sin 15^\circ = \frac{\sqrt{6} - \sqrt{2}}{4}$ using the difference formula for sine and the fact that $\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$.

2) Verify again that $\cos \frac{\pi}{12} = \cos 15^\circ = \frac{\sqrt{6} + \sqrt{2}}{4}$ using the difference formula for cosine and the fact that $\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$.

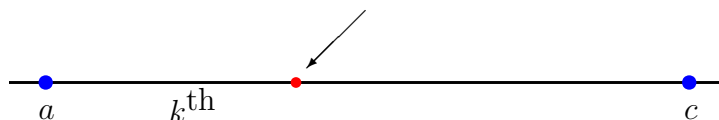
3) Verify again that $\sin \frac{5\pi}{12} = \sin 75^\circ = \frac{\sqrt{6} + \sqrt{2}}{4}$ using the sum formula for sine and the fact that $\frac{5\pi}{12} = \frac{\pi}{4} + \frac{\pi}{6}$.

4) Verify again that $\cos \frac{5\pi}{12} = \cos 75^\circ = \frac{\sqrt{6} - \sqrt{2}}{4}$ using the sum formula for cosine and the fact that $\frac{5\pi}{12} = \frac{\pi}{4} + \frac{\pi}{6}$.

5) Given the sine and cosine values for $\frac{\pi}{12}$ and $\frac{5\pi}{12}$ (equivalently, 15° and 75°), find the sine and cosine values for $\frac{7\pi}{12}$, $\frac{11\pi}{12}$, $\frac{13\pi}{12}$, $\frac{17\pi}{12}$, $\frac{19\pi}{12}$, and $\frac{23\pi}{12}$ (equivalently, 105° , 165° , 195° , 255° , 285° , and 345°).

6) In order to find the value that is k^{th} , $0 \leq k \leq 1$, of the way from a to c , we use the formula

$$a(1 - k) + ck. \quad (1)$$



(Notice that, for $k = 0$, the formula gives a , that for $k = 1$, the formula gives c , and that, for $k = \frac{1}{2}$, the formula gives $\frac{a+c}{2}$, the average of a and c .) Use formula (1) to find the following values.

a) $\frac{2}{3}$ of the way from 0 to $\frac{\pi}{12}$

b) $\frac{1}{3}$ of the way from $\frac{5\pi}{12}$ to $\frac{\pi}{2}$

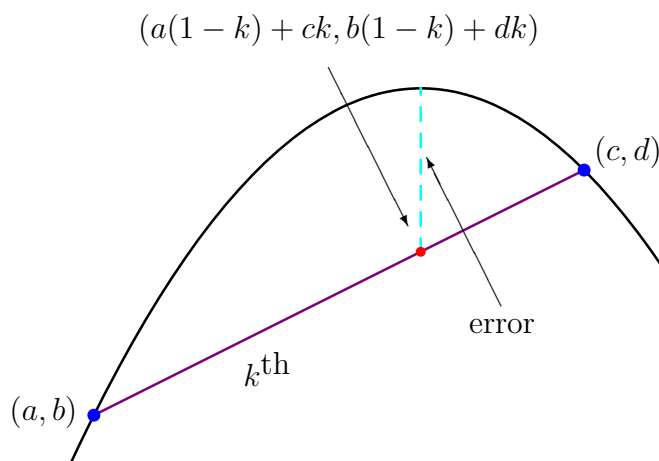
7) We may generalize the formula in (1) to points. In order to find the value that is k^{th} , $0 \leq k \leq 1$, of the way along the line segment from (a, b) to (c, d) , we use the formula

$$(a(1 - k) + ck, b(1 - k) + dk). \quad (2)$$

Use formula (2) to find the following values. In each case, plot the points on a graph to visually verify the results.

- a) $\frac{2}{3}$ of the way from $(0, 0)$ to $(3, 6)$ b) $\frac{1}{2}$ of the way from $\left(\frac{\pi}{6}, \frac{1}{2}\right)$ to $\left(\frac{\pi}{2}, 1\right)$

8) We may use formula (2) to approximate a function between known points on its graph.



a) Since $\sin 0 = 0$ and $\sin \frac{\pi}{12} = \frac{\sqrt{6} - \sqrt{2}}{4}$, we can approximate the value of $\sin \frac{\pi}{18}$ (or $\sin 10^\circ$) by finding the point on the line segment that is $\frac{2}{3}$ of the way from $(0, 0)$ to $\left(\frac{\pi}{12}, \frac{\sqrt{6} - \sqrt{2}}{4}\right)$, using equation (2). (You verified that $\frac{\pi}{18}$ was $\frac{2}{3}$ of the way from 0 to $\frac{\pi}{12}$ in problem 6a) above.) Compare the y -coordinate of your answer to your calculator's value for $\sin \frac{\pi}{18}$, and calculate the absolute value of the difference in the two results, rounded to four decimal places.

b) Since $\cos 0 = 1$ and $\cos \frac{\pi}{12} = \frac{\sqrt{6} + \sqrt{2}}{4}$, we can approximate the value of $\cos \frac{\pi}{18}$ (or $\cos 10^\circ$) by finding the point on the line segment that is $\frac{2}{3}$ of the way from $(0, 1)$ to $\left(\frac{\pi}{12}, \frac{\sqrt{6} + \sqrt{2}}{4}\right)$, using equation (2). Compare the y -coordinate of your answer to your calculator's value for $\cos \frac{\pi}{18}$, and calculate the absolute value of the difference in the two results, rounded to four decimal places.

c) Since $\sin \frac{5\pi}{12} = \frac{\sqrt{6} + \sqrt{2}}{4}$ and $\sin \frac{\pi}{2} = 1$, we can approximate the value of $\sin \frac{4\pi}{9}$ (or $\sin 80^\circ$) by finding the point on the line segment that is $\frac{1}{3}$ of the way from $\left(\frac{5\pi}{12}, \frac{\sqrt{6} + \sqrt{2}}{4}\right)$ to $\left(\frac{\pi}{2}, 1\right)$, using equation (2). (You verified that $\frac{4\pi}{9}$ was $\frac{1}{3}$ of the way from $\frac{5\pi}{12}$ to $\frac{\pi}{2}$ in problem 6b) above.) Compare the y -coordinate of your answer to your calculator's value for $\sin \frac{4\pi}{9}$, and calculate the absolute value of the difference in the two results, rounded to four decimal places.

d) Since $\cos \frac{5\pi}{12} = \frac{\sqrt{6} - \sqrt{2}}{4}$ and $\cos \frac{\pi}{2} = 0$, we can approximate the value of $\cos \frac{4\pi}{9}$ (or $\cos 80^\circ$) by finding the point on the line segment that is $\frac{1}{3}$ of the way from $\left(\frac{5\pi}{12}, \frac{\sqrt{6} - \sqrt{2}}{4}\right)$ to $\left(\frac{\pi}{2}, 0\right)$, using equation (2). Compare the y -coordinate of your answer to your calculator's value for $\cos \frac{4\pi}{9}$, and calculate the absolute value of the difference in the two results, rounded to four decimal places.

4.2 Sines, Cosines, and Tangents from the Unit Circle

In a previous chapter, we correctly defined the sine and cosine by taking a point (x, y) on the terminal ray of the angle θ at a distance of $r = \sqrt{x^2 + y^2}$ from the origin, and let

$$\sin \theta = \frac{y}{r} \quad \text{and} \quad \cos \theta = \frac{x}{r}.$$

This allowed us to use our reference triangles to calculate the different values for the sine and cosine. However, we can simplify this process by fixing the radius of all of our points to $r = 1$, so that all of our points fall on the unit circle. If we fix $r = 1$, then

$$\sin \theta = y \quad \text{and} \quad \cos \theta = x.$$

The Unit Circle

If we fix the radius to 1, then the x - and y - coordinates of points on the unit circle become the cosine and sine, respectively, of the angles whose terminal rays intersect the points, as shown in Figure 2.

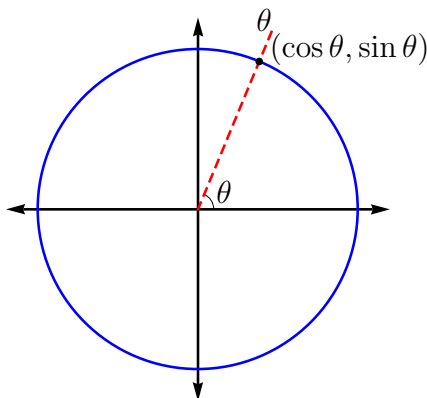


Figure 2

This is a great tool for helping to remember when the sine and cosine are positive or negative – we just have to remember in which quadrant the angle falls! Also, we just have to remember a few coordinates from the first quadrant (really just $0 \leq \theta \leq \frac{\pi}{4}$), and we can find the other points by reflecting across the axes and diagonals. The complete unit circle, showing the coordinates for all points on the circle at integer multiples of $\frac{\pi}{12} = 15^\circ$, appears in Figure 3. The animations **unitcircle.nb** and **unitcircle.mov** give the decimal x - and y -coordinates for points on the unit circle at integer degree increments.

Lastly, defining the sine and cosine in terms of the unit circle gives us a useful way of generating a continuous graph of both the sine and the cosine functions.

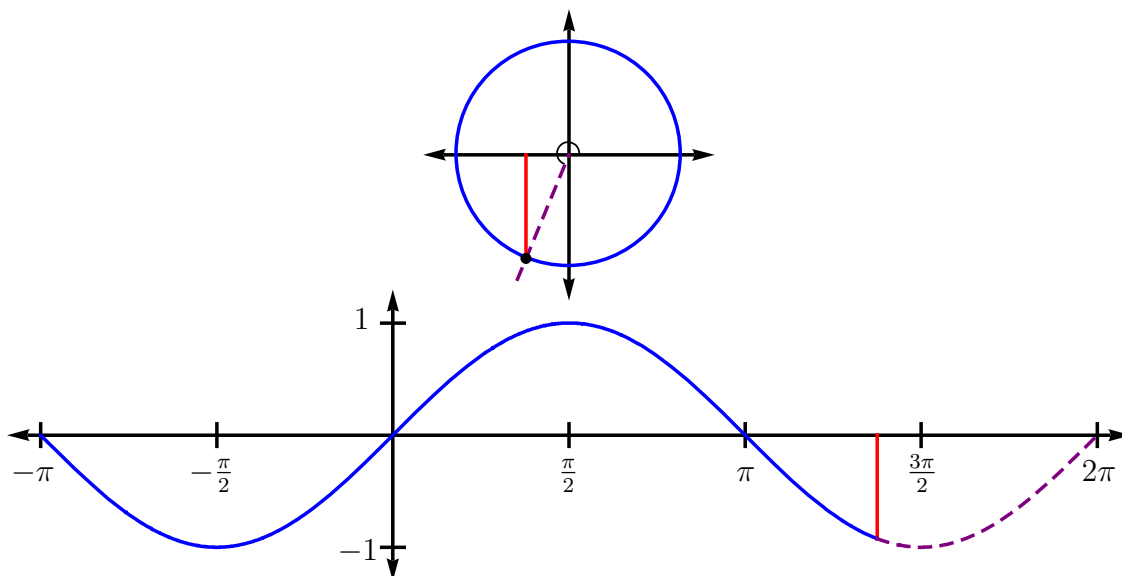


Figure 4: The sine function.

The Cosine of an Angle θ

In order to draw the cosine function, think about a point moving around the unit circle that corresponds to the angle (or arc length) θ , and plot the angle on the horizontal axes versus the x -value of the point on the vertical axes. The process is illustrated in Figure 5, and a full animation, **cosine.mov** or **cosine.nb**, of the point moving from $-\pi$ to 2π can be found on the class webpage.

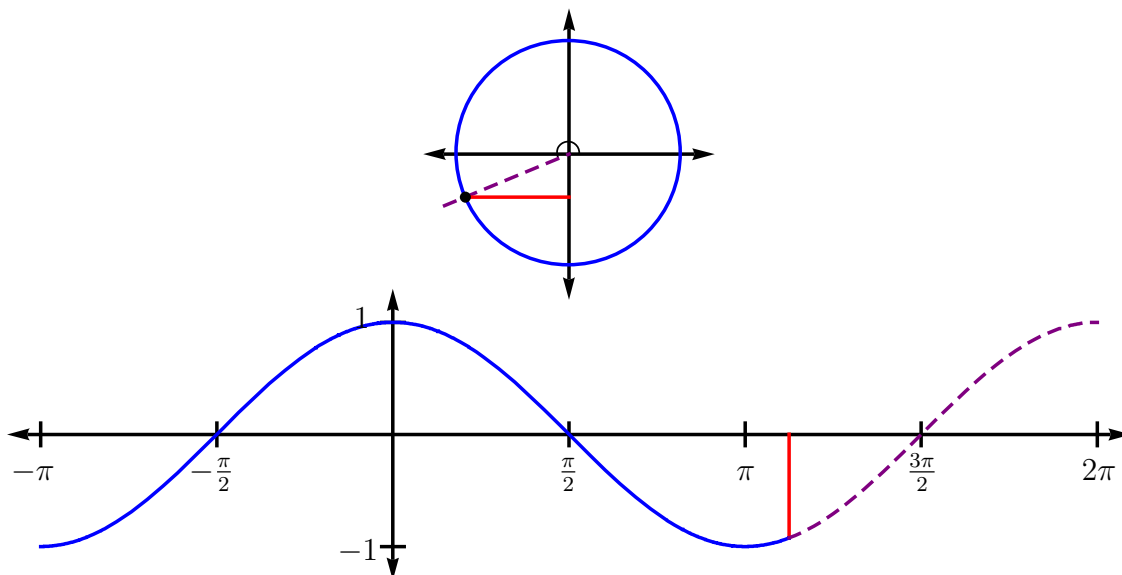


Figure 5: The cosine function.

As with the sine function, we see that the cosine function follows the footprints that we plotted in Figure 1, and that $\cos \theta$ is periodic with period 2π and amplitude 1.

The Tangent of an Angle θ

As mentioned at the beginning of this section, in the unit circle, we have

$$\sin \theta = y \quad \text{and} \quad \cos \theta = x.$$

Therefore, the slope of the line through the origin $(0, 0)$ and through the point (x, y) on the unit circle is

$$m = \frac{y - 0}{x - 0} = \frac{y}{x} = \frac{\sin \theta}{\cos \theta} = \tan \theta.$$

In order to draw the tangent function, think about a point moving around the unit circle that corresponds to the angle (or arc length) θ , and plot the angle on the horizontal axes versus the slope of the line through the origin and the point on the vertical axes. The process is illustrated in Figure 6, and a full animation, **tan1.mov** or **tan1.nb**, of the point moving from $-\pi$ to 2π can be found on the class webpage. Note that the concept of an amplitude is

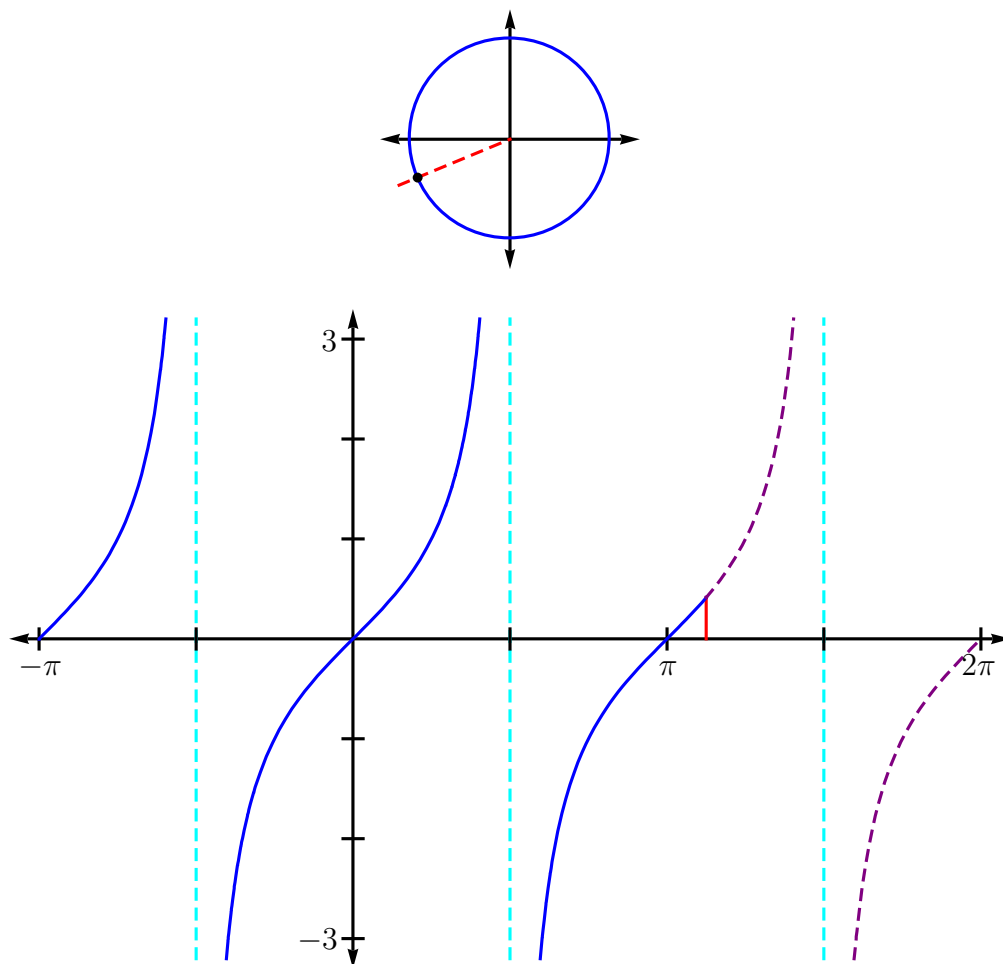
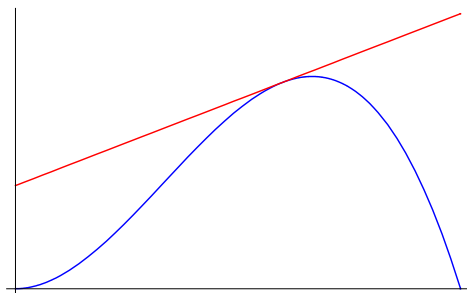


Figure 6: The tangent function.

no longer valid here, since the tangent ranges from $-\infty$ to ∞ . Also note that although the sine and the cosine have period 2π , the tangent has period π .

4.2 Exercises

- 1) The unit circle in Figure 3 gives us the exact sine and cosine values for angles of radian measure $0, \frac{\pi}{12}, \dots, \frac{\pi}{2}$. Find the exact tangent values for angles of radian measure $0, \frac{\pi}{12}, \dots, \frac{\pi}{2}$, if they exist.
- 2) Find the exact cotangent values for angles of radian measure $0, \frac{\pi}{12}, \dots, \frac{\pi}{2}$, if they exist.
- 3) Find the exact secant values for angles of radian measure $0, \frac{\pi}{12}, \dots, \frac{\pi}{2}$, if they exist.
- 4) Find the exact cosecant values for angles of radian measure $0, \frac{\pi}{12}, \dots, \frac{\pi}{2}$, if they exist.
- 5) The graphs of the sine and cosine functions in Figures 4 and 5 with respect to θ radians are drawn to scale, meaning that the distance from 0 to 1 is the same on both axes. A *tangent* line is a line at a point of a curve that reflects the general direction of the curve at that point. We do not know how to calculate the slope of a tangent line, but we do know how to calculate the slope of a line through the points (x_1, y_1) and (x_2, y_2) : $m = \frac{y_2 - y_1}{x_2 - x_1}$.

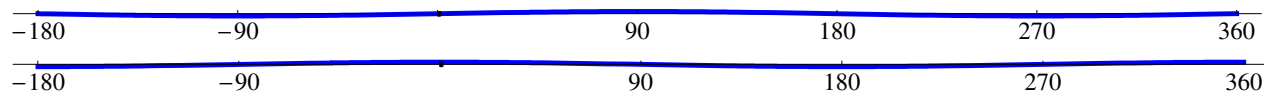


- a) Estimate the slope of the tangent line to $\sin \theta$ at $\theta = 0$ radians by finding the slope of the line through the points $(0, 0)$ and $(h, \sin h)$ for $h = 10^{-6}$ radians. Compare this value to the value of $\cos 0$.
- b) Estimate the slope of the tangent line to $\sin \theta$ at $\theta = \frac{\pi}{2}$ radians by finding the slope of the line through the points $\left(\frac{\pi}{2}, 1\right)$ and $\left(\frac{\pi}{2} + h, \sin\left(\frac{\pi}{2} + h\right)\right)$ for $h = 10^{-6}$ radians. Compare this value to the value of $\cos \frac{\pi}{2}$.
- c) Estimate the slope of the tangent line to $\sin \theta$ at $\theta = \pi$ radians by finding the slope of the line through the points $(\pi, 0)$ and $(\pi + h, \sin(\pi + h))$ for $h = 10^{-6}$ radians. Compare this value to the value of $\cos \pi$.
- d) Estimate the slope of the tangent line to $\cos \theta$ at $\theta = 0$ radians by finding the slope of the line through the points $(0, 1)$ and $(h, \cos h)$ for $h = 10^{-6}$ radians. Compare this value to the value of $\sin 0$.

e) Estimate the slope of the tangent line to $\cos \theta$ at $\theta = \frac{\pi}{2}$ radians by finding the slope of the line through the points $\left(\frac{\pi}{2}, 0\right)$ and $\left(\frac{\pi}{2} + h, \cos\left(\frac{\pi}{2} + h\right)\right)$ for $h = 10^{-6}$ radians. Compare this value to the value of $\sin \frac{\pi}{2}$.

f) Estimate the slope of the tangent line to $\cos \theta$ at $\theta = -\frac{\pi}{2}$ radians by finding the slope of the line through the points $\left(-\frac{\pi}{2}, 0\right)$ and $\left(-\frac{\pi}{2} + h, \cos\left(-\frac{\pi}{2} + h\right)\right)$ for $h = 10^{-6}$ radians. Compare this value to the value of $\sin\left(-\frac{\pi}{2}\right)$.

6) The graphs of the sine and cosine functions, shown below at top and at bottom, respectively, with respect to ϕ degrees, are drawn to scale.



a) Estimate the slope of the tangent line to $\sin \phi$, where ϕ is in degrees, at $\phi = 0^\circ$ by finding the slope of the line through the points $(0, 0)$ and $(h, \sin h)$ for $h = 10^{-6}^\circ$. Compare this value to the value of $\cos 0^\circ$.

b) Estimate the slope of the tangent line to $\cos \phi$, where ϕ is in degrees, at $\phi = 90^\circ$ by finding the slope of the line through the points $(90, 0)$ and $(90 + h, \cos(90 + h))$ for $h = 10^{-6}^\circ$. Compare this value to the value of $\sin 90^\circ$.

4.3 Combining the Functions

Curious things begin to happen when we start adding, subtracting, multiplying and dividing the sine and cosine functions.

Identity for the Sine of a Doubled Angle

Suppose that we multiply the sine and cosine functions, by multiplying the individual function values at each value of θ . This process is illustrated in Figure 7, and a full animation, **sinecosine.nb** or **sinecosine.mov**, of the process with θ ranging from $-\pi$ to 2π can be found on the class webpage. Notice that the resulting function

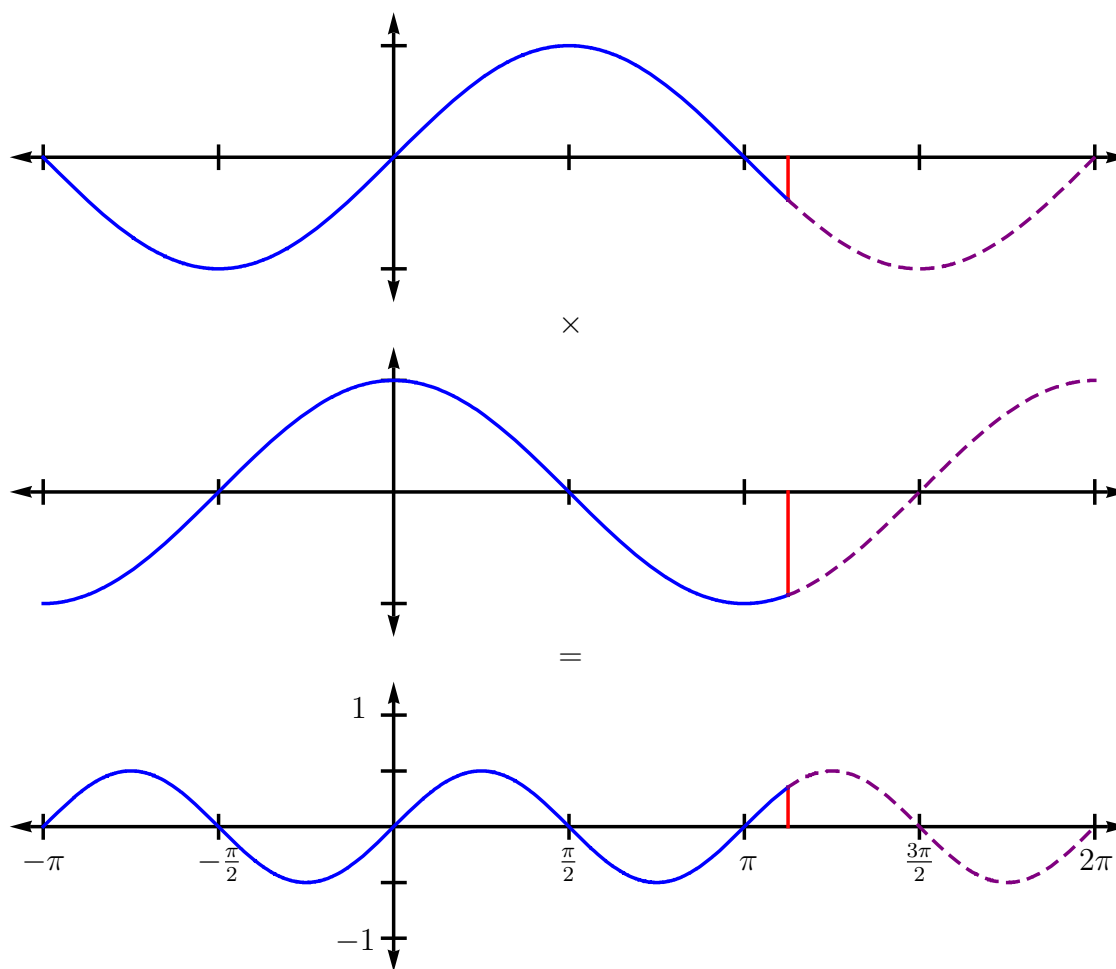


Figure 7: The values from the sine function at top are multiplied by the values of the cosine function in the middle, resulting in the periodic function at the bottom.

- is 0 whenever either of the sine or cosine functions is 0,
- has maximums and minimums when the sine and cosine are $\pm \frac{1}{\sqrt{2}}$, and

- follows the sign rules for multiplication; that is, positive when both have the same sign, and negative when the signs are different.

If the resulting curve looks familiar, it should. It is a sine function, like the one at the top of Figure 7, although with a period of π instead of 2π , and with amplitude $\frac{1}{2}$ instead of 1. In algebra, we learned that the graph of a function

$$af(cx)$$

is just the graph of $f(x)$ stretched horizontally by factor of $\frac{1}{c}$ and vertically by a factor of a . Since the amplitude of the new curve is $\frac{1}{2}$, then in this case, $a = \frac{1}{2}$. Also, since the new period is π , then in this case,

$$\pi = \frac{1}{c} \cdot 2\pi,$$

and so,

$$c = \frac{2\pi}{\pi} = 2.$$

Thus, the formula for our new curve would be

$$\frac{1}{2} \sin(2\theta),$$

and we have a new identity for the sine of twice an angle:

$$\sin(2\theta) = 2 \sin \theta \cos \theta. \quad (3)$$

We can verify equation (3) algebraically, using our sum formula for sine:

$$\sin(2\theta) = \sin(\theta + \theta) = \sin \theta \cos \theta + \cos \theta \sin \theta$$

$$\sin(2\theta) = 2 \sin \theta \cos \theta.$$

Time Out!

The following observation has nothing to do with the above identities, but everything to do with the sine and cosine functions if you proceed on to calculus. Looking at Figure 7 where the top curve is the sine function and the middle curve is the cosine function, did you notice that the sine has maximums and minimums at values of θ where the cosine function is zero, and vice versa? Also, the graphs are given in terms of radians, and drawn to scale. A **tangent** line is a line at a point of a curve that reflects the general direction of the curve at that point. Did you notice that the slope of a tangent line on the sine function is equal to the value of the cosine function, and that the slope of a tangent line on the cosine is the opposite of the value of the sine function? This only happens when we consider the sine and cosine functions in terms of radians, another reason to use them in our work.

Time In!

Example: Verify that $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$ using the double-angle identity for sine.

Since $\frac{1}{2} \cdot \frac{\pi}{3} = \frac{\pi}{6}$, then we can use the sine and cosine values of this angle in our formula:

$$2 \sin \frac{\pi}{6} \cos \frac{\pi}{6} = 2 \left(\frac{1}{2} \right) \left(\frac{\sqrt{3}}{2} \right) = \frac{\sqrt{3}}{2}.$$

Example: Verify that $\sin \frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$ using the double-angle identity for sine.

Since $\frac{1}{2} \cdot \frac{4\pi}{3} = \frac{2\pi}{3}$, then we can use the sine and cosine values of this angle in our formula:

$$2 \sin \frac{2\pi}{3} \cos \frac{2\pi}{3} = 2 \left(\frac{\sqrt{3}}{2} \right) \left(-\frac{1}{2} \right) = -\frac{\sqrt{3}}{2}.$$

Identity for the Cosine of a Doubled Angle

We can find other combinations of the sine and cosine functions that will yield other periodic functions. If we take the cosine function values (shown as a dashed curve at top in Figure 8) and square them, we get the nonnegative curve with period π shown at the top of Figure 8. Likewise, if we square the sine function values (shown as a dashed curve in the middle in Figure 8), we get a similar nonnegative curve with period π shown in the middle of Figure 8. The bottom curve shows the top values minus the middle values. A full animation, **squaresdiff.nb** or **squaresdiff.mov**, of the process is available on the class webpage.

Again, this curve should look familiar. It is a cosine function with amplitude 1, but with period π , or $\cos(2\theta)$. We have basically shown graphically that $\cos(2\theta)$ is the difference of $\cos^2 \theta$ and $\sin^2 \theta$, although we can prove it algebraically using the sum identity for cosines:

$$\begin{aligned} \cos(2\theta) &= \cos(\theta + \theta) = \cos \theta \cos \theta - \sin \theta \sin \theta \\ \cos(2\theta) &= \cos^2 \theta - \sin^2 \theta. \end{aligned} \tag{4}$$

Using the Pythagorean identities, we can get a couple of other variations of this double-angle identity. Since

$$\sin^2 \theta + \cos^2 \theta = 1$$

for all θ , then

$$\cos^2 \theta = 1 - \sin^2 \theta \quad \text{and} \quad \sin^2 \theta = 1 - \cos^2 \theta,$$

so

$$\cos(2\theta) = (1 - \sin^2 \theta) - \sin^2 \theta = 1 - 2\sin^2 \theta \tag{5}$$

and

$$\cos(2\theta) = \cos^2 \theta - (1 - \cos^2 \theta) = 2\cos^2 \theta - 1. \tag{6}$$

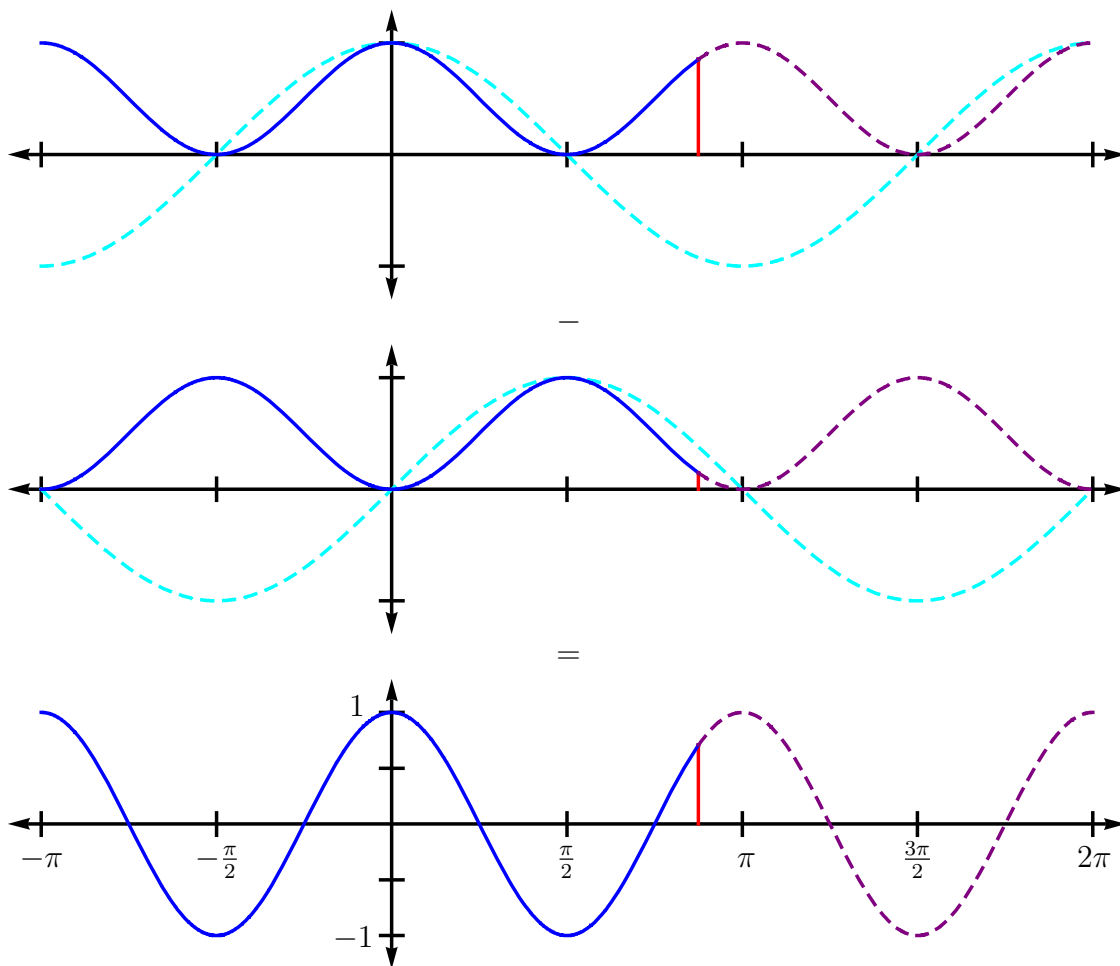


Figure 8: The values from the cosine function at top are squared, as are the sine function values in the middle. The top values minus the middle values give the resulting periodic curve at bottom.

The equations in (3), (4), (5), and (6) are known as the **double-angle** identities for sine and cosine.

Example: Verify that $\cos \frac{\pi}{3} = \frac{1}{2}$ using all of the versions of the double-angle identity for cosine.

Since $\frac{1}{2} \cdot \frac{\pi}{3} = \frac{\pi}{6}$, then we can use the sine and cosine values of this angle in our formulas:

$$\begin{aligned} \left(\cos \frac{\pi}{6}\right)^2 - \left(\sin \frac{\pi}{6}\right)^2 &= \left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{1}{2}\right)^2 = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}, \\ 2\left(\cos \frac{\pi}{6}\right)^2 - 1 &= 2\left(\frac{\sqrt{3}}{2}\right)^2 - 1 = 2\left(\frac{3}{4}\right) - 1 = \frac{3}{2} - 1 = \frac{1}{2}, \text{ and} \end{aligned}$$

$$1 - 2 \left(\sin \frac{\pi}{6} \right)^2 = 1 - 2 \left(\frac{1}{2} \right)^2 = 1 - 2 \left(\frac{1}{4} \right) = 1 - \frac{1}{2} = \frac{1}{2}.$$

Example: Verify that $\cos \frac{4\pi}{3} = -\frac{1}{2}$ using all of the versions of the double-angle identity for cosine.

Since $\frac{1}{2} \cdot \frac{4\pi}{3} = \frac{2\pi}{3}$, then we can use the sine and cosine values of this angle in our formulas:

$$\begin{aligned} \left(\cos \frac{2\pi}{3} \right)^2 - \left(\sin \frac{2\pi}{3} \right)^2 &= \left(-\frac{1}{2} \right)^2 - \left(\frac{\sqrt{3}}{2} \right)^2 = \frac{1}{4} - \frac{3}{4} = -\frac{1}{2}, \\ 2 \left(\cos \frac{2\pi}{3} \right)^2 - 1 &= 2 \left(-\frac{1}{2} \right)^2 - 1 = 2 \left(\frac{1}{4} \right) - 1 = \frac{1}{2} - 1 = -\frac{1}{2}, \text{ and} \\ 1 - 2 \left(\sin \frac{2\pi}{3} \right)^2 &= 1 - 2 \left(\frac{\sqrt{3}}{2} \right)^2 = 1 - 2 \left(\frac{3}{4} \right) = 1 - \frac{3}{2} = -\frac{1}{2}. \end{aligned}$$

We can use equations (5) and (6) to develop **half-angle** identities for sine and cosine. Solving equation (5) for $\sin^2 \theta$, we get

$$\sin^2 \theta = \frac{1 - \cos(2\theta)}{2},$$

or equivalently,

$$\sin^2 \frac{\theta}{2} = \frac{1 - \cos \theta}{2}. \quad (7)$$

Solving equation (6) for $\cos^2 \theta$, we get

$$\cos^2 \theta = \frac{1 + \cos(2\theta)}{2},$$

or equivalently,

$$\cos^2 \frac{\theta}{2} = \frac{1 + \cos \theta}{2}. \quad (8)$$

The identities in (7) and (8) are also sometimes called the **power-reducing** identities, since we can replace the squared trigonometric functions with non-squared cosines. When actually solving for $\sin \frac{\theta}{2}$ or $\cos \frac{\theta}{2}$, taking the square root of each side will generate both a positive and negative answer. We must take care to select the answer with the correct sign based upon which quadrant the angle lies.

Example: Find the sine and cosine values of $\frac{\pi}{8} = 22.5^\circ$.

Since $2 \left(\frac{\pi}{8} \right) = \frac{\pi}{4}$, then we can use the cosine value of this angle in our formulas. Using the half-angle identity for sine, we get

$$\sin^2 \frac{\pi}{8} = \frac{1 - \cos \frac{\pi}{4}}{2} = \frac{1 - \frac{\sqrt{2}}{2}}{2} = \frac{2 - \sqrt{2}}{2} \cdot \frac{1}{2} = \frac{2 - \sqrt{2}}{4}.$$

Since $\frac{\pi}{8}$ is in the first quadrant, then

$$\sin \frac{\pi}{8} = \frac{\sqrt{2 - \sqrt{2}}}{2} \approx 0.3827.$$

Using the half-angle identity for cosine, we get

$$\cos^2 \frac{\pi}{8} = \frac{1 + \cos \frac{\pi}{4}}{2} = \frac{1 + \frac{\sqrt{2}}{2}}{2} = \frac{2 + \sqrt{2}}{2} \cdot \frac{1}{2} = \frac{2 + \sqrt{2}}{4}.$$

Since $\frac{\pi}{8}$ is in the first quadrant, then

$$\cos \frac{\pi}{8} = \frac{\sqrt{2 + \sqrt{2}}}{2} \approx 0.9239.$$

You may verify both of these exact solutions using your calculator's values for $\sin \frac{\pi}{8}$ and $\cos \frac{\pi}{8}$.

4.3 Exercises

1) Verify that $\sin \frac{\pi}{2} = \sin 90^\circ = 1$ using the double-angle identity for sine and the sine and cosine function values for $\frac{\pi}{4} = 45^\circ$.

2) Verify that $\cos \frac{\pi}{2} = \cos 90^\circ = 0$ using all of the double-angle identities for cosine and the sine and cosine function values for $\frac{\pi}{4} = 45^\circ$.

3) Verify that $\sin \frac{\pi}{4} = \sin 45^\circ = \frac{\sqrt{2}}{2}$ using the half-angle identity for sine and the cosine of $\frac{\pi}{2} = 90^\circ$.

4) Verify that $\cos \frac{\pi}{4} = \cos 45^\circ = \frac{\sqrt{2}}{2}$ using the half-angle identity for cosine and the cosine of $\frac{\pi}{2} = 90^\circ$.

5) Verify that $\sin \frac{\pi}{12} = \sin 15^\circ = \frac{\sqrt{6} - \sqrt{2}}{4}$ using the half-angle identity for sine and the cosine of $\frac{\pi}{6} = 30^\circ$.

6) Verify that $\cos \frac{\pi}{12} = \cos 15^\circ = \frac{\sqrt{6} + \sqrt{2}}{4}$ using the half-angle identity for cosine and the cosine of $\frac{\pi}{6} = 30^\circ$.

7) Verify that $\sin \frac{\pi}{6} = \sin 30^\circ = \frac{1}{2}$ using the double-angle identity for sine and the sine and cosine function values for $\frac{\pi}{12} = 15^\circ$.

8) Verify that $\cos \frac{\pi}{6} = \cos 30^\circ = \frac{\sqrt{3}}{2}$ using all of the double-angle identities for cosine and the sine and cosine function values for $\frac{\pi}{12} = 15^\circ$.

9) Verify that $\sin \frac{5\pi}{12} = \sin 75^\circ = \frac{\sqrt{6} + \sqrt{2}}{4}$ using the half-angle identity for sine and the cosine of $\frac{5\pi}{6} = 150^\circ$.

10) Verify that $\cos \frac{5\pi}{12} = \cos 75^\circ = \frac{\sqrt{6} - \sqrt{2}}{4}$ using the half-angle identity for cosine and the cosine of $\frac{5\pi}{6} = 150^\circ$.

11) Find the amplitude and period (in radians) of the following sine and cosine functions. Verify your answers graphically.

- | | |
|---|---|
| a) $3 \sin(5\theta)$ | b) $\frac{\cos(4\theta)}{2}$ |
| c) $\cos\left(\frac{3}{2}\theta\right)$ | d) $2 \sin\left(\frac{4\theta}{5}\right)$ |
| e) $\frac{1}{2} \sin(2\pi\theta)$ | f) $\frac{\pi}{2} \cos(\pi\theta)$ |

12) Find the period (in radians) of the following tangent functions. Verify your answers graphically.

- | | |
|---|--|
| a) $3 \tan(4\theta)$ | b) $\tan\left(\frac{\theta}{6}\right)$ |
| c) $\tan\left(\frac{3}{2}\theta\right)$ | d) $2 \tan(\pi\theta)$ |

13) Use the double-angle identities for sine and cosine to develop a double-angle identity for $\tan(2\theta)$ in terms of $\tan \theta$.

14) Use the half-angle identities for sine and cosine to develop a half-angle identity for $\tan^2 \frac{\theta}{2}$ in terms of $\cos \theta$.

4.4 Graphs of the Tangent and Cotangent Functions

Since the tangent is the ratio of the sine to the cosine – that is,

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

for all values of θ where $\cos \theta \neq 0$ – we can actually use the graphs of the sine function and cosine function to generate the graph of the tangent function. Likewise with the cotangent, since the cotangent is the ratio of the cosine to the sine – that is,

$$\cot \theta = \frac{\cos \theta}{\sin \theta}$$

for all values of θ where $\sin \theta \neq 0$.

The Tangent from the Sine and Cosine

We have already generated the graph of the tangent function (see Figure 6) using the unit circle. If we take the function values for the sine function and divide them by the function values for the cosine function, then the resulting curve is again the graph of the tangent function. This process is illustrated in Figure 9, and a full animation, **tan2.nb** or **tan2.mov**, is available on the class website. Notice that the resulting function

- is 0 wherever the sine function is 0,
- is undefined wherever the cosine function is 0, and
- obeys the sign rules; that is, it is positive where the sine and cosine functions have the same sign, and is negative when the signs are different.

The tangent function is a completely different brand of trigonometric function. While the sine and cosine are continuous and defined for all values of θ , the tangent is discontinuous and even undefined in places. Also, while the sine and cosine are each 2π -periodic, the tangent function is π -periodic. The sine and cosine functions increase and decrease in value, while the tangent function is always increasing from left to right. And finally, there is no use talking about the “amplitude” of the tangent function, since it shoots off to both ∞ and $-\infty$.

The Cotangent from the Sine and Cosine

If we reverse the numerator and denominator from the previous example, and take the function values for the cosine function and divide them by the function values for the sine function, then the resulting curve is the graph of the cotangent function. This process is illustrated in Figure 10, and a full animation, **cot.nb** or **cot.mov**, is available on the class website. Notice that the resulting function

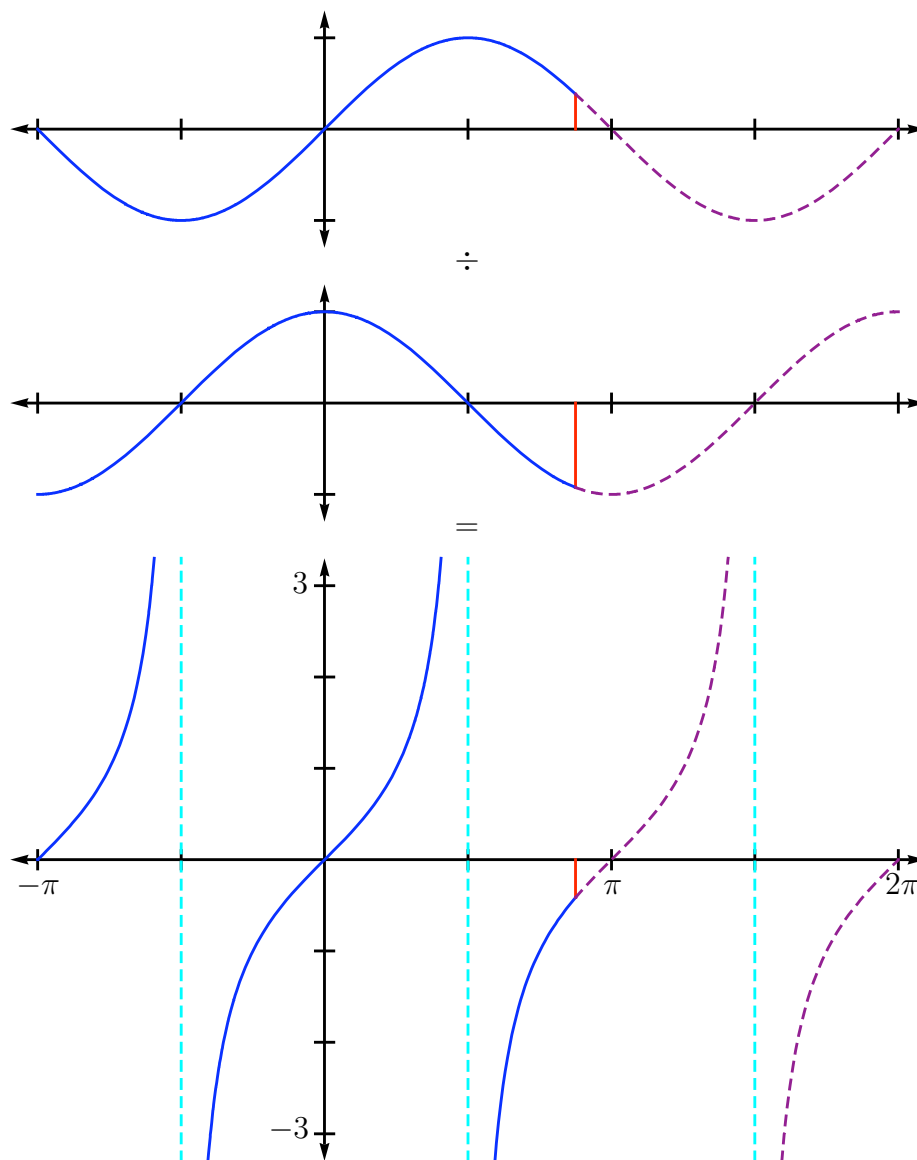


Figure 9: The values from the sine function at top are divided by the values from the cosine function in the middle to generate the graph of the tangent function.

- is 0 wherever the cosine function is 0,
- is undefined wherever the sine function is 0, and
- obeys the sign rules; that is, it is positive where the top and middle functions have the same sign, and is negative when the signs are different.

The cotangent function is like the tangent function, with some notable differences. It is, like the tangent, π -periodic and unbounded in value, but instead of always increasing like the tangent, the cotangent is always decreasing. When graphing the cotangent function on a graphing calculator, it is important to note that most graphing calculators do not have a “COT” button. We have two options:

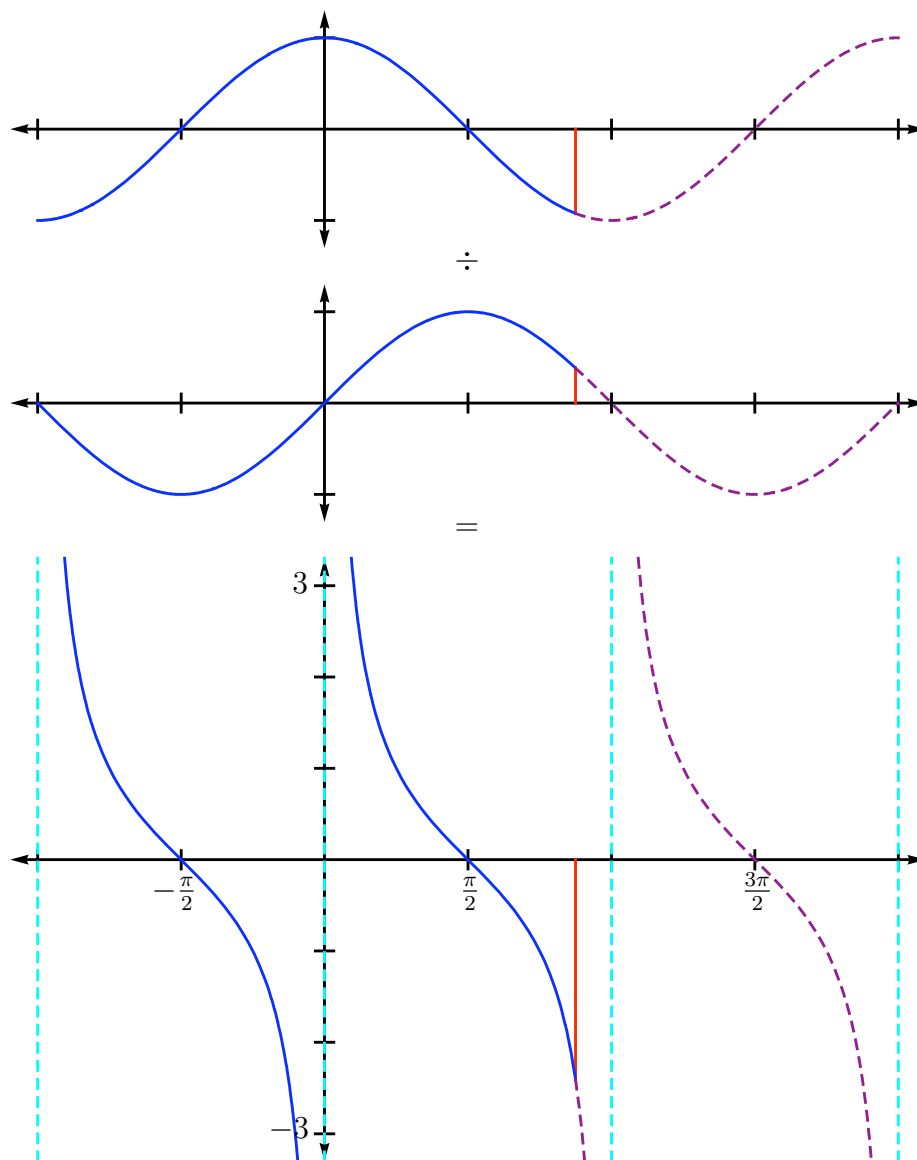


Figure 10: The values from the cosine function at top are divided by the values from the sine function in the middle to generate the graph of the cotangent function.

- the “COT” function is available in the catalog of the calculator, or
- remember that $\cot \theta = \frac{1}{\tan \theta}$ where defined.

4.4 Exercises

In problems 1) through 10), graph each pair of curves, with θ measured in radians, and determine whether the functions are identically equal. If you determine that $f(\theta) = g(\theta)$, then verify the identity algebraically by making the one function look like the other one.

1) $f(\theta) = \tan(-\theta)$, $g(\theta) = -\tan \theta$

2) $f(\theta) = \cot(-\theta)$, $g(\theta) = \cot \theta$

3) $f(\theta) = \tan \theta$, $g(\theta) = \frac{1}{\tan(\frac{\pi}{2} - \theta)}$

4) $f(\theta) = \cot \theta$, $g(\theta) = \frac{1}{\tan \theta}$

5) $f(\theta) = \tan(2\theta)$, $g(\theta) = \frac{2 \tan \theta}{1 + \tan^2 \theta}$

6) $f(\theta) = 1 - \tan^2 \theta$, $g(\theta) = \frac{2 \tan \theta}{\tan(2\theta)}$

7) $f(\theta) = \tan^2 \theta$, $g(\theta) = \frac{1 - \cos(2\theta)}{1 + \cos(2\theta)}$

8) $f(\theta) = \cot^2 \theta$, $g(\theta) = \frac{1 - \sin(2\theta)}{1 + \sin(2\theta)}$

9) $f(\theta) = \tan\left(\theta + \frac{\pi}{4}\right)$, $g(\theta) = \frac{1 + \tan \theta}{1 - \tan \theta}$

10) $f(\theta) = \tan\left(\theta - \frac{\pi}{4}\right)$, $g(\theta) = \frac{\tan \theta - 1}{1 + \tan \theta}$

11) The graphs of the tangent and cotangent functions in Figures 9 and 10 with respect to θ radians are drawn to scale, meaning that the distance from 0 to 1 is the same on both axes.

a) Estimate the slope of the tangent line to the graph of $\tan \theta$ at $\theta = 0$ radians by finding the slope of the line through the points $(0, 0)$ and $(h, \tan h)$ for $h = 10^{-6}$ radians.

b) Estimate the slope of the tangent line to the graph of $\tan \theta$ at $\theta = \frac{\pi}{4}$ radians by finding the slope of the line through the points $\left(\frac{\pi}{4}, 1\right)$ and $\left(\frac{\pi}{4} + h, \tan\left(\frac{\pi}{4} + h\right)\right)$ for $h = 10^{-6}$ radians.

c) Estimate the slope of the tangent line to the graph of $\cot \theta$ at $\theta = \frac{\pi}{2}$ radians by finding the slope of the line through the points $\left(\frac{\pi}{2}, 0\right)$ and $\left(\frac{\pi}{2} + h, \cot\left(\frac{\pi}{2} + h\right)\right)$ for $h = 10^{-6}$ radians.

d) Estimate the slope of the tangent line to the graph of $\cot \theta$ at $\theta = \frac{\pi}{4}$ radians by finding the slope of the line through the points $\left(\frac{\pi}{4}, 1\right)$ and $\left(\frac{\pi}{4} + h, \cot\left(\frac{\pi}{4} + h\right)\right)$ for $h = 10^{-6}$ radians.

4.5 The Graphs of the Secant and Cosecant Functions

The tangent and cotangent functions resembled each other, but looked nothing like the sine and cosine. Next, we will develop the graphs of the secant and cosecant functions, which, as we will find, look nothing like any of the previous trigonometric functions (although they will resemble each other.) Since the secant is the reciprocal of the cosine – that is,

$$\sec \theta = \frac{1}{\cos \theta}$$

for all θ such that $\cos \theta \neq 0$ – we may generate its graph from the graph of the cosine function. Likewise, since the cosecant is the reciprocal of the sine – that is,

$$\csc \theta = \frac{1}{\sin \theta}$$

for all θ such that $\sin \theta \neq 0$ – we may generate its graph from the graph of the sine function.

The Secant from the Cosine Function

If we take the reciprocal of the function values for the cosine function, then the resulting curve is the graph of the secant function. This process is illustrated in Figure 11, and a full animation, **sec.nb** or **sec.mov**, is available on the class website. Notice that the resulting function

- is never 0, and is actually never between -1 and 1 ,
- is undefined wherever the cosine function is 0, and
- is positive where the cosine function is positive, and is negative where the cosine is negative.

The secant function is 2π -periodic like the cosine function, but while the cosine function is bounded (by -1 and 1), the secant is unbounded in value. The secant has vertical asymptotes every π units in the same locations as the tangent, but while the tangent is always increasing, the secant switches back and forth between increasing and decreasing.

When graphing the secant function on a graphing calculator, it is important to note that most graphing calculators do not have a “SEC” button. We have two options:

- the “SEC” function is available in the catalog of the calculator, or
- remember that $\sec \theta = \frac{1}{\cos \theta}$ where defined.

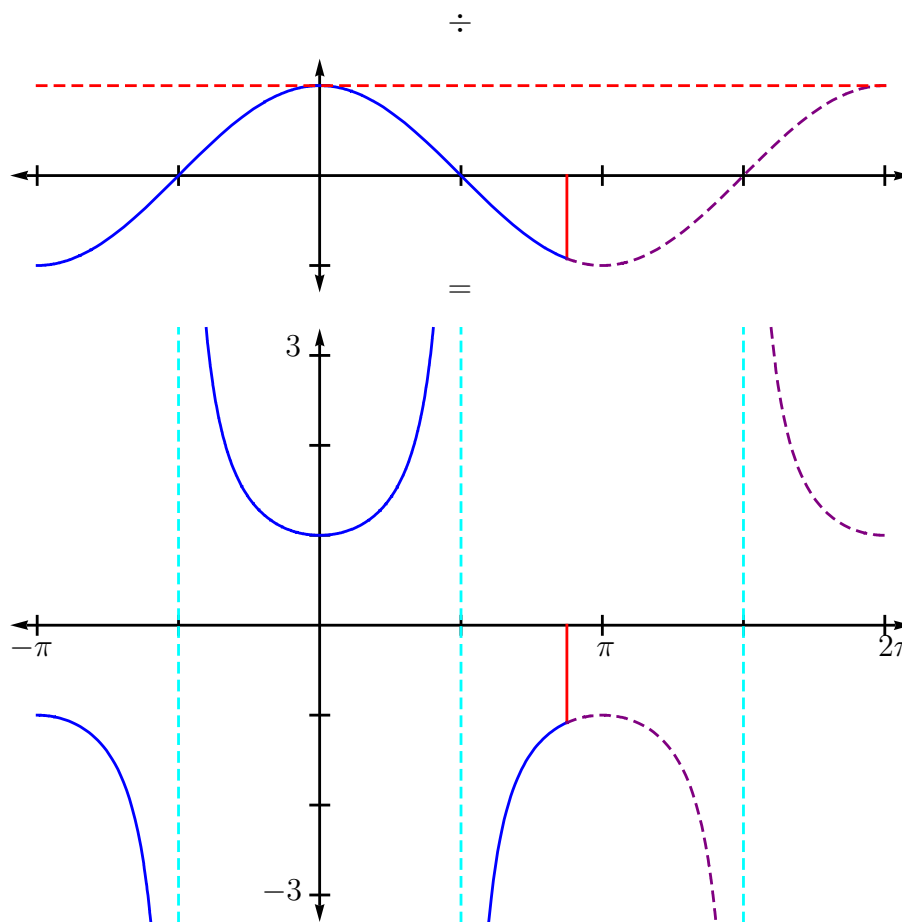


Figure 11: The red dashed line is the constant function 1. Dividing this function by the values from the cosine function at top gives the graph of the secant function.

The Cosecant from the Sine

If we take the reciprocal of the function values for the sine function, then the resulting curve is the graph of the cosecant function. This process is illustrated in Figure 12, and a full animation, **csc.nb** or **csc.mov**, is available on the class website. Notice that the resulting function

- is never 0, and is actually never between -1 and 1 ,
- is undefined wherever the sine function is 0, and
- is positive where the sine function is positive, and is negative where the sine is negative.

The cosecant function is 2π -periodic like the sine function, but while the sine function is bounded (by -1 and 1), the cosecant is unbounded in value. The secant has vertical asymptotes every π units in the same locations as the cotangent, but while the cotangent is always decreasing, the cosecant switches back and forth between increasing and decreasing.

When graphing the cosecant function on a graphing calculator, it is important to note that most graphing calculators do not have a “CSC” button. We have two options:

- the “CSC” function is available in the catalog of the calculator, or

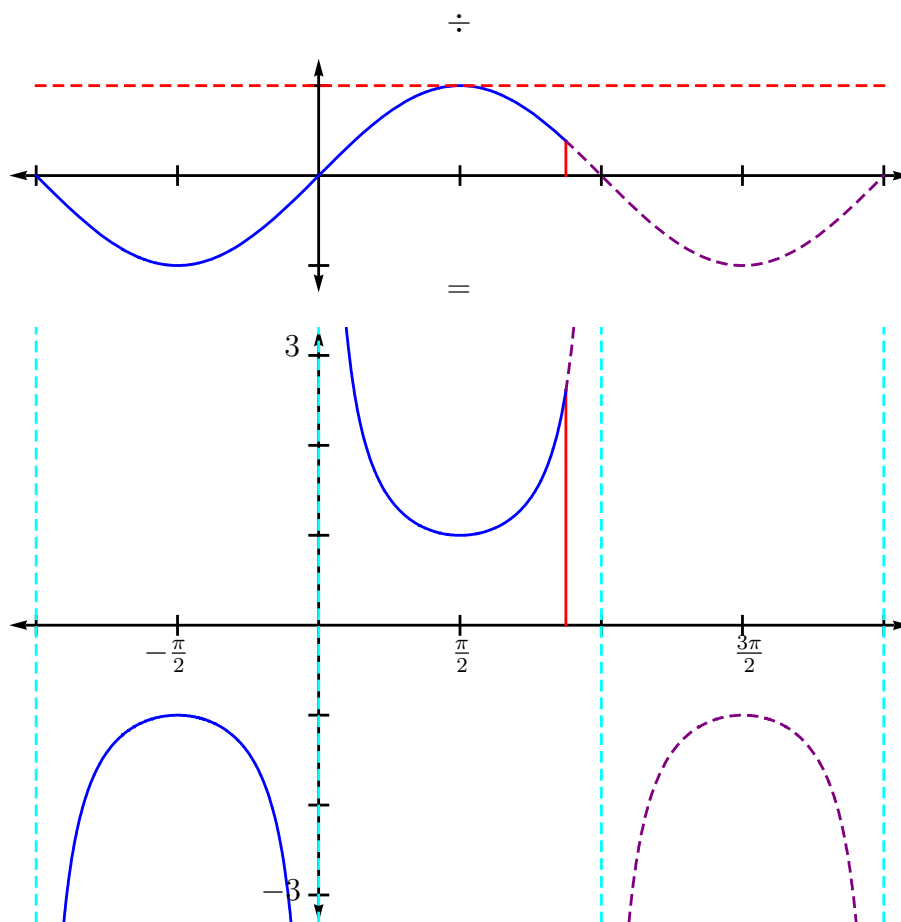


Figure 12: The red dashed line is the constant function 1. Dividing this function by the values from the sine function at top gives the graph of the cosecant function.

- remember that $\csc \theta = \frac{1}{\sin \theta}$ where defined.

4.5 Exercises

In problems 1) through 10), graph each pair of curves, with θ measured in radians, and determine whether the functions are identically equal. If you determine that $f(\theta) = g(\theta)$, then verify the identity algebraically by making the one function look like the other one.

- 1) $f(\theta) = \sec(-\theta)$, $g(\theta) = \sec \theta$
- 2) $f(\theta) = \csc(-\theta)$, $g(\theta) = \csc \theta$
- 3) $f(\theta) = \sec \theta$, $g(\theta) = \frac{1}{\sin(\frac{\pi}{2} - \theta)}$
- 4) $f(\theta) = \csc \theta$, $g(\theta) = \frac{1}{\sin \theta}$
- 5) $f(\theta) = \sec(2\theta)$, $g(\theta) = \frac{1}{\cos^2 \theta - \sin^2 \theta}$
- 6) $f(\theta) = \sec \theta$, $g(\theta) = \frac{2 \csc(2\theta)}{\csc \theta}$
- 7) $f(\theta) = \sec^2 \theta$, $g(\theta) = \tan^2 \theta + 1$
- 8) $f(\theta) = \csc^2 \theta$, $g(\theta) = \frac{2}{1 - \cos(2\theta)}$
- 9) $f(\theta) = \sec\left(\theta + \frac{\pi}{2}\right)$, $g(\theta) = \csc \theta$
- 10) $f(\theta) = \sec\left(\theta + \frac{\pi}{2}\right)$, $g(\theta) = \csc(\theta - \pi)$

11) The graphs of the secant and cosecant functions in Figures 11 and 12 with respect to θ radians are drawn to scale, meaning that the distance from 0 to 1 is the same on both axes.

a) Estimate the slope of the tangent line to $\sec \theta$ at $\theta = 0$ radians by finding the slope of the line through the points $(0, 1)$ and $(h, \sec h)$ for $h = 10^{-6}$ radians.

b) Estimate the slope of the tangent line to $\sec \theta$ at $\theta = \frac{\pi}{4}$ radians by finding the slope of the line through the points $\left(\frac{\pi}{4}, \sqrt{2}\right)$ and $\left(\frac{\pi}{4} + h, \sec\left(\frac{\pi}{4} + h\right)\right)$ for $h = 10^{-6}$ radians.

c) Estimate the slope of the tangent line to $\csc \theta$ at $\theta = \frac{\pi}{2}$ radians by finding the slope of the line through the points $\left(\frac{\pi}{2}, 1\right)$ and $\left(\frac{\pi}{2} + h, \csc\left(\frac{\pi}{2} + h\right)\right)$ for $h = 10^{-6}$ radians.

d) Estimate the slope of the tangent line to $\csc \theta$ at $\theta = \frac{\pi}{4}$ radians by finding the slope of the line through the points $\left(\frac{\pi}{4}, \sqrt{2}\right)$ and $\left(\frac{\pi}{4} + h, \csc\left(\frac{\pi}{4} + h\right)\right)$ for $h = 10^{-6}$ radians.

4.6 Exact trigonometric values: possible or impossible?

With most of our previous work with functions $f(x)$, we have had the option of evaluating them for any value of x for which they were defined (polynomials, rational functions, etc.) Is this possible with trigonometric functions? Since all of them are based upon the sine and cosine functions, we will just focus on answering the question for these two functions.

Previously in this chapter, we were able to establish exact values of the sine and cosine functions for every angle $\frac{\pi}{12}n = 15^\circ n$, where n is an integer. Finding the exact sine or cosine value of any angle θ seems like a tall order, so let's start with one that should be easier. Can we find the exact value of, for example, $\sin 5^\circ$?

Our sum and difference identities are of no use to us, since the sum or difference of two integer multiples of 15° is also an integer multiple of 15° . We could use the half-angle identities to calculate the exact value of $\sin 7.5^\circ$, but to calculate $\sin 5^\circ$, we would need a third-angle identity. Let's see if we can develop that identity.

To develop the half-angle identities, we started by developing the double-angle identities, so it makes sense to start here by developing triple-angle identities. Note that

$$\begin{aligned}\sin(3\theta) &= \sin(2\theta + \theta) = \sin(2\theta)\cos\theta + \cos(2\theta)\sin\theta \\ \sin(3\theta) &= (2\sin\theta\cos\theta)\cos\theta + (2\cos^2\theta - 1)\sin\theta \\ \sin(3\theta) &= 4\sin\theta\cos^2\theta - \sin\theta.\end{aligned}$$

Using the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$, we can get $\sin(3\theta)$ completely in terms of $\sin\theta$:

$$\begin{aligned}\sin(3\theta) &= 4\sin\theta(1 - \sin^2\theta) - \sin\theta \\ \sin(3\theta) &= 3\sin\theta - 4\sin^3\theta.\end{aligned}\tag{9}$$

Likewise, for $\cos(3\theta)$, we have

$$\begin{aligned}\cos(3\theta) &= \cos(2\theta + \theta) = \cos(2\theta)\cos\theta - \sin(2\theta)\sin\theta \\ \cos(3\theta) &= (1 - 2\sin^2\theta)\cos\theta - (2\sin\theta\cos\theta)\sin\theta \\ \cos(3\theta) &= \cos\theta - 4\sin^2\theta\cos\theta \\ \cos(3\theta) &= \cos\theta - 4(1 - \cos^2\theta)\cos\theta \\ \cos(3\theta) &= 4\cos^3\theta - 3\cos\theta.\end{aligned}\tag{10}$$

In order to have general formulas for the sine and cosine of one-third of an angle, we would need to solve equations (9) and (10), respectively, for $\sin\theta$ and $\cos\theta$, respectively. This is where our method starts to fall apart – although we can solve quadratic equations every time using the quadratic formula, cubic equations are generally harder to solve. Using *Mathematica*TM to solve equation (9) for $\sin\theta$, we are able to find three solutions, two of which are extraneous, and one valid solution involving the imaginary number i :

$$\sin\theta = \frac{-1 + i\sqrt{3} - (1 + i\sqrt{3})(-\sin(3\theta) + i\cos(3\theta))^{\frac{2}{3}}}{4(-\sin(3\theta) + i\cos(3\theta))^{\frac{1}{3}}},$$

which is problematic. We do not yet know how to raise complex numbers to rational powers like $\frac{1}{3}$ and $\frac{2}{3}$. Solving for $\cos \theta$ in equation (10) yields similar results. Even if we substitute $\theta = 5^\circ = \frac{\pi}{36}$ into equations (9) and (10) and try to get an answer for just one $\sin 5^\circ$ or $\cos 5^\circ$, the situation does not improve.

It would appear that we will not be able to calculate sines and cosines exactly for any angle. In Chapter 5, after we have discussed the ties between complex numbers and the sine and cosine, we will discuss how our calculators are able to quickly calculate approximate (up to as many digits as necessary) values for the sine and cosine functions.

4.6 Exercises

1) Use the half-angle identities to find the exact value of the sine and cosine of $\frac{\pi}{24} = 7.5^\circ$. Check your answer by generating a decimal approximation of your answer and comparing to your calculator's decimal approximation to $\sin \frac{\pi}{24}$ and $\cos \frac{\pi}{24}$.

2) Use the half-angle identities and the solution to problem 1) above to calculate the value of the sine and cosine of $\frac{\pi}{48} = 3.75^\circ$, rounded to 8 decimal places. Compare your answer to your calculator's decimal approximation of $\sin \frac{\pi}{48}$ and $\cos \frac{\pi}{48}$.

3) Use the half-angle identities and the solution to problem 2) above to find the value of the sine and cosine of $\frac{\pi}{96} = 1.875^\circ$, rounded to 8 decimal places. Compare your answer to your calculator's decimal approximation to $\sin \frac{\pi}{96}$ and $\cos \frac{\pi}{96}$.

4) Use the half-angle identities and the solution to problem 3) above to find the value of the sine and cosine of $\frac{\pi}{192} = 0.9375^\circ$, rounded to 8 decimal places. Compare your answer to your calculator's decimal approximation to $\sin \frac{\pi}{192}$ and $\cos \frac{\pi}{192}$.

5) Use the results from problems 1) through 4) above and the sum and difference identities to find the approximate sine and cosine of the following angles, rounded to 8 decimal places. Compare your answer to your calculator's decimal approximation of the sine and cosine values of the angles.

a) $\frac{\pi}{16} = 11.25^\circ$

b) $\frac{\pi}{32} = 5.625^\circ$

c) $\frac{5\pi}{96} = 9.375^\circ$

d) $\frac{7\pi}{192} = 6.5625^\circ$

4.7 Inverse Trigonometric Functions

The inverse of a function $f(x)$ is found by exchanging the “ x ” and “ y ” in the formula, and solving (if possible) for the “ y ” variable. Graphically, the inverse of a function $f(x)$ is its reflection across the line $y = x$. See the example in Figure 13. The curve at left is a function (passes the vertical line test – has exactly one y -value for each x -value). However, its inverse, the reflection across the line $y = x$, is not a function.

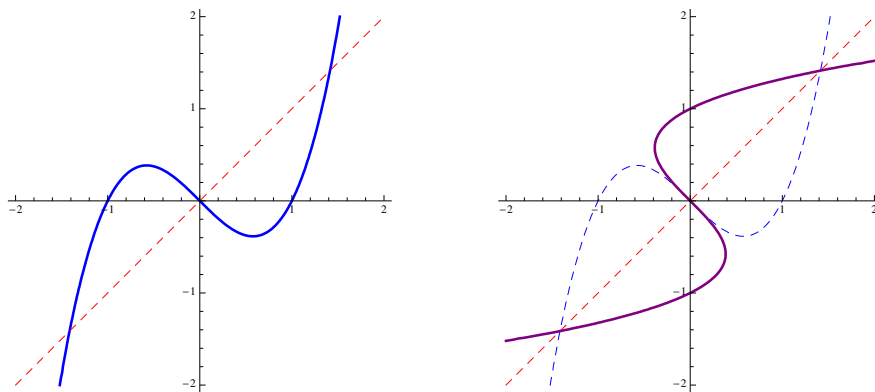


Figure 13: The curve at left is a function, but its inverse shown on the right is not. The curve at left is not one-to-one.

In order for a function $f(x)$ to have an inverse that is also a function, $f(x)$ must be a *one-to-one* function, meaning that not only does $x = y$ imply that $f(x) = f(y)$ (it is a function), but also $f(x) = f(y)$ implies $x = y$. A function that is one-to-one has exactly one x -value for each y -value. In Figure 13, some y -values on the curve at left (like $y = 0$) have as many as three x -values associated with them, so it is not one-to-one.

In the case where a function $f(x)$ is one-to-one, we typically use the notation $f^{-1}(x)$ to indicate its inverse function. The notation is poorly conceived – we typically read the exponent “ -1 ” to mean the reciprocal, and in general, $f^{-1}(x) \neq \frac{1}{f(x)}$ – but widely accepted. Just be careful, and pay attention to the context in which it is being used. The inverse can be thought of as “undoing” the original function. In fact, the following are necessarily true of an inverse function:

$$f(f^{-1}(x)) = x \quad \text{and} \quad f^{-1}(f(x)) = x$$

for all x in which both the original and inverse are defined.

Inverse of the Sine Function

An inspection of the graph of the sine function, shown on the left in Figure 14, reveals that the sine function is not one-to-one (fails the horizontal line test), so we will not be able to find a true inverse sine function. However, if we were to define the function

$$f(\theta) = \sin \theta, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

(shown in Figure 14 between the two vertical red dashed lines), then $f(\theta)$ would be one-to-one, and its inverse function $f^{-1}(\theta)$ would be the reflection across the diagonal line $y = \theta$, shown on the right in Figure 14. Many books (and calculators) will use the notation “ $\sin^{-1} y$ ” to refer to this function, although it is not really the inverse of the entire sine function. For this reason, many books abandon the inverse notation altogether, and use the notation “ $\arcsin y$ ” to refer to this function. I suggest you be familiar with both styles of notation, just in case, but I will use the notation “ $\arcsin y$ ” from this point on.

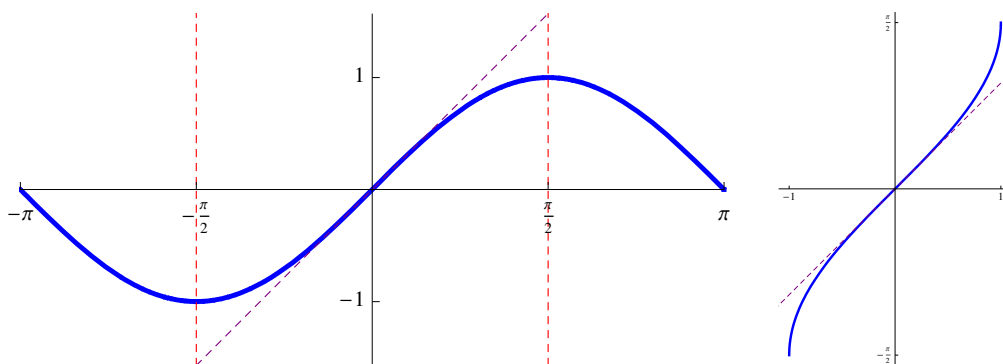


Figure 14: The curve at left is the sine function, with the red dashed lines restricting the domain to $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. The curve at right is the inverse function $\arcsin \theta$.

Example: Find $\arcsin\left(-\frac{1}{2}\right)$.

$$\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6} \quad \text{since } -\frac{\pi}{2} \leq -\frac{\pi}{6} \leq \frac{\pi}{2} \text{ and } \sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$$

Example: Find $\arcsin\left(\sin \frac{\pi}{4}\right)$.

$$\arcsin\left(\sin \frac{\pi}{4}\right) = \arcsin\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4} \quad \text{since } -\frac{\pi}{2} \leq \frac{\pi}{4} \leq \frac{\pi}{2} \text{ and } \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

Notice that for $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, we have that $\arcsin(\sin \theta) = \theta$.

Example: Find $\arcsin\left(\sin \frac{5\pi}{4}\right)$.

$$\arcsin\left(\sin \frac{5\pi}{4}\right) = \arcsin\left(-\frac{1}{\sqrt{2}}\right) = -\frac{\pi}{4} \quad \text{since } -\frac{\pi}{2} \leq -\frac{\pi}{4} \leq \frac{\pi}{2} \text{ and } \sin\left(-\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$

Notice that since $\theta = \frac{5\pi}{4}$ is not in the range $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, then $\arcsin(\sin \theta) \neq \theta$. We may have to make an adjustment if we are actually looking for angles outside the range of the inverse sine.

Inverse of the Cosine Function

As before, an inspection of the graph of the cosine function, shown on the left in Figure 15, reveals that the cosine function is not one-to-one either, so we will not be able to find a true inverse cosine function. However, if we were to define the function

$$f(\theta) = \cos \theta, \quad 0 \leq \theta \leq \pi$$

(shown in Figure 15 between the two vertical red dashed lines), then $f(\theta)$ would be one-to-one, and its inverse function $f^{-1}(\theta)$ would be the reflection across the diagonal line $y = \theta$, shown on the right in Figure 15. As before, both notations “ $\cos^{-1} y$ ” and “ $\arccos y$ ” are used, but I will tend to use the notation “ $\arccos y$ ” from this point on.

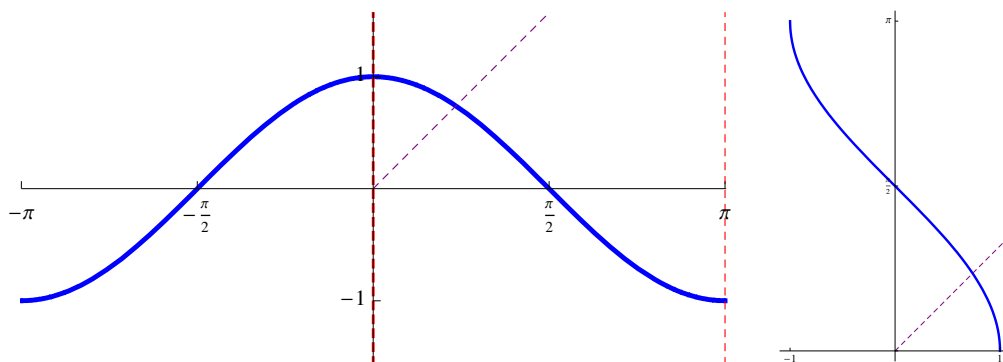


Figure 15: The curve at left is the cosine function, with the red dashed lines restricting the domain to $0 \leq \theta \leq \pi$. The curve at right is the inverse function $\arccos \theta$.

Example: Find $\arccos\left(-\frac{1}{2}\right)$.

$$\arccos\left(-\frac{1}{2}\right) = \frac{2\pi}{3} \quad \text{since } 0 \leq \frac{2\pi}{3} \leq \pi \text{ and } \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$$

Example: Find $\arccos\left(\cos \frac{\pi}{6}\right)$.

$$\arccos\left(\cos \frac{\pi}{6}\right) = \arccos\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6} \quad \text{since } 0 \leq \frac{\pi}{6} \leq \pi \text{ and } \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

Notice that for $0 \leq \theta \leq \pi$, we have that $\arccos(\cos \theta) = \theta$.

Example: Find $\arccos\left(\cos\left(-\frac{\pi}{4}\right)\right)$.

$$\arccos\left(\cos\left(-\frac{\pi}{4}\right)\right) = \arccos\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4} \quad \text{since } 0 \leq \frac{\pi}{4} \leq \pi \text{ and } \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

Notice that since $\theta = -\frac{\pi}{4}$ is not in the range $0 \leq \theta \leq \pi$, then $\arccos(\cos \theta) \neq \theta$. We may have to make an adjustment if we are actually looking for angles outside the range of the inverse cosine.

The Inverse of the Tangent Function

As before, we can look at the graph of the tangent function at left in Figure 16 and tell that it is not one-to-one. However, if we limit the domain to $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, then the limited function is one-to-one. The inverse function of the limited function is shown at right in Figure 16.

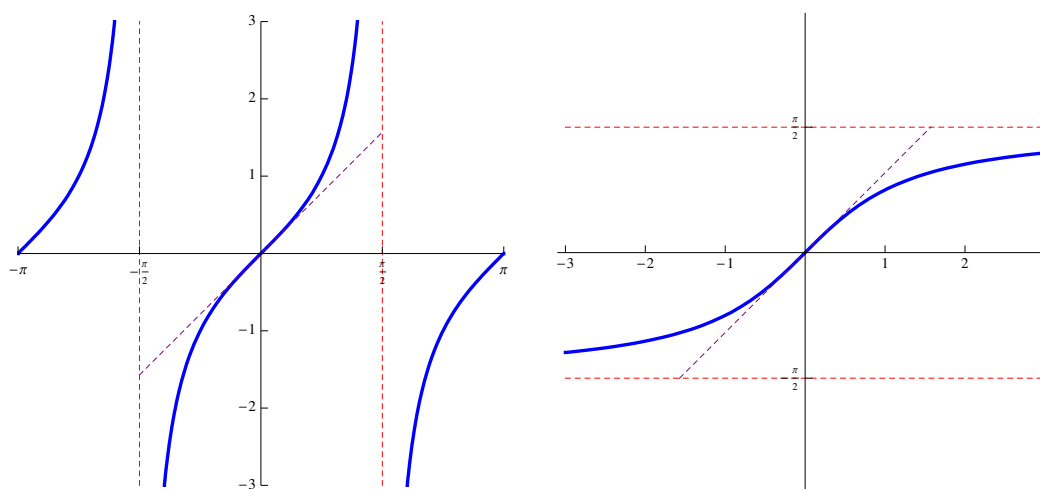


Figure 16: The curve at left is the tangent function, with the red dashed lines restricting the domain to $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. The curve at right is the inverse function $\arctan \theta$.

Example: Find $\arctan(-\sqrt{3})$.

$$\arctan(-\sqrt{3}) = -\frac{\pi}{3} \quad \text{since } -\frac{\pi}{2} < -\frac{\pi}{3} < \frac{\pi}{2} \text{ and } \tan\left(-\frac{\pi}{3}\right) = -\sqrt{3}$$

Example: Find $\arctan\left(\tan \frac{\pi}{6}\right)$.

$$\arctan\left(\tan \frac{\pi}{6}\right) = \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6} \quad \text{since } -\frac{\pi}{2} < \frac{\pi}{6} < \frac{\pi}{2} \text{ and } \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$$

Notice that for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, we have that $\arctan(\tan \theta) = \theta$.

Example: Find $\arctan\left(\tan \frac{3\pi}{4}\right)$.

$$\arctan\left(\tan \frac{3\pi}{4}\right) = \arctan(-1) = -\frac{\pi}{4} \quad \text{since } -\frac{\pi}{2} < -\frac{\pi}{4} < \frac{\pi}{2} \text{ and } \tan\left(-\frac{\pi}{4}\right) = -1$$

Notice that since $\theta = \frac{3\pi}{4}$ is not in the range $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, then $\arctan(\tan \theta) \neq \theta$. We may have to make an adjustment if we are actually looking for angles outside the range of the inverse tangent.

The Other Inverse Trigonometric Functions

You may be spotting a pattern – none of the trigonometric functions are one-to-one. In order to manufacture so-called “inverse” functions, we have to limit the domain of the original trigonometric function. The remaining trigonometric functions are reciprocals of the other trigonometric functions,

$$\cot \theta = \frac{1}{\tan \theta} \quad \sec \theta = \frac{1}{\cos \theta} \quad \csc \theta = \frac{1}{\sin \theta},$$

for all values of θ for which they are defined. Therefore, it makes sense to limit the domains of the cotangent, secant, and cosecant to match the tangent, cosine, and sine, respectively. The only slight exception is with the cotangent, because the function value “0” occurs at both $\theta = -\frac{\pi}{2}$ and $\theta = \frac{\pi}{2}$, and we need to include it once, but only once. Therefore, we arbitrarily include $\theta = \frac{\pi}{2}$ in the domain of the truncated cotangent, and hence, the range of the inverse cotangent. It does cause some odd discontinuities with the inverse functions, but we will see that this is probably the best way to handle the inverses in application. The curves for the inverse functions of the restricted-domain trigonometric functions are shown in Figures 17, 18, and 19.

Suppose that we are solving a trigonometric equation, and we end up with something like

$$\cot \theta = \text{blah, blah, blah, } \dots$$

We would usually use the inverse key above our trigonometric function to get a numerical solution for θ . However, very few calculators have a “COT” button, not to mention an inverse cotangent button. But if we draw upon the fact that, for $\theta \neq 0$,

$$\cot \theta = \frac{1}{\tan \theta},$$

then

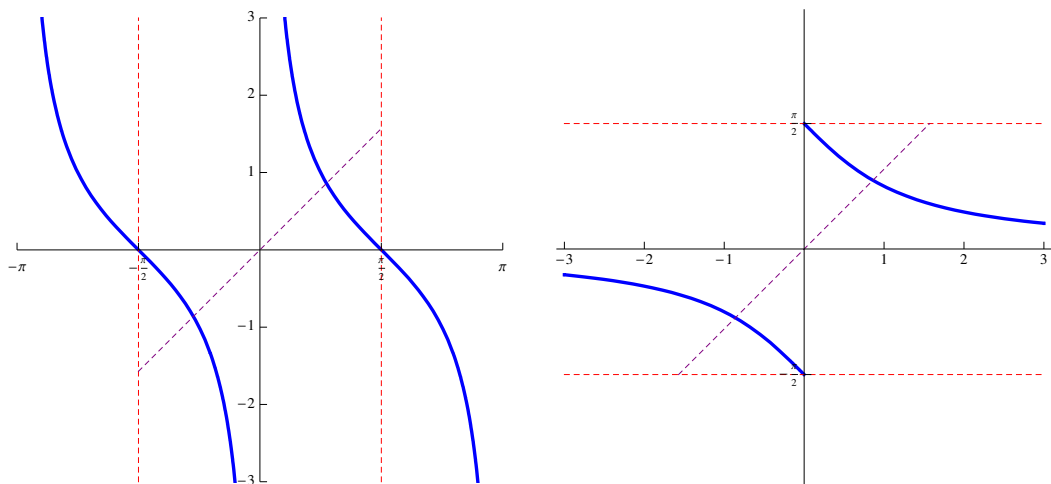


Figure 17: The curve at left is the cotangent function, with the red dashed lines restricting the domain to $-\frac{\pi}{2} < \theta \leq \frac{\pi}{2}$. The curve at right is the inverse function $\operatorname{arccot} y$ or $\cot^{-1} y$.

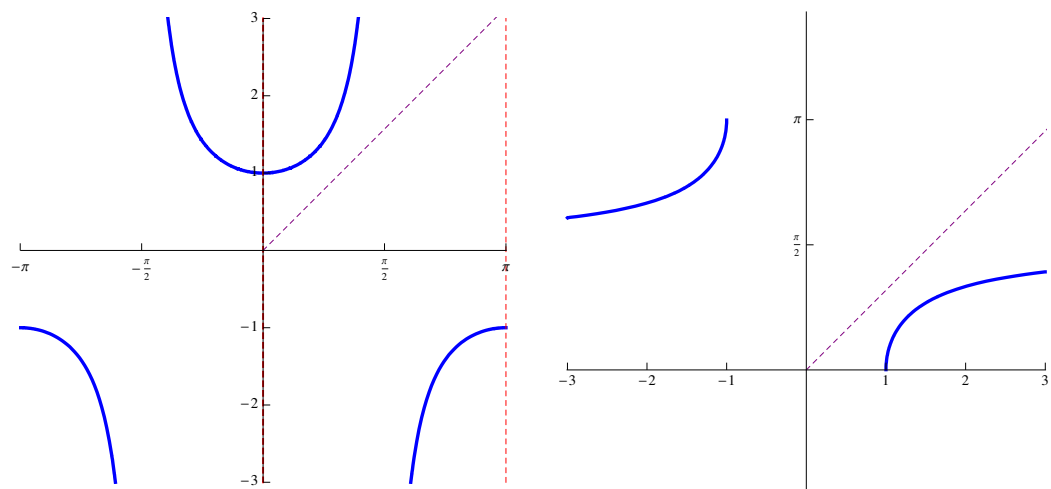


Figure 18: The curve at left is the secant function, with the red dashed lines restricting the domain to $0 \leq \theta \leq \pi$. The curve at right is the inverse function $\operatorname{arcsec} y$ or $\sec^{-1} y$.

$$\frac{1}{\tan \theta} = \text{blah, blah, blah, } \dots$$

$$\tan \theta = \frac{1}{\text{blah, blah, blah, } \dots}, \text{ and}$$

$$\theta = \arctan \left(\frac{1}{\text{blah, blah, blah, } \dots} \right).$$

We may summarize this technique in the following theorem.

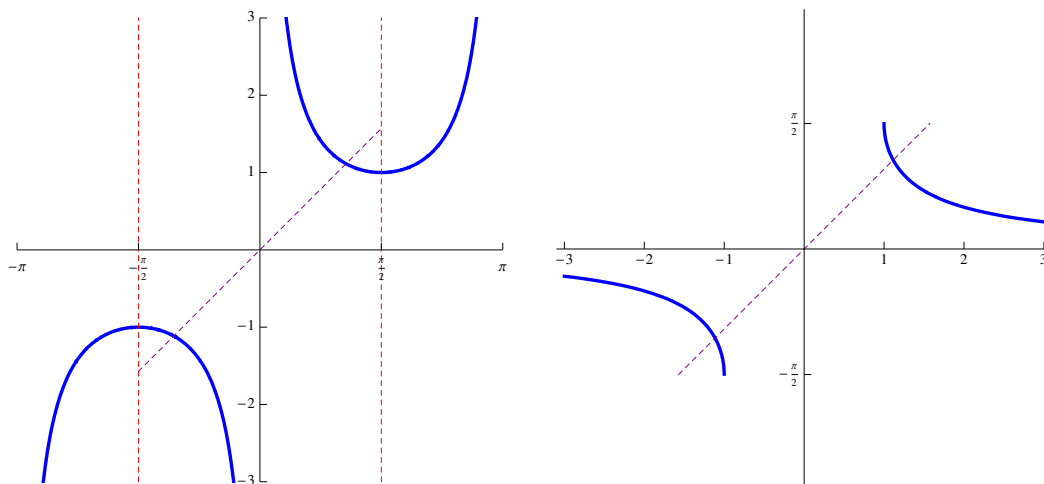


Figure 19: The curve at left is the cosecant function, with the red dashed lines restricting the domain to $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. The curve at right is the inverse function $\operatorname{arccsc} y$ or $\csc^{-1} y$.

Theorem: For any value of y for which the inverse cotangent, secant, and cosecant, respectively, are defined, we have

$$\operatorname{arccot} y = \arctan \left(\frac{1}{y} \right), \quad \operatorname{arcsec} y = \arccos \left(\frac{1}{y} \right), \quad \operatorname{arccsc} y = \arcsin \left(\frac{1}{y} \right).$$

Example: Find $\operatorname{arccot} \sqrt{3}$.

$$\operatorname{arccot} \sqrt{3} = \arctan \left(\frac{1}{\sqrt{3}} \right) = \frac{\pi}{3}$$

Example: Find $\operatorname{arcsec} (-\sqrt{2})$.

$$\operatorname{arcsec} (-\sqrt{2}) = \arccos \left(-\frac{1}{\sqrt{2}} \right) = \frac{3\pi}{4}$$

4.7 Exercises

In problems 1) through 14), calculate the following exactly.

- | | |
|---|---|
| 1) $\arccos\left(-\frac{\sqrt{3}}{2}\right)$ | 2) $\arcsin\frac{\sqrt{2}}{2}$ |
| 3) $\arctan\frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}}$ | 4) $\operatorname{arccot}\frac{\sqrt{2}-\sqrt{6}}{\sqrt{2}+\sqrt{6}}$ |
| 5) $\operatorname{arccsc} 2$ | 6) $\operatorname{arcsec}(-2)$ |
| 7) $\arctan\left(\tan\frac{\pi}{4}\right)$ | 8) $\arctan\left(\tan\frac{5\pi}{4}\right)$ |
| 9) $\arccos(\cos 1)$ | 10) $\arccos(\cos(-1))$ |
| 11) $\arcsin(\sin 2)$ | 12) $\arcsin(\sin 4)$ |
| 13) $\arctan(\tan 2)$ | 14) $\arctan(\tan 4)$ |

In problems 15) through 24), use your calculator to find an approximation to each of the following, accurate to four decimal places.

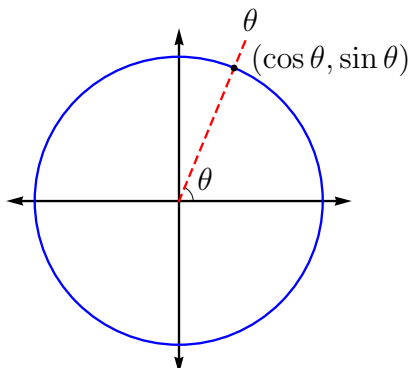
- | | |
|-------------------------------------|-------------------------------------|
| 15) $\arccos(-0.8)$ | 16) $\arcsin 0.15$ |
| 17) $\arctan 5$ | 18) $\operatorname{arccot}(-100)$ |
| 19) $\operatorname{arccsc} 4$ | 20) $\operatorname{arcsec}(-4)$ |
| 21) $\operatorname{arcsec}(\cos 2)$ | 22) $\arcsin(\sin 4)$ |
| 23) $\operatorname{arccot}(\tan 2)$ | 24) $\operatorname{arccsc}(\sin 4)$ |

25) Consider the following statement: For all $-1 \leq y \leq 1$, we have $\sin(\arcsin y) = y$. Either explain why this statement is true, or provide a counterexample showing that the statement is false.

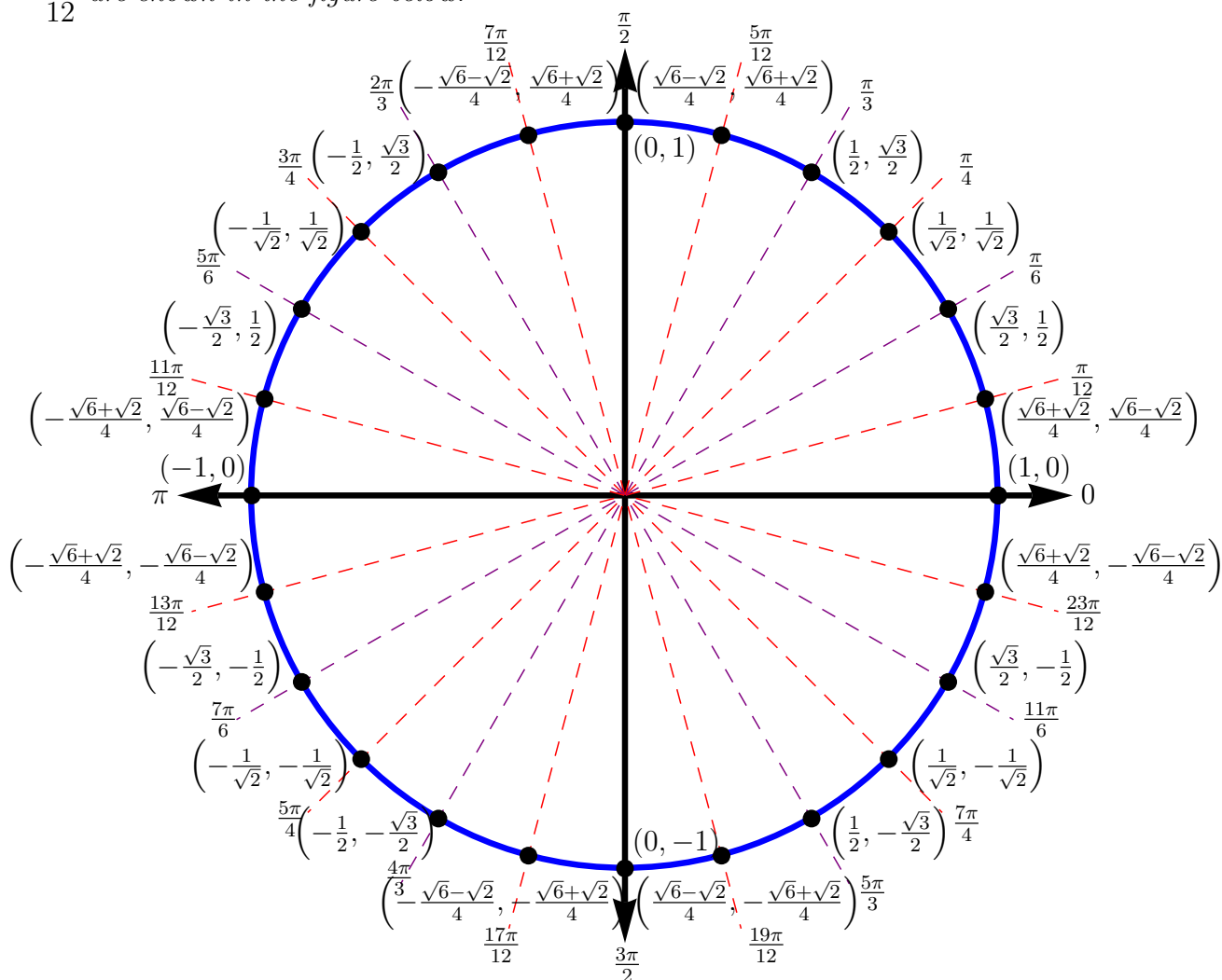
26) Consider the following statement: For all real θ , we have $\arcsin(\sin \theta) = \theta$. Either explain why this statement is true, or provide a counterexample showing that the statement is false.

4.8 Review: What have we learned (or relearned)?

Theorem: In a unit circle, the coordinates of a point on the circle corresponding to the angle θ (or equivalently, arc length θ) are $(\cos \theta, \sin \theta)$.



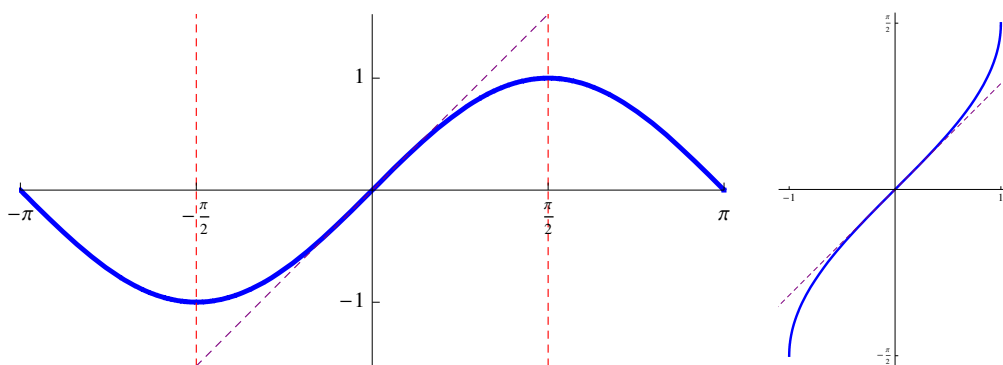
Exact Values of Sines and Cosines of $\frac{\pi}{12}n$ The exact cosines and sines of multiples of $\frac{\pi}{12}$ are shown in the figure below.



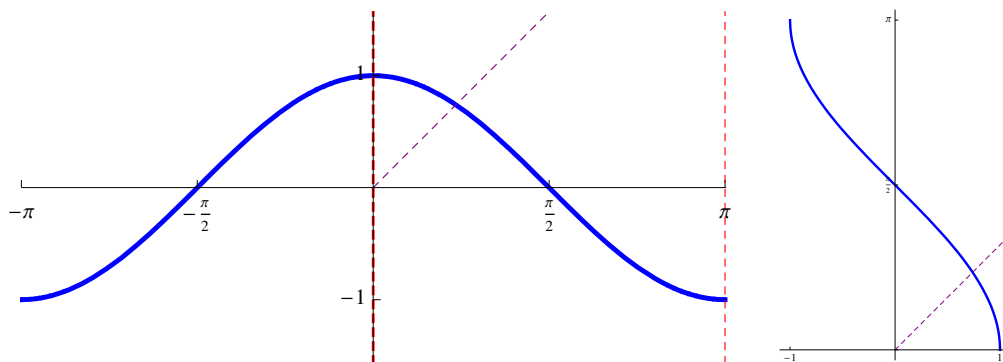
Definition: One-to-one Functions A function $f(x)$ is said to be one-to-one if $f(a) = f(b)$ implies that $a = b$. Graphically, a one-to-one function passes a horizontal line test.

Definition: Inverse Function The inverse of a function is the result of exchanging the dependent and independent variables. If the original function $f(x)$ is one-to-one, then the inverse is also a function, denoted $f^{-1}(x)$.

The Graph of the Sine Function and Its Inverse At left is the function $\sin \theta$, which is 2π -periodic and has amplitude 1. If we restrict the domain of $y = \sin \theta$ to $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, then that function is one-to-one and has the inverse $\theta = \arcsin y = \sin^{-1} y$, shown at right.



The Graph of the Cosine Function and Its Inverse At left is the function $\cos \theta$, which is 2π -periodic and has amplitude 1. If we restrict the domain of $y = \cos \theta$ to $0 \leq \theta \leq \pi$, then that function is one-to-one and has the inverse $\theta = \arccos y = \cos^{-1} y$, shown at right.



Theorem: Double-Angle Identities

$$\begin{aligned}\sin(2\theta) &= 2 \sin \theta \cos \theta \\ \cos(2\theta) &= 2 \cos^2 \theta - 1\end{aligned}$$

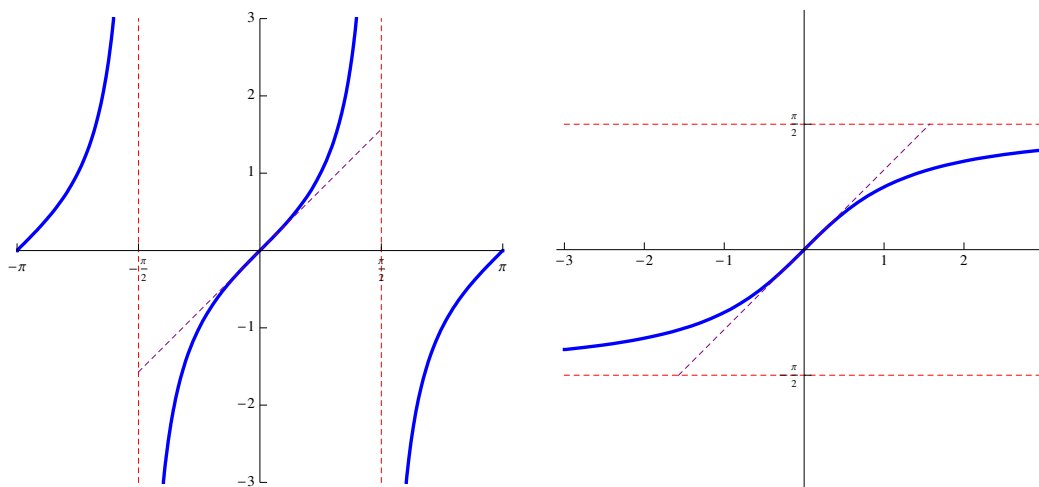
$$\begin{aligned}\cos(2\theta) &= \cos^2 \theta - \sin^2 \theta \\ \cos(2\theta) &= 1 - 2 \sin^2 \theta\end{aligned}$$

Theorem: The functions $a \sin(c\theta)$ and $a \cos(c\theta)$ are each $\frac{2\pi}{c}$ -periodic with amplitude a .

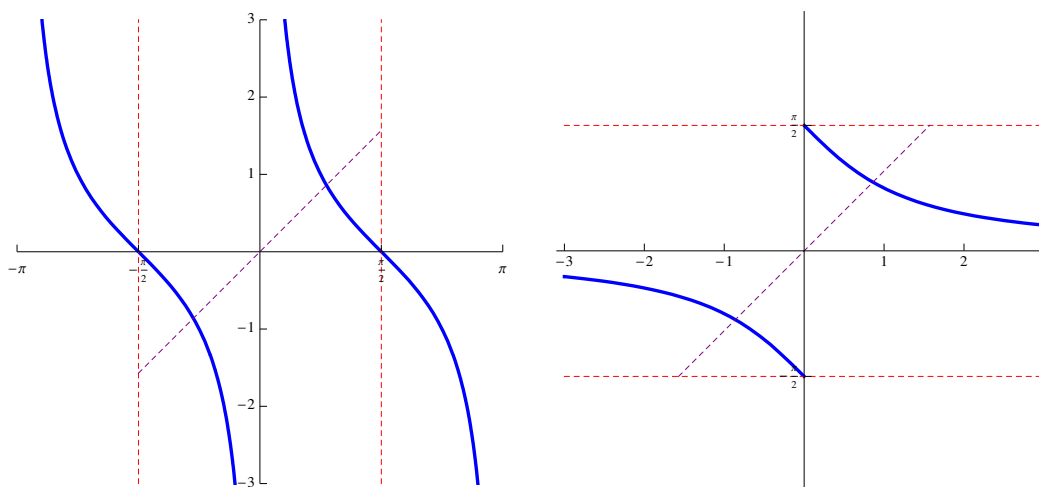
Theorem: Half-Angle Identities (or Power-Reducing Identities)

$$\sin^2\left(\frac{\theta}{2}\right) = \frac{1 - \cos \theta}{2} \qquad \cos^2\left(\frac{\theta}{2}\right) = \frac{1 + \cos \theta}{2}$$

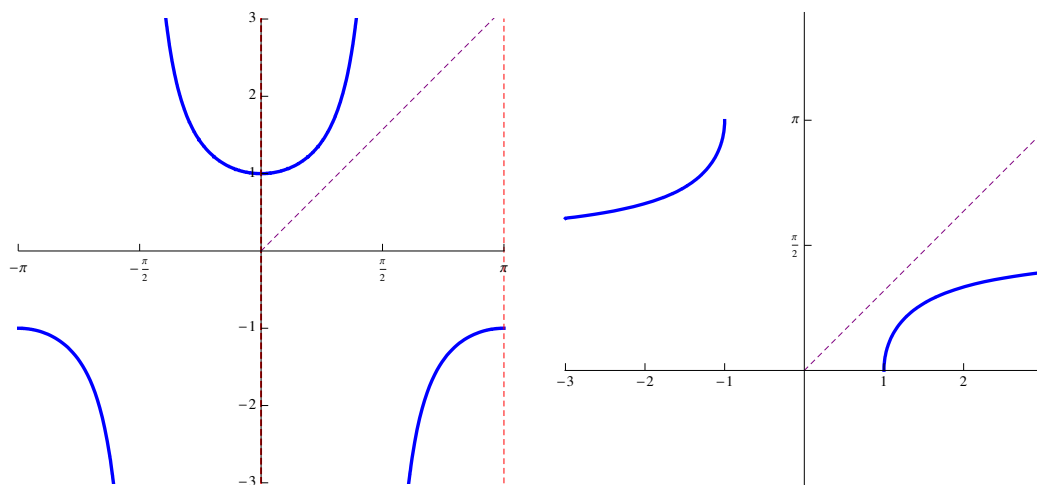
The Graph of the Tangent Function and Its Inverse *At left is the function $\tan \theta$, which is π -periodic. If we restrict the domain of $y = \tan \theta$ to $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, then that function is one-to-one and has the inverse $\theta = \arctan y = \tan^{-1} y$, shown at right.*



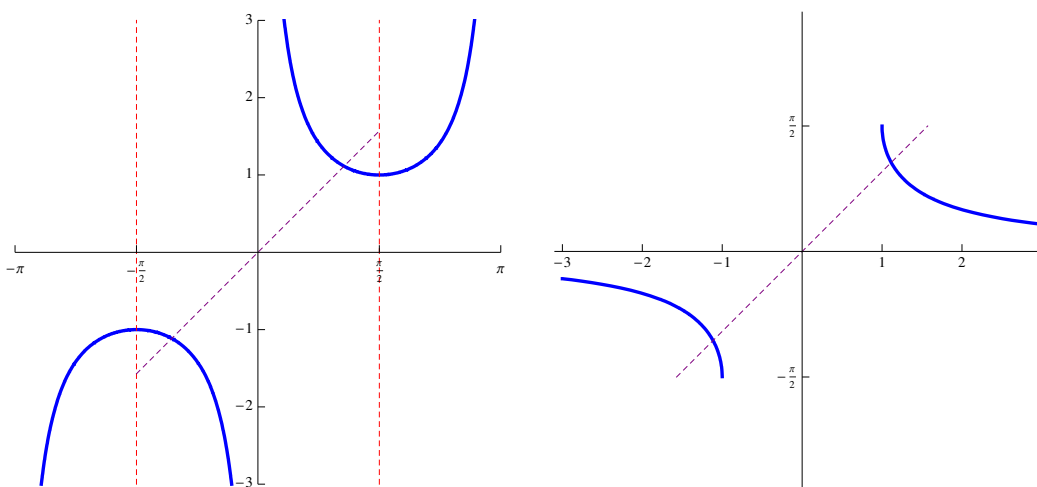
The Graph of the Cotangent Function and Its Inverse *At left is the function $\cot \theta$, which is π -periodic. If we restrict the domain of $y = \cot \theta$ to $-\frac{\pi}{2} < \theta \leq \frac{\pi}{2}$, then that function is one-to-one and has the inverse $\theta = \arctan y = \cot^{-1} y$, shown at right.*



The Graph of the Secant Function and Its Inverse At left is the function $\sec \theta$, which is 2π -periodic. If we restrict the domain of $y = \sec \theta$ to $0 \leq \theta \leq \pi$, then that function is one-to-one and has the inverse $\theta = \operatorname{arcsec} y = \sec^{-1} y$, shown at right.



The Graph of the Cosecant Function and Its Inverse At left is the function $\csc \theta$, which is 2π -periodic. If we restrict the domain of $y = \csc \theta$ to $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, then that function is one-to-one and has the inverse $\theta = \operatorname{arcsin} y = \sin^{-1} y$, shown at right.

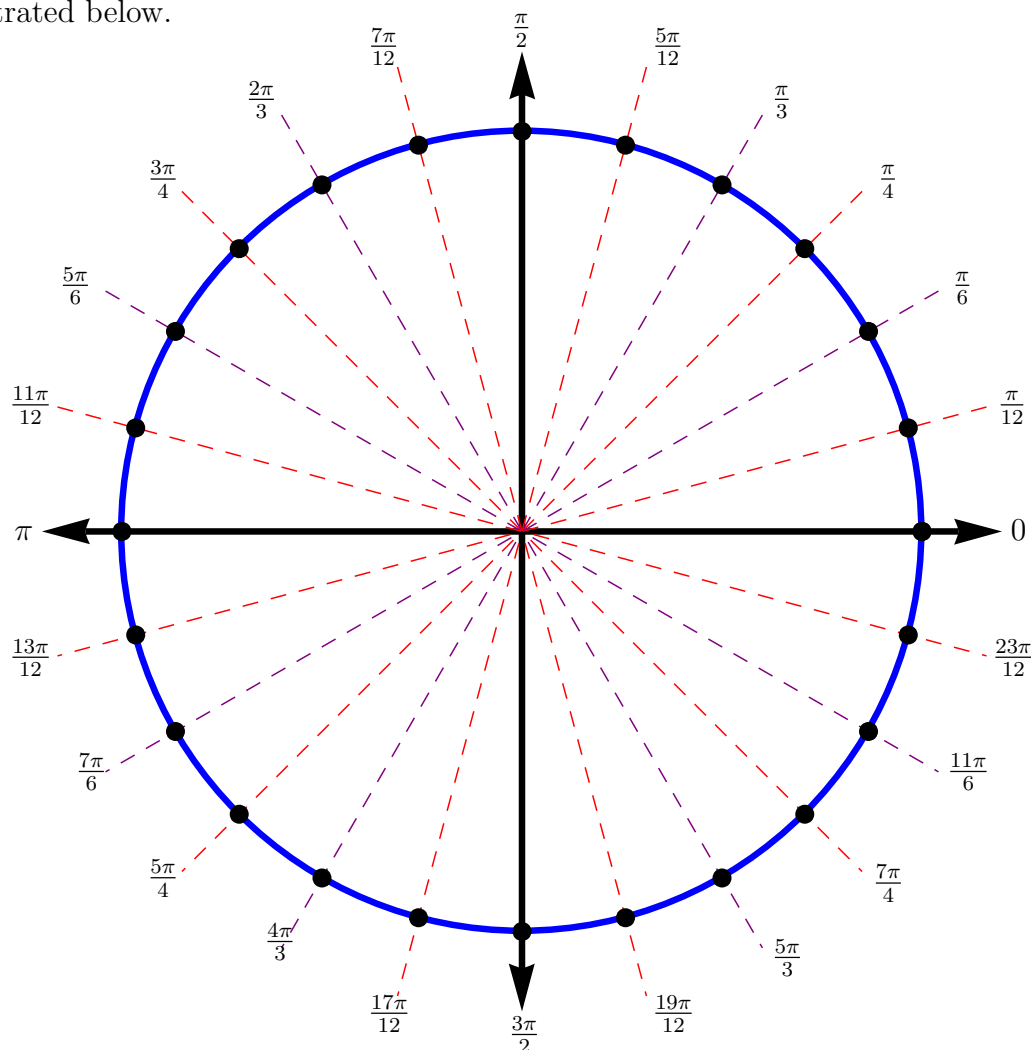


Theorem: For any value of y for which the inverse cotangent, secant, and cosecant, respectively, are defined, we have

$$\operatorname{arccot} y = \arctan \left(\frac{1}{y} \right), \quad \operatorname{arcsec} y = \arccos \left(\frac{1}{y} \right), \quad \operatorname{arccsc} y = \arcsin \left(\frac{1}{y} \right).$$

Review Exercises

- 1) Find the coordinates for the points on the unit circle occurring at $15^\circ = \frac{\pi}{12}$ intervals, as illustrated below.



- 2) Verify that $\sin \frac{5\pi}{12} = \sin 75^\circ = \frac{\sqrt{6} + \sqrt{2}}{4}$ using the difference formula for sine and the fact that $\frac{5\pi}{12} = \frac{2\pi}{3} - \frac{\pi}{4}$.
- 3) Verify that $\cos \frac{5\pi}{12} = \cos 75^\circ = \frac{\sqrt{6} - \sqrt{2}}{4}$ using the difference formula for cosine and the fact that $\frac{5\pi}{12} = \frac{3\pi}{4} - \frac{\pi}{3}$.
- 4) Verify that $\sin \frac{7\pi}{12} = \sin 105^\circ = \frac{\sqrt{6} + \sqrt{2}}{4}$ using the sum formula for sine and the fact that $\frac{7\pi}{12} = \frac{\pi}{3} + \frac{\pi}{4}$.

15) Find the exact value of each of the following.

a) $\arcsin\left(-\frac{\sqrt{2}}{2}\right)$

b) $\arcsin\left(\sin\left(-\frac{3\pi}{4}\right)\right)$

c) $\cos\left(\arccos\frac{1}{2}\right)$

d) $\arctan\left(-\sqrt{3}\right)$

e) $\operatorname{arcsec} 2$

f) $\operatorname{arccsc} \frac{2}{\sqrt{3}}$

16) Use your calculator to find an approximation to each of the following, accurate to four decimal places.

a) $\arcsin(-0.8)$

b) $\arccos 0.15$

c) $\arctan 25$

d) $\operatorname{arccot} \frac{\pi}{4}$

e) $\operatorname{arccsc}(-3)$

f) $\operatorname{arcsec} \pi$

Chapter 5: Complex Numbers and Trigonometry

MATH 117: Trigonometry

5.1 Introduction

Most survivors of high-school algebra know about imaginary and complex numbers. We define the value i to be the (non-real) value such that $i^2 = -1$, or loosely, $i = \sqrt{-1}$. Of course, the very notion that we can square a number and get a -1 is ridiculous, and since i can not possibly be real, we say that it is an **imaginary** number. We typically learn about imaginary numbers right before we start to solve quadratics of the form $ax^2 + bx + c = 0$ using the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

If the discriminant $b^2 - 4ac < 0$, then our solutions to the quadratic will contain an imaginary number. Numbers of the form $a + bi$, where a and b are real numbers, are called **complex** numbers, and can be solutions to quadratic equations. Notice that the set of real numbers is included in the set of complex numbers (when $b = 0$).

We can perform all of the basic operations on complex numbers:

Addition:

$$\begin{aligned}(1 + 2i) + (3 + 4i) &= (1 + 3) + (2 + 4)i \\ &= 4 + 6i\end{aligned}$$

Subtraction:

$$\begin{aligned}(1 + 2i) - (3 + 4i) &= (1 + 2i) + (-3 - 4i) \\ &= (1 - 3) + (2 - 4)i \\ &= -2 - 2i\end{aligned}$$

Multiplication:

$$\begin{aligned}(1 + 2i)(3 + 4i) &= 3 + 4i + 6i + 8i^2 \\ &= (3 - 8) + (4 + 6)i \\ &= -5 + 10i\end{aligned}$$

Division:

$$\begin{aligned}\frac{1 + 2i}{3 + 4i} &= \frac{1 + 2i}{3 + 4i} \cdot \frac{3 - 4i}{3 - 4i} \\ &= \frac{3 - 4i + 6i - 8i^2}{9 - 12i + 12i - 16i^2} \\ &= \frac{3 + 8 + (-4 + 6)i}{9 + 16} \\ &= \frac{11 + 2i}{25} = \frac{11}{25} + \frac{2}{25}i\end{aligned}$$

Notice that in the division problem we multiplied by the same number in the numerator and denominator. The **complex conjugate** of $a + bi$, denoted $\overline{a + bi}$, is $a - bi$ and the number we used is the complex conjugate of the denominator, effectively converting the denominator to a real number. Also, notice that the multiplicative inverse of a complex number is a scaled

version of its conjugate:

$$\frac{1}{a+bi} = \frac{1}{a+bi} \cdot \frac{a-bi}{a-bi} = \frac{1}{a^2+b^2}(a-bi).$$

Also, we can equate complex numbers to points in a plane by letting the real part of the complex number be the first coordinate of the point and letting the real coefficient of the imaginary part be the second coordinate; that is,

$$a+bi \iff (a,b),$$

as demonstrated in Figure 1.

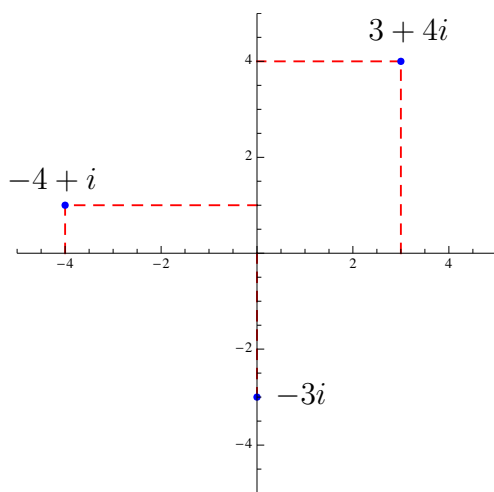


Figure 1

If we place the unit circle (that we used to develop the sine and cosine function in Chapter 4) in the complex plane, then instead of worrying about the point $(\cos \theta, \sin \theta)$ on the circle, we can think of the complex number $\cos \theta + i \sin \theta$, shown in Figure 2. In fact, if we take any complex number $a+bi$, and find the angle θ of the ray from the origin that passes through the point, then we can specify the complex number exactly by giving the angle θ and the distance from the origin. We call this distance the **modulus** of a complex number, given by

$$|a+bi| = \sqrt{a^2+b^2} \text{ or } \sqrt{(a+bi)(\overline{a+bi})}.$$

Thus, the complex number $a+bi$ has a new representation, called the **trigonometric form** of the complex number,

$$r(\cos \theta + i \sin \theta)$$

where $r = \sqrt{a^2+b^2}$ is the modulus of the complex number and θ is in the quadrant containing the number $a+bi$ so that $\tan \theta = \frac{b}{a}$. We call the original format of the complex number $a+bi$ the **rectangular form** of the complex number.

This new representation has its advantages, one of which we will illustrate in the next example.

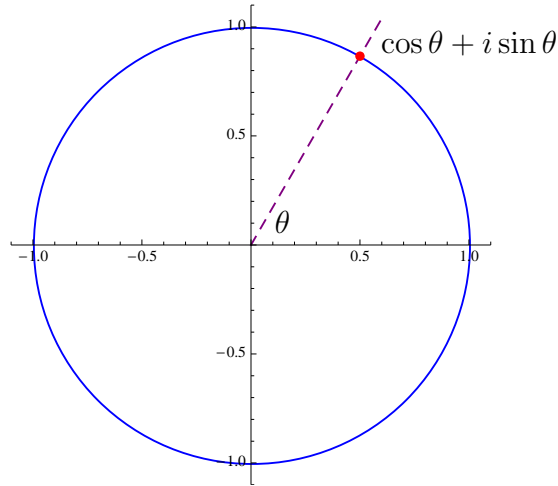


Figure 2

Example: Let $u = 1 + \sqrt{3}i$ and $v = i$. Find the trigonometric form of each number. Find the product of u and v and find its trigonometric form as well.

For the complex number u , we have

$$\begin{aligned} r_u &= \sqrt{1^2 + (\sqrt{3})^2} = 2 \quad \text{and} \quad \tan \theta_u = \frac{\sqrt{3}}{1} = \sqrt{3} \\ \theta_u &= \frac{\pi}{3}, \end{aligned}$$

since u is in the first quadrant. Then

$$u = 1 + \sqrt{3}i = 2 \left[\cos \left(\frac{\pi}{3} \right) + i \sin \left(\frac{\pi}{3} \right) \right].$$

For the complex number v , we have

$$\begin{aligned} r_v &= \sqrt{0^2 + 1^2} = 1 \quad \text{and} \quad \tan \theta_v = \frac{1}{0} = \text{undefined} \\ \theta_v &= \frac{\pi}{2}, \end{aligned}$$

since v is on the positive imaginary axis. Then

$$v = i = \cos \left(\frac{\pi}{2} \right) + i \sin \left(\frac{\pi}{2} \right).$$

The product uv is

$$uv = (1 + \sqrt{3}i)(i) = i + \sqrt{3}i^2 = -\sqrt{3} + i.$$

Then

$$\begin{aligned} r_{uv} &= \sqrt{(-\sqrt{3})^2 + 1^2} = 2 \quad \text{and} \quad \tan \theta_{uv} = \frac{1}{-\sqrt{3}} \\ \theta_{uv} &= \left(-\frac{\pi}{6} \right) + \pi = \frac{5\pi}{6}, \end{aligned}$$

since $\arctan \left(-\frac{\pi}{\sqrt{3}} \right)$ is in the fourth quadrant, and our complex number is in the second quadrant. Therefore,

$$uv = 2 \left[\cos \left(\frac{5\pi}{6} \right) + i \sin \left(\frac{5\pi}{6} \right) \right].$$

Notice that $r_{uv} = r_u r_v = 2(1)$ and that $\theta_{uv} = \theta_u + \theta_v = \frac{\pi}{3} + \frac{\pi}{2}$. The solution is illustrated in Figure 3.

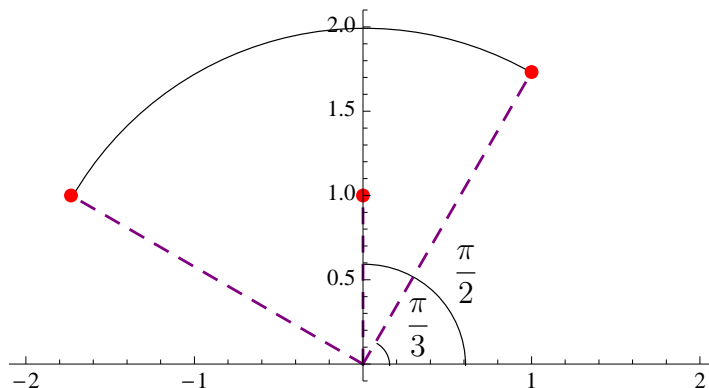


Figure 3

We can, in fact, prove that this will always happen, regardless of the r -values and the θ 's. Let

$$u = r_u(\cos \theta_u + i \sin \theta_u) \quad \text{and} \quad v = r_v(\cos \theta_v + i \sin \theta_v).$$

Then

$$\begin{aligned} uv &= [r_u(\cos \theta_u + i \sin \theta_u)] [r_v(\cos \theta_v + i \sin \theta_v)] \\ uv &= r_u r_v [\cos \theta_u \cos \theta_v - \sin \theta_u \sin \theta_v + i (\sin \theta_u \cos \theta_v + \cos \theta_u \sin \theta_v)] \\ uv &= r_u r_v [\cos(\theta_u + \theta_v) + i \sin(\theta_u + \theta_v)]. \end{aligned} \tag{1}$$

Therefore, once we have two complex numbers in trigonometric form, we can multiply them by multiplying their moduli and adding their angles.

While this is a nice property, writing “ $\cos \theta + i \sin \theta$ ” all of the time is bothersome. We could create a short-hand function notation for “ $\cos \theta + i \sin \theta$ ”; in fact, many books will use “ $\text{cis } \theta$ ” as short-hand for “ $\cos \theta + i \sin \theta$ ”. However, we can find a more accurate mathematical statement for “ $\cos \theta + i \sin \theta$ ”, based on the fact that adding the angles when we multiply is reminiscent of exponents.

5.1 Exercises

1) Perform the following complex number operations.

a) $(3 - 2i) + (-1 + 3i)$

b) $(-2 - 8i) - (-3 - 7i)$

c) $(4 - i) - \left(2 + \frac{1}{2}i\right)$

d) $\left(\frac{5}{2} - \frac{4}{3}i\right) - \left(\frac{3}{2} + \frac{2}{3}i\right)$

e) $(3 + i)(-2 - 5i)$

f) $\left(2 - \frac{1}{2}i\right) \left(\frac{3}{2} + 4i\right)$

g) $\frac{7 - i}{1 + 2i}$

h) $\frac{1 + 2i}{2 - i}$

2) Convert the following complex numbers from rectangular form to trigonometric form.

a) $1 + i$

b) $\sqrt{2} - \sqrt{2}i$

c) $\frac{1}{4} - \frac{\sqrt{3}}{4}i$

d) $-2\sqrt{3} + 2i$

3) Convert the following complex numbers from trigonometric form to rectangular form.

a) $\sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right)$

b) $2 \left(\cos \left(-\frac{\pi}{4} \right) + i \sin \left(-\frac{\pi}{4} \right) \right)$

c) $\frac{1}{2} \left(\cos \left(-\frac{\pi}{3} \right) + i \sin \left(-\frac{\pi}{3} \right) \right)$

d) $4 \left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right)$

4) Verify that the formula in equation (1) works by multiplying the following pairs of complex numbers two different ways: first, by multiplying the numbers in rectangular form, and second, by converting the numbers to trigonometric form, applying equation (1), and converting back to rectangular form.

a) $1 + i$ and $\sqrt{2} - \sqrt{2}i$

b) $\frac{1}{4} - \frac{\sqrt{3}}{4}i$ and $-2\sqrt{3} + 2i$

c) i and i

d) -1 and $-i$

5) Let $u = 2 \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right)$ and $v = \sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right)$.

a) Convert both u and v into rectangular form and calculate $\frac{1}{u}$ and $\frac{1}{v}$. Convert both of your answers back into trigonometric form.

b) Speculate as to a formula for the trigonometric form of $\frac{1}{z}$ where $z = r(\cos \theta + i \sin \theta)$.

c) Prove that your formula for the trigonometric form of $\frac{1}{z}$ where $z = r(\cos \theta + i \sin \theta)$ is valid for all nonzero z .

d) Use your formula for the trigonometric form of $\frac{1}{z}$ and equation (1) to prove that

$$\frac{u}{v} = \frac{r_u}{r_v} [\cos(\theta_u - \theta_v) + i \sin(\theta_u - \theta_v)],$$

where $u = r_u (\cos \theta_u + i \sin \theta_u)$ and $v = r_v (\cos \theta_v + i \sin \theta_v)$.

5.2 Series Representations of Functions

Most survivors of high-school algebra are familiar with the definition of a^n for a nonzero base a and rational number n . If n is a whole number, then

$$a^n = a \cdot a \cdot \dots \cdot a \text{ (} n \text{ factors of } a\text{),}$$

with the convention that $a^0 = 1$. If the exponent is a negative integer, then

$$a^{-n} = \frac{1}{a^n} \text{ for } n > 0.$$

If the exponent is a rational number, then for integer m and positive integer n ,

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m.$$

You probably also remember the **exponent rules**:

$$a^m a^n = a^{m+n}$$

$$\frac{a^m}{a^n} = a^{m-n}$$

$$(a^m)^n = a^{mn}.$$

Your high-school teacher may not have made a big deal out of the fact that we really never define a^n for all real numbers n (for example, how are we supposed to interpret 2^π ?). We can define exponential values for irrational exponents, but it requires a bit of calculus, so we will pass on the notion for now. However, we can think about a continuous exponential function

$$f(x) = a^x, \text{ for } a > 0,$$

defined explicitly for rational x and taking “in-between” values for irrational x . Figure 4 shows exponential functions with $a = 2$ and $a = 3$ on the left and in the middle, respectively.

If we look at the tangent line (that just touches the curve and gives the “slope” of the curve at that instant – again, some calculus at work here) to $y = 2^x$ at $(0, 1)$, shown on the left in Figure 4, we see that the slope of the tangent line is less than 1, as evidenced by the fact that it intersects the x -axis to the left of -1 . If we look at the tangent line to $y = 3^x$ at $(0, 1)$, shown in the middle in Figure 4, we see that the slope of the tangent line is more than 1, as evidenced by the fact that it intersects the x -axis to the right of -1 . (See the movie **basesandslope.mov** on the course webpage for a nice animation of how the curve and the tangent line change as the base changes.) It stands to reason that there must be some value between 2 and 3 such that the slope of the tangent line to the exponential function of that base at $(0, 1)$ would be equal to 1. That magical value is denoted e , after the Swiss mathematician Leon Euler (pronounced “oiler”) that first introduced the constant. The constant is irrational, with

$$e \approx 2.7182818284590452354 \dots$$

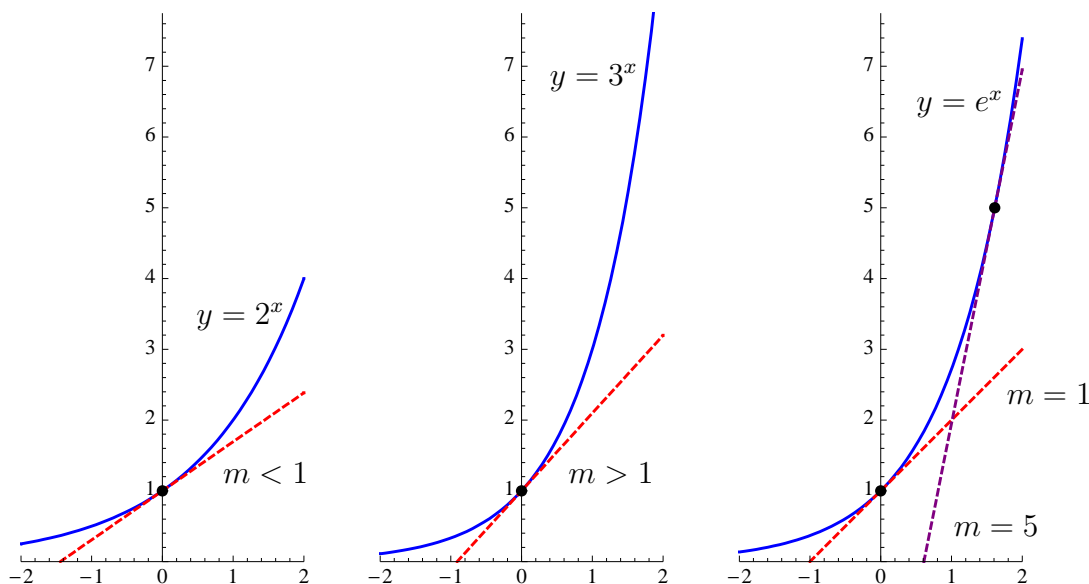


Figure 4

The graph of $y = e^x$ is shown on the right in Figure 4. It is interesting to note that the tangent line at any point on the curve has a slope equal to the y -value of the point.

Perhaps the even cooler property of this curve is that we can add up an infinite number of terms of the form

$$a_n x^n, \text{ for } n = 0, 1, \dots$$

to build this curve exactly. We define n **factorial**, denoted $n!$, for positive integer n by

$$n! = n(n-1) \cdots (1),$$

with $0!$ defined to be 1. Then

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots \quad (2)$$

Again, proving this conclusively is a calculus topic, but you can probably be convinced graphically, by graphing $y = e^x$ along with polynomial approximations to the exponential function with successively more terms. The graph of $y = e^x$ along with a degree 4 approximation is shown in Figure 5. A full animation **expapprox.mov** showing the convergence of the polynomials to $y = e^x$ as we add more terms is on the class webpage. Notice that the approximation is initially only valid at $x = 0$, but as we add more terms, the approximation gets closer to the curve over a larger interval.

We can build some of our old trigonometric functions with infinite series as well. The sine function can be restated as

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \dots \quad (3)$$

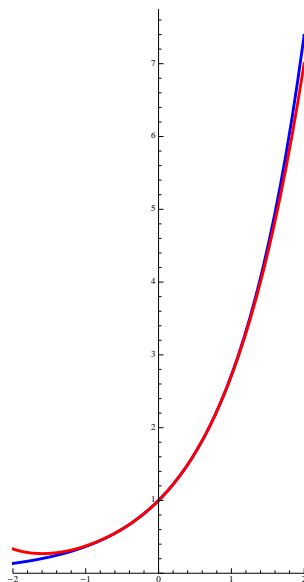


Figure 5: The blue curve is $y = e^x$. The red curve is the degree 4 polynomial approximation.

The graph of $y = \sin x$ along with a degree 7 approximation is shown in Figure 6. A full animation **sinapprox.mov** showing the convergence of the polynomials to $y = \sin x$ as we add more terms is on the class webpage. The cosine function can be restated as

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \dots \quad (4)$$

The graph of $y = \cos x$ along with a degree 6 approximation is shown in Figure 7. A full animation **cosapprox.mov** showing the convergence of the polynomials to $y = \cos x$ as we add more terms is on the class webpage.

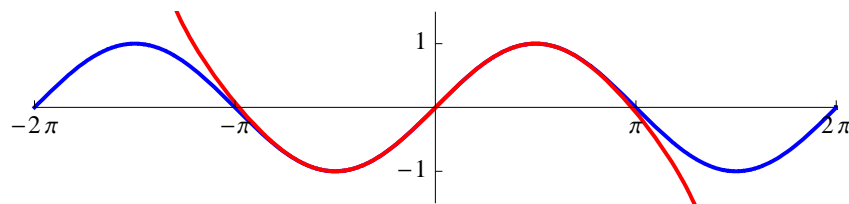


Figure 6: The blue curve is $y = \sin x$. The red curve is the degree 7 polynomial approximation.

As cool as it is to be able to use series to describe exponential, sine, and cosine functions, the coolest is yet to come.

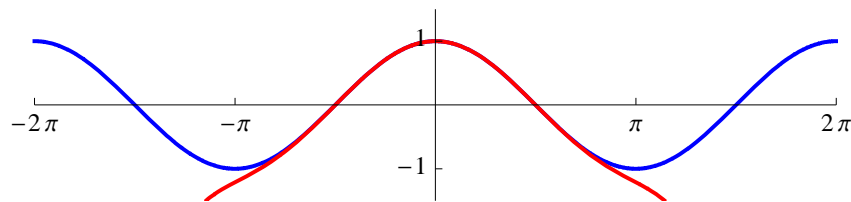


Figure 7: The blue curve is $y = \cos x$. The red curve is the degree 6 polynomial approximation.

5.2 Exercises

- 1) Use successively higher degree approximations of e^x using the formula in equation (2) to approximate e^3 . For the n^{th} term approximation, $n = 0, 1, 2, 3, 4, 5, 6$, calculate the error in your approximation compared to your calculator's decimal value of e^3 .
- 2) Use successively higher degree approximations of e^x using the formula in equation (2) to approximate e^{-3} . For the n^{th} term approximation, $n = 0, 1, 2, 3, 4, 5, 6$, calculate the error in your approximation compared to your calculator's decimal value of e^{-3} .
- 3) Use successively higher degree approximations of $\sin x$ using the formula in equation (3) to approximate $\sin \frac{3\pi}{4}$. For the n^{th} term approximation, $n = 0, 1, 2, 3, 4, 5, 6$, calculate the error in your approximation compared to your calculator's decimal value of $\sin \frac{3\pi}{4}$.
- 4) Use successively higher degree approximations of $\sin x$ using the formula in equation (3) to approximate $\sin \left(-\frac{3\pi}{4}\right)$. For the n^{th} term approximation, $n = 0, 1, 2, 3, 4, 5, 6$, calculate the error in your approximation compared to your calculator's decimal value of $\sin \left(-\frac{3\pi}{4}\right)$.
- 5) Use successively higher degree approximations of $\cos x$ using the formula in equation (4) to approximate $\cos \frac{5\pi}{6}$. For the n^{th} term approximation, $n = 0, 1, 2, 3, 4, 5, 6$, calculate the error in your approximation compared to your calculator's decimal value of $\cos \frac{5\pi}{6}$.
- 6) Use successively higher degree approximations of $\cos x$ using the formula in equation (4) to approximate $\cos \left(-\frac{5\pi}{6}\right)$. For the n^{th} term approximation, $n = 0, 1, 2, 3, 4, 5, 6$, calculate the error in your approximation compared to your calculator's decimal value of $\cos \left(-\frac{5\pi}{6}\right)$.

5.3 Euler's Formula

In the homework for the previous section, we substituted values for x in the series representations of e^x , $\sin x$, and $\cos x$ to get approximate function values and to prove some interesting infinite sum results. Now I want us to try something that may seem like folly at first – replace $\sin \theta$ and $\cos \theta$ in “ $\cos \theta + i \sin \theta$ ” with their series representations from equations (3) and (4), respectively.

$$\begin{aligned}\cos \theta + i \sin \theta &= \left(1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} - \frac{\theta^6}{720} + \dots\right) + i \left(\theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} - \frac{\theta^7}{5040} + \dots\right) \\ &= 1 + i\theta - \frac{\theta^2}{2} - i\frac{\theta^3}{6} + \frac{\theta^4}{24} + i\frac{\theta^5}{120} - \frac{\theta^6}{720} - i\frac{\theta^7}{5040} + \dots\end{aligned}$$

We would like to remove the imaginary coefficients. Notice that

$$\begin{aligned}i^0 &= 1, & i^1 &= i, & i^2 &= -1, & i^3 &= i^2 i = -i, \\ i^4 &= i^2 i^2 = (-1)(-1) = 1, & i^5 &= i^4 i = i, & i^6 &= i^4 i^2 = -1, & i^7 &= i^4 i^3 = -i, \text{ etc.}\end{aligned}$$

Then

$$\begin{aligned}\cos \theta + i \sin \theta &= 1 + i\theta + (-1)\frac{\theta^2}{2} + (-i)\frac{\theta^3}{6} + \frac{\theta^4}{24} + i\frac{\theta^5}{120} + (-1)\frac{\theta^6}{720} + (-i)\frac{\theta^7}{5040} + \dots \\ &= 1 + i\theta + i^2\frac{\theta^2}{2} + i^3\frac{\theta^3}{6} + i^4\frac{\theta^4}{24} + i^5\frac{\theta^5}{120} + i^6\frac{\theta^6}{720} + i^7\frac{\theta^7}{5040} + \dots \\ &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \frac{(i\theta)^7}{7!} + \dots = e^{i\theta},\end{aligned}$$

which is just the coolest! Not only do we have a nice, compact way of saying “ $\cos \theta + i \sin \theta$ ”, but we did not have to make it up – it’s actually equal! We are also now in a very ironic situation – we still do not know exactly how to interpret the base e raised to an irrational number like π , but we do know how to interpret e raised to a complex power like $i\pi$. In fact,

$$e^{i\pi} = \cos \pi + i \sin \pi = -1 + i(0) = -1.$$

The equation

$$e^{i\theta} = \cos \theta + i \sin \theta \tag{5}$$

is known as **Euler's formula**.

It is important to note that most (if not all) of the trigonometric identities that we struggled and took pages to prove in earlier chapters are now just a consequence of our exponent rules and complex number multiplication.

Example: Use Euler's formula to verify the sum formulas for sine and cosine.

Consider $e^{i(\alpha+\beta)}$.

$$e^{i(\alpha+\beta)} = e^{i\alpha+i\beta} = e^{i\alpha}e^{i\beta}$$

$$\cos(\alpha + \beta) + i \sin(\alpha + \beta) = (\cos \alpha + i \sin \alpha)(\cos \beta + i \sin \beta)$$

$$\cos(\alpha + \beta) + i \sin(\alpha + \beta) = (\cos \alpha \cos \beta - \sin \alpha \sin \beta) + i (\sin \alpha \cos \beta + \cos \alpha \sin \beta)$$

Then, since the complex numbers are equal only if the real parts are equal and the imaginary parts are equal, then we have the two results

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

and

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta.$$

Example: Use Euler's formula to generate triple-angle identities for sine and cosine.

Consider $e^{i(3\theta)} = (e^{i\theta})^3$. Then

$$\begin{aligned} \cos(3\theta) + i \sin(3\theta) &= (\cos \theta + i \sin \theta)^3 \\ &= (\cos \theta)^3 + 3(\cos \theta)^2(i \sin \theta) + 3(\cos \theta)(i \sin \theta)^2 + (i \sin \theta)^3 \\ &= \cos^3 \theta + 3i \cos^2 \theta \sin \theta - 3 \cos \theta \sin^2 \theta - i \sin^3 \theta \\ &= \cos^3 \theta - 3 \cos \theta \sin^2 \theta + i (3 \cos^2 \theta \sin \theta - \sin^3 \theta) \end{aligned}$$

Therefore, since the real part on the left must equal the real part on the right, and the same is true for the imaginary parts, we have

$$\cos(3\theta) = \cos^3 \theta - 3 \cos \theta \sin^2 \theta \quad \text{and} \quad \sin(3\theta) = 3 \cos^2 \theta \sin \theta - \sin^3 \theta.$$

Taking Euler's formula into account, we now have a neat way to write complex numbers when expressing them in trigonometric form:

$$r (\cos \theta + i \sin \theta) = r e^{i\theta}.$$

We will continue to call this format the trigonometric form of the complex number, although occasionally, to be more specific, we will refer to this format as the **complex exponential** form of the complex number.

5.3 Exercises

1) Write the following complex numbers, expressed in the original trigonometric form, in the new complex exponential form.

a) $4 \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right)$

b) $\frac{1}{2} \left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right)$

c) $2 \left(\cos \frac{7\pi}{3} + i \sin \frac{7\pi}{3} \right)$

d) $20 \left(\cos \frac{\pi}{12} + i \sin \frac{\pi}{12} \right)$

2) Convert the following complex numbers from rectangular form to trigonometric form using our new complex exponential format.

a) $1 + i$

b) $\sqrt{2} - \sqrt{2}i$

c) $\frac{1}{4} - \frac{\sqrt{3}}{4}i$

d) $-2\sqrt{3} + 2i$

3) Convert the following complex numbers from trigonometric form to rectangular form.

a) $\sqrt{2}e^{i\frac{\pi}{4}}$

b) $2e^{-i\frac{\pi}{4}}$

c) $\frac{1}{2}e^{-i\frac{\pi}{3}}$

d) $4e^{i\frac{5\pi}{6}}$

4) Verify that the exponent rules work with the complex exponential by multiplying the following pairs of complex numbers two different ways: first, by multiplying the numbers in rectangular form, and second, by converting the numbers to complex exponential form, applying the exponent rules, and converting back to rectangular form.

a) $1 + i$ and $\sqrt{2} - \sqrt{2}i$

b) $\frac{1}{4} - \frac{\sqrt{3}}{4}i$ and $-2\sqrt{3} + 2i$

c) i and i

d) -1 and $-i$

5) Let $u = 2e^{i\frac{\pi}{3}}$ and $v = \sqrt{2}e^{i\frac{\pi}{4}}$.

a) Convert both u and v into rectangular form and calculate $\frac{1}{u}$ and $\frac{1}{v}$. Convert both of your answers back into complex exponential form.

b) Speculate as to a formula for the complex exponential form of $\frac{1}{z}$ where $z = re^{i\theta}$. Is the formula consistent with our exponent rules?

c) Prove that your formula for the complex exponential form of $\frac{1}{z}$ where $z = re^{i\theta}$ is valid for all nonzero z .

d) Use exponent rules to prove that

$$\frac{u}{v} = \frac{r_u}{r_v} e^{i(\theta_u - \theta_v)},$$

where $u = r_u e^{i\theta_u}$ and $v = r_v e^{i\theta_v}$.

6) Using Euler's Formula, calculate the modulus of $e^{i\theta}$ in terms of the angle θ .

5.4 DeMoivre's Formula

Euler's formula has several consequences for the way we do mathematics. It has allowed us to convert complex-number multiplication problems into real-number multiplication and real-number addition problems, and visa versa; that is,

$$\text{if } u = r_u e^{i\theta_u} \text{ and } v = r_v e^{i\theta_v}, \text{ then } uv = r_u r_v e^{i(\theta_u + \theta_v)}. \quad (6)$$

But, it can also allow us to solve some rather difficult equations. Consider the equation $x^3 = 1$. Clearly, $x = 1$ is a solution, but the Fundamental Theorem of Algebra tells us that this degree-three equation should have three complex solutions, and $x = 1$ is only one of them. What other two numbers, possibly complex, can we raise to the third power and still get 1?

To answer this question, consider what happens in equation (6) if we let $z = r e^{i\theta}$ and replace both u and v with z . Then

$$z^2 = r^2 e^{i(2\theta)},$$

which is consistent with the exponent rules:

$$z^2 = (r e^{i\theta})^2 = r^2 e^{2(i\theta)} = r^2 e^{i(2\theta)}.$$

It is not hard to show that for positive integer n and a complex number $z = r e^{i\theta}$, we have

$$z^n = r^n e^{in\theta} = r^n [\cos(n\theta) + i \sin(n\theta)].$$

This is called **DeMoivre's Formula**, and it provides us with a very quick way of calculating whole number powers of complex numbers.

Example: Calculate $(1 + \sqrt{3}i)^5$ by direct multiplication and by using DeMoivre's Formula.

Direct multiplication:

$$\begin{aligned} (1 + \sqrt{3}i)^5 &= (1 + \sqrt{3}i)(1 + \sqrt{3}i)(1 + \sqrt{3}i)(1 + \sqrt{3}i)(1 + \sqrt{3}i) \\ &= (1 - 3 + \sqrt{3}i + \sqrt{3}i)(1 - 3 + \sqrt{3}i + \sqrt{3}i)(1 + \sqrt{3}i) \\ &= (-2 + 2\sqrt{3}i)(-2 + 2\sqrt{3}i)(1 + \sqrt{3}i) \\ &= (4 - 12 - 4\sqrt{3}i - 4\sqrt{3}i)(1 + \sqrt{3}i) \\ &= (-8 - 8\sqrt{3}i)(1 + \sqrt{3}i) \\ &= -8 + 24 - 8\sqrt{3}i - 8\sqrt{3}i \\ &= 16 - 16\sqrt{3}i \end{aligned}$$

DeMoivre's Formula: First, we convert $1 + \sqrt{3}i$ into its complex exponential form.

$$r = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{4} = 2 \quad \text{and} \quad \theta = \frac{\pi}{3} \text{ since } \cos \frac{\pi}{3} = \frac{1}{2} \text{ and } \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

Then $1 + \sqrt{3}i = 2e^{i\frac{\pi}{3}}$, and so

$$\begin{aligned} (1 + \sqrt{3}i)^5 &= (2e^{i\frac{\pi}{3}})^5 = 2^5 e^{i\frac{5\pi}{3}} = 32 \left(\cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3} \right) \\ &= 32 \left(\frac{1}{2} + i \left(-\frac{\sqrt{3}}{2} \right) \right) \\ &= 16 - 16\sqrt{3}i \end{aligned}$$

How does this help us solve equations like $x^3 = 1$? First, realize that $1 = e^{i(2\pi k)}$ for any integer k , so if $z = re^{i\theta}$, then

$$z = z(1) = re^{i\theta} e^{i(2\pi k)} = re^{i(\theta+2\pi k)}.$$

If we take the n^{th} -root of both sides, for positive integer n , we get

$$z^{\frac{1}{n}} = (re^{i(\theta+2\pi k)})^{\frac{1}{n}} = r^{\frac{1}{n}} e^{i\frac{\theta+2\pi k}{n}} = r^{\frac{1}{n}} \left[\cos \left(\frac{\theta}{n} + \frac{2\pi k}{n} \right) + i \sin \left(\frac{\theta}{n} + \frac{2\pi k}{n} \right) \right].$$

This result is consistent with DeMoivre's Formula, since if we raise $z^{\frac{1}{n}}$ to the n^{th} power, we get

$$\begin{aligned} (z^{\frac{1}{n}})^n &= (r^{\frac{1}{n}})^n \left[\cos n \left(\frac{\theta}{n} + \frac{2\pi k}{n} \right) + i \sin n \left(\frac{\theta}{n} + \frac{2\pi k}{n} \right) \right] \\ &= r [\cos(\theta + 2\pi k) + i \sin(\theta + 2\pi k)] \\ &= r (\cos \theta + i \sin \theta) = re^{i\theta} = z. \end{aligned}$$

We are able to find distinct n^{th} -roots for $k = 0, 1, \dots, n-1$.

Example: Find all of the solutions to the equation $x^3 = 1$.

Taking the 3rd-root of both sides we get

$$\begin{aligned} x &= 1^{\frac{1}{3}} \\ &= 1^{\frac{1}{3}} \left[\cos \left(0 + \frac{2\pi k}{3} \right) + i \sin \left(0 + \frac{2\pi k}{3} \right) \right], \text{ for } k = 0, 1, 2 \\ &= \cos 0 + i \sin 0, \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}, \cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3} \\ &= 1, -\frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} - i\frac{\sqrt{3}}{2}. \end{aligned}$$

The reader may verify by hand that each of the solutions satisfy the given equation. The solutions, known as the **third roots of unity**, are graphed in Figure 8. Notice that if you increase each angle by a factor of 3, you end up back at $z = 1$. This process is shown in the animation **thirdrootsunity.mov**.

We are now able to calculate the n^{th} roots of some very exotic numbers.

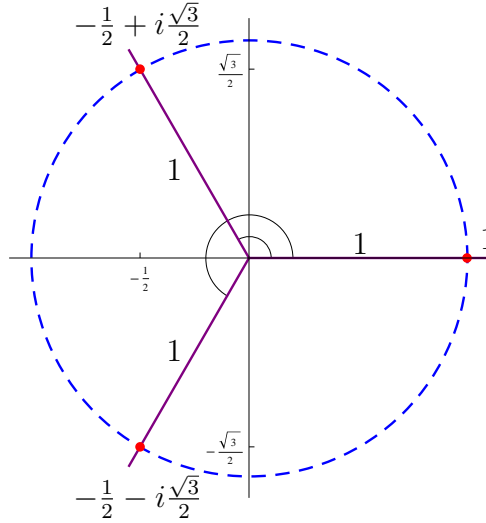


Figure 8: The third roots of unity, equally spaced on the unit circle.

Example: Find all of the fourth roots of -4 .

First, we put -4 into its complex exponential form.

$$r = \sqrt{(-4)^2 + 0^2} = \sqrt{16} = 4 \quad \text{and} \quad \theta = \pi \text{ since } \cos \pi = \frac{-4}{4} = -1 \text{ and } \sin \pi = \frac{0}{4} = 0$$

Then $-4 = 4e^{i\pi}$, and so

$$\begin{aligned} (-4)^{\frac{1}{4}} &= 4^{\frac{1}{4}} \left[\cos \left(\frac{\pi}{4} + \frac{2\pi k}{4} \right) + i \sin \left(\frac{\pi}{4} + \frac{2\pi k}{4} \right) \right], \text{ for } k = 0, 1, 2, 3 \\ &= \sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right), \sqrt{2} \left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right), \\ &\quad \sqrt{2} \left(\cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4} \right), \sqrt{2} \left(\cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4} \right) \\ &= \sqrt{2} \left(\frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \right), \sqrt{2} \left(-\frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \right), \\ &\quad \sqrt{2} \left(-\frac{\sqrt{2}}{2} - i \frac{\sqrt{2}}{2} \right), \sqrt{2} \left(\frac{\sqrt{2}}{2} - i \frac{\sqrt{2}}{2} \right) \\ &= 1 + i, -1 + i, -1 - i, 1 - i. \end{aligned}$$

We can quickly verify the validity of these answers:

$$\begin{aligned} (1 + i)^4 &= (1 - 1 + 2i)(1 - 1 + 2i) = (2i)^2 = -4, \\ (-1 + i)^4 &= (1 - 1 - 2i)(1 - 1 - 2i) = (-2i)^2 = -4, \\ (-1 - i)^4 &= (1 - 1 + 2i)(1 - 1 + 2i) = (2i)^2 = -4, \\ (1 - i)^4 &= (1 - 1 - 2i)(1 - 1 - 2i) = (-2i)^2 = -4. \end{aligned}$$

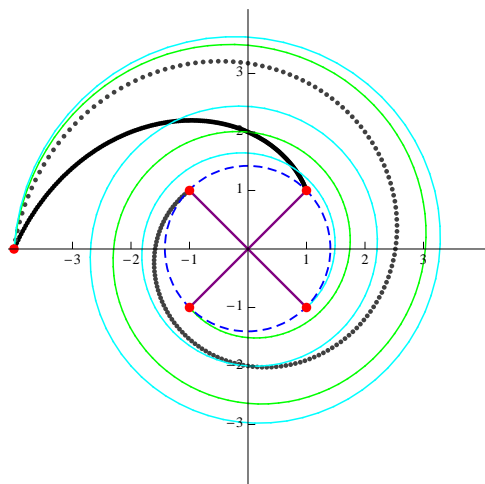


Figure 9: The four fourth roots of -4 , and the rotational paths when raising each to the fourth power.

The solutions are shown in Figure 9, along with the path the solutions would take if we raised them to the k^{th} power, for $1 \leq k \leq 4$.

5.4 Exercises

1) Calculate each of the following exactly using two methods: a) direct multiplication in rectangular form, and b) converting to trigonometric form, using DeMoivre's Formula, and converting back to rectangular form.

a) $(1 - i)^3$

b) $\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)^6$

c) $(-2i)^8$

d) $\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)^4$

e) 2^5

f) $\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^3$

2) Verify that the three third roots of unity 1 , $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$, and $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$ from the first example all satisfy the equation $x^3 = 1$.

3) Verify that the four fourth roots of unity 1 , i , -1 , and $-i$ all satisfy the equation $x^4 = 1$.

4) Find each solution to the following equations, in rectangular form if complex.

a) $x^6 = 1$

b) $x^6 = -1$

c) $x^4 = 16$

d) $x^3 = i$

- 5) Find all of the eighth roots of unity; that is, all values of $1^{\frac{1}{8}}$.
- 6) Find all third roots of $-i$; that is, all values of $(-i)^{\frac{1}{3}}$.
- 7) Use the fact that the roots of unity are equally spaced around the unit circle to find vertices needed to inscribe a regular dodecagon (equilateral 12-gon) inside a unit circle.

5.5 CORDIC Algorithm

Back in Chapter 4, we discovered the hard way that we can calculate the sine and cosine values for only a small portion of the infinite values in the interval $[0, 2\pi)$. So how is it that our calculators can give us numerical results for any value that we input? Calculators use an algorithm to calculate numerical values for sine and cosine, an algorithm based upon complex numbers, called CORDIC (**CO**ordinate **R**otation **D**igital **C**omputer). The basic idea was eluded to in the exercises from section 4.6, using the half-angle identities to find sine and cosine values halfway between the known values and using our sum and difference identities to calculate the values for more and more inputs. The only difference is that the CORDIC algorithm is more computationally efficient for binary machines like computers and calculators. Complex numbers (or, more accurately, two real numbers which serve as the real and imaginary parts of complex numbers) are used because

1. as shown in Section 5.3, complex number multiplication effectively replaces the need for the sum and difference identities, and
2. we may calculate both the sine and cosine values, hence all of the trigonometric function values, at the same time.

This section will illustrate the complex number ideas programmed into computers and calculators to calculate trigonometric function values for real number radian measures up to a certain decimal precision. The code used to demonstrate the effectiveness of the algorithms will be written in *Mathematica*TM, although the ideas could certainly be translated into any computer language. First, we will develop the algorithm with complex numbers on the unit circle in the complex plane. After using that algorithm to develop the basic idea, we will illustrate the actual CORDIC algorithm.

The Basic Idea

Recall from Section 5.1 that complex numbers of the form $\cos \theta + i \sin \theta$ fall somewhere on the unit circle in the complex plane, and from Section 5.3 that by Euler's formula, they are equal to $e^{i\theta}$. Then each value of θ corresponds to exactly one complex number on the unit circle, where the real part is the cosine of θ and the imaginary part is the sine of θ . Thus, if we know the sine and cosine values for θ_1 and θ_2 , then from equation (6), we can quickly use complex multiplication to calculate the sine and cosine values for $\theta_1 \pm \theta_2$:

$$(\cos \theta_1 + i \sin \theta_1) (\cos(\pm\theta_2) + i \sin(\pm\theta_2)) = e^{i\theta_1} e^{\pm i\theta_2} = e^{i(\theta_1 \pm \theta_2)} = \cos(\theta_1 \pm \theta_2) + i \sin(\theta_1 \pm \theta_2).$$

Notice that

$$\cos(\pm\theta_2) + i \sin(\pm\theta_2) = \cos \theta_2 \pm i \sin(\theta_2),$$

so both adding and subtracting an angle will still involve complex number multiplication. When subtracting an angle, we will simply multiply by the conjugate.

Complex multiplication is the basic idea that the CORDIC algorithm is built on. We will start with building our own “unit circle” version.

Step 1 (Initialization): We will need to generate a table of known sine and cosine values. We can start with the values for $\frac{\pi}{2}$ and use the half-angle identities to generate numerical values. This would generally be done initially and then used every time the function is called. We should also be sure to perform the calculations up to the precision of the machine on which you are working. The *Mathematica*TM code for generating the table is

```
vals = Table[{N[Pi/2^(i + 1), 20], 0, 0}, {i, 0, 39}];
vals[[1, 3]] = 1;
Do[{vals[[i, 2]] = N[Sqrt[(1 + vals[[i - 1, 2]])/2], 40];
    vals[[i, 3]] = N[Sqrt[(1 - vals[[i - 1, 2]])/2], 40];},
    {i, 2, 40}]
```

Note the use of the half-angle identities for cosine and sine to calculate the table values “vals[[i, 2]]” and “vals[[i, 3]]”, respectively. The first 20 rows of values generated, rounded to 15 decimal places, are shown in Table 1.

Table 1: Approximate Cosine and Sine Values for $\frac{\pi}{2} (2^{-k})$ Radians, $k = 0, \dots, 19$

Radians	Cosine	Sine
1.570796326794897	0.000000000000000	1.000000000000000
0.785398163397448	0.707106781186548	0.707106781186548
0.392699081698724	0.923879532511287	0.382683432365090
0.196349540849362	0.980785280403230	0.195090322016128
0.098174770424681	0.995184726672197	0.098017140329561
0.049087385212341	0.998795456205172	0.049067674327418
0.024543692606170	0.999698818696204	0.024541228522912
0.012271846303085	0.999924701839145	0.012271538285720
0.006135923151543	0.999981175282601	0.006135884649154
0.003067961575771	0.999995293809576	0.003067956762966
0.001533980787886	0.999998823451702	0.001533980186285
0.000766990393943	0.999999705862882	0.000766990318743
0.000383495196971	0.999999926465718	0.000383495187571
0.000191747598486	0.999999981616429	0.000191747597311
0.000095873799243	0.999999995404107	0.000095873799096
0.000047936899621	0.999999998851027	0.000047936899603
0.000023968449811	0.999999999712757	0.000023968449808
0.000011984224905	0.999999999928189	0.000011984224905
0.000005992112453	0.999999999982047	0.000005992112453
0.000002996056226	0.999999999995512	0.000002996056226

Time out!

There are a couple of interesting things to point out about Table 1.

- Notice that most of the cosine values in Table 1 are approximately 1. This will motivate the calculations in the real version of the CORDIC algorithm.
- This has no effect on our work here, but is a simple curiosity: notice that the radian measure of the angles and their sine values are virtually equal for small, positive radian values. This fact will come in handy later in your mathematical careers.

Time in!

Step 2 (Framing the angle): We start by putting the angle we are working with in the range $-\pi < \theta \leq \pi$.

- If $\theta \leq -\pi$, then add $\frac{\pi}{2}$ until $-\pi < \theta \leq \pi$, keeping track of the number of times it is added.
- If $\theta > \pi$, then subtract $\frac{\pi}{2}$ until $-\pi < \theta \leq \pi$, keeping track of the number of times it is subtracted.

The *Mathematica*TM code for this step is

```

tmod = N[t, 30]; count = 0;
While[tmod <= - Pi, (count = count - 1; tmod = tmod + Pi/2;)];
While[tmod > Pi, (count = count + 1; tmod = tmod - Pi/2;)];

```

Example: Suppose that the input angle is $\frac{7\pi}{4}$. Since this angle is greater than π , then

$$\mathbf{tmod} = \frac{7\pi}{4} - \frac{\pi}{2} = \frac{5\pi}{4} \quad \text{and} \quad \mathbf{count} = 0 + 1 = 1.$$

Since this new angle is still greater than π , then

$$\mathbf{tmod} = \frac{5\pi}{4} - \frac{\pi}{2} = \frac{3\pi}{4} \quad \text{and} \quad \mathbf{count} = 1 + 1 = 2.$$

Step 3 (Rotations): Our goal is to progressively accumulate the necessary angle to rotate our given angle from 0, while simultaneously rotating the point on the unit circle that corresponds to 0 radians, $1 + 0i$, the same amount. We start with an initial accumulated angle of 0, and step through each row of our generated table.

- If our given angle minus the accumulated angle is greater than or equal to 0, then we multiply our current complex number by the first complex number $a + bi$ in our table and add our table angle to the accumulated angle.
- If our given angle minus the accumulated angle is less than 0, then we multiply by the conjugate of the first complex number $a + bi$ in our table and subtract our table angle from the accumulated angle.
- Repeat these steps for each row of the table.

The *Mathematica*TM code for this step is

```

acc = 0; curr = 1;
Do[(temp = tmod - acc;
  If[temp >= 0,
    (curr = curr*(vals[[i, 2]] + I*vals[[i, 3]]);
    acc = acc + vals[[i, 1]]);
    (curr = curr*(vals[[i, 2]] - I*vals[[i, 3]]);
    acc = acc - vals[[i, 1]]);)],
{i, 1, 40}];

```

Example (continued): With the input angle $\frac{7\pi}{4}$, we actually work with the angle **tmod** = $\frac{3\pi}{4}$ and set **count** to 2. With the initial accumulated angle **acc** = 0 and the current complex number **curr** = $1 + 0i$, we subtract our current accumulated angle from **tmod**:

$$\mathbf{temp} = \frac{3\pi}{4} - 0 = \frac{3\pi}{4}.$$

Since **temp** ≥ 0 , we multiply by the $\cos \theta + i \sin \theta$ in the first row of Table 1, and add the first row angle to the accumulated angle:

$$\mathbf{curr} = (1 + 0i)(0 + 1i) = i \quad \text{and} \quad \mathbf{acc} = 0 + \frac{\pi}{2} = \frac{\pi}{2}.$$

We then repeat the process for each row of the table. In the second pass, we subtract our current accumulated angle from **tmod**:

$$\mathbf{temp} = \frac{3\pi}{4} - \frac{\pi}{2} = \frac{\pi}{4}.$$

Since **temp** ≥ 0 , then we multiply **curr** by the $\cos \theta + i \sin \theta$ in the second row of Table 1, and add the second row angle to the accumulated angle:

$$\mathbf{curr} \approx i(0.70710678 + 0.70710678i) = -0.70710678 + 0.70710678i \quad \text{and} \quad \mathbf{acc} = \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}.$$

It is important to note that although the accumulated angle is now exactly equal to our adjusted angle, the algorithm does not stop – it always runs through each row of the table, for speed reasons. The repeated condition checking to see if the angle has been accumulated would take longer than just running through the rows, and since we are dealing with decimal approximations, it would only rarely stop the algorithms mid-run. The next row will over-shoot the given angle by $\frac{\pi}{8}$, but the rest of the rows will pull the accumulated angle back to $\frac{3\pi}{4}$.

Step 4 (Unframing the angle): We just need to multiply by i for each $\frac{\pi}{2}$ we subtracted in Step 1, or likewise, multiply by $-i$ or, equivalently, divide by i , for each $\frac{\pi}{2}$ we had to add. The *Mathematica*TM code for this step is

```
curr = curr*I ^ count;
```

Example (continued): After going through all of the rows of the table, we would have

$$\mathbf{curr} \approx -0.70710678 + 0.70710678i.$$

Since we subtracted $\frac{\pi}{2}$ from our initial angle twice (**count**= 2), then we need to multiply by i twice to get the complex number that corresponds to our original angle:

$$\mathbf{curr} \approx (-0.70710678 + 0.70710678i)i^2 = (-0.70710678 + 0.70710678i)(-1)$$

$$\mathbf{curr} \approx 0.70710678 - 0.70710678i$$

Step 5 (Output of results): The cosine of the given angle will be the real part of our resulting complex number and the sine of the given angle will be the imaginary part of the complex number. The various *Mathematica*TM print commands are shown below.

```
s = N[Im[curr], 8];
c = N[Re[curr], 8];
places[x_, n_] := (num = Floor[Log[10, Abs[x]]];
  Return[N[x, n + num + 1]]);
Print["The sine of ", t, " is ", places[s, 8], "."];
Print["The cosine of ", t, " is ", places[c, 8], "."];
Print["The tangent of ", t, " is ", places[s/c, 8], "."];
Print["The secant of ", t, " is ", places[1/c, 8], "."];
Print["The cosecant of ", t, " is ", places[1/s, 8], "."];
Print["The cotangent of ", t, " is ", places[c/s, 8], "."];
```

(The **places** function is there to give us the desired number of decimal places. *Mathematica*TM works on significant digits rather than decimal places, so we have to adjust the number of significant digits we ask for to get the requested number of decimal places.)

Example (continued): Notice that the real part of **curr** gives the numerical value for $\cos \frac{7\pi}{4}$ and that the imaginary part gives the numerical value for $\sin \frac{7\pi}{4}$.

The *Mathematica*TM file **CORDIC.nb** on the class webpage contains all of the code that has been discussed here.

The Real CORDIC Algorithm

While the algorithm we just developed is very efficient itself, the calculations taking place are not. Computers and calculators work more efficiently when performing operations on decimals that involve only powers of two. For example, “0.1” is a one-digit decimal number to us, but to a calculator,

$$0.1_{\text{ten}} = 0.0001100110011 \dots_{\text{two}} = 0.\overline{00011}_{\text{two}},$$

a repeating, non-terminating decimal, since the positions now are valued at one-half, one-fourth, one-eighth, etc., all powers of two in the denominator, none of which will be divisible by 5. The result is that the computer has to truncate the decimal at some level, which introduces some

rounding error, albeit a very small amount. If we just work with fractions whose denominators are positive powers of 2, then the binary decimal representation is all zeroes except for the closing digit, which is one. Also, if we multiply two such numbers, we get another fraction whose denominator is a positive power of 2. Hence, multiplying two numbers like that is just a shift of the position of the one.

Also, to a computer, a complex number is really just a pair of real numbers. If your computer or calculator can do operations with complex numbers, it's only because someone taught it how to: given the four numbers a, b, c, d where a and c are real parts of two complex numbers and b and d are imaginary parts of two complex numbers, then since

$$(a + bi)(c + di) = ac - bd + (ad + bc)i,$$

the real part of the product can be defined as $ac - bd$ and the imaginary part as $ad + bc$. If in our algorithm, we replace all of the cosine values with “1” (most were nearly 1 anyway) and replace the sine values with fractions with denominators that are positive powers of two, then the real part of the solution is $1 - bd$ (a shift of the position of the “1” and an addition) and the imaginary part is $d + c$ (an addition). Thus, no multiplications need to be performed – additions and shifts are carried out much more efficiently by our binary machines. The table values are no longer on the unit circle, but on a vertical line through $1 + 0i$, as illustrated in Figure 10. The corresponding angles are not the same, and have to be calculated ahead of time. Also, since the moduli of the numbers is no longer 1, we have to divide by the product of the moduli of the complex numbers in the table, called the **CORDIC gain**, which is approximately 1.64676026. However, the basic idea is the same as the unit-circle version we illustrated earlier.

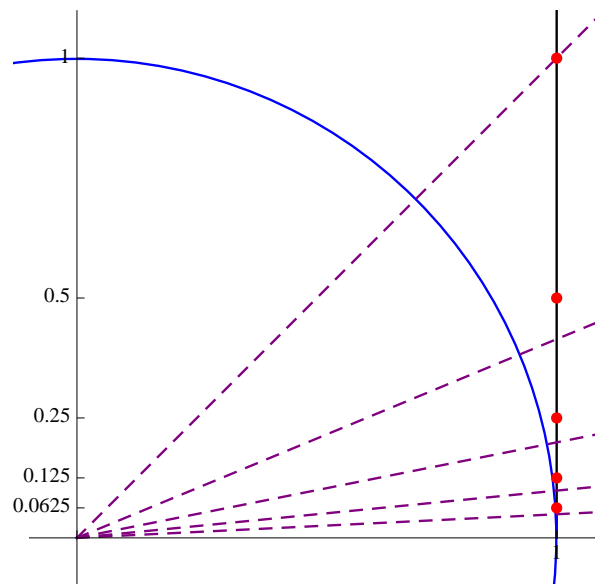


Figure 10: The CORDIC table values that take advantage of binary calculations.

5.5 Exercises

1) Perform Step 2 of the CORDIC algorithm on the following angles to get an angle θ in the range $-\pi < \theta \leq \pi$. Give the value of **tmod** and **count** at the end of the step for each angle.

a) $\frac{7\pi}{6}$

b) $\frac{23\pi}{12}$

c) $-\frac{8\pi}{3}$

d) 3

2) Using exact values for the first three rows of Table 1 (given below), perform Step 3 of our unit-circle version of the CORDIC algorithm on the following angles, converted to decimals, to find the third-row CORDIC approximation of **tmod**. Then perform Step 4 to unframe the angle. Give the final value of **curr**, with the real and imaginary parts rounded to 8 decimal places. Compare to your calculator's values for the cosine and sine of the angle, respectively.

Radians	Cosine	Sine
$\frac{\pi}{2}$	0	1
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{8}$	$\frac{\sqrt{2 + \sqrt{2}}}{2}$	$\frac{\sqrt{2 - \sqrt{2}}}{2}$

a) $\frac{7\pi}{6}$

b) $\frac{23\pi}{12}$

c) $-\frac{8\pi}{3}$

d) 3

3) Using the *Mathematica*TM file **CORDIC.nb** on the class webpage, calculate all of the trigonometric function values for the following angles. Compare to the computer's values for the trigonometric functions at each angle.

a) $\frac{7\pi}{6}$

b) $\frac{23\pi}{12}$

c) $-\frac{8\pi}{3}$

d) 3

4) The numerical precision and the depth of the table used in the CORDIC algorithm become extremely important as we put in values that are close to the locations of vertical asymptotes with the tangent, cotangent, secant, or cosecant functions; that is, multiples of $\frac{\pi}{2}$. Use the *Mathematica*TM file **CORDIC.nb** on the class webpage to calculate all of the trigonometric function values for the following angles. Compare to the computer's values for the trigonometric functions at each angle.

a) $\pi = 0.0001$

b) $\frac{\pi}{2} + 0.00001$

c) $-\frac{\pi}{2} + 0.001$

d) 0.000001

- 5) While the unit-circle version of CORDIC that we developed uses the half-angle identities to generate points on the unit circle associated with the angles $\frac{\pi}{2} \left(2^{-k} \right)$ radians for integer $k \geq 0$, the actual binary-efficient version of CORDIC uses the point $(0, 1)$ for $\frac{\pi}{2}$ and points $1 + 2^{-k}i$ for integer $k \geq 0$ for the rest of the table (illustrated in Figure 10). What are the angles associated with these points for $k = 0, \dots, 10$, in radians, rounded to six decimal places? Compare each to the analogous angle in the unit-circle version, given in Table 1.
- 6) Verify that the CORDIC gain is approximately 1.64676026 by multiplying the moduli of the complex numbers $1 + 2^{-k}i$ for $k = 0, \dots, 14$.

5.6 Review: What have we learned (or relearned)?

Definition: Complex Conjugate The complex conjugate of $a + bi$, denoted $\overline{a + bi}$, is $a - bi$.

Definition: Modulus of a Complex Number The modulus of a complex number $a + bi$, denoted $|a + bi|$, is given by

$$|a + bi| = \sqrt{(a + bi)(\overline{a + bi})} = \sqrt{a^2 + b^2}.$$

Definition: Trigonometric Form of a Complex Number The trigonometric form of the complex number $a + bi$ is given by

$$r(\cos \theta + i \sin \theta),$$

where $r = \sqrt{a^2 + b^2}$ (the modulus of $a + bi$) and θ is in the quadrant of the complex plane containing $a + bi$ so that $\tan \theta = \frac{b}{a}$.

Theorem: Suppose that we have two complex numbers in trigonometric form,

$$u = r_u(\cos \theta_u + i \sin \theta_u) \quad \text{and} \quad v = r_v(\cos \theta_v + i \sin \theta_v).$$

Then the trigonometric form of the product uv is given by

$$uv = r_u r_v [\cos(\theta_u + \theta_v) + i \sin(\theta_u + \theta_v)].$$

Thus, the modulus of the product is the product of the moduli, and the angle in the trigonometric form of the product is the sum of the two angles.

Definition: Series Representations of Basic Functions

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \dots$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \dots$$

Theorem: Euler's Formula For any real number θ , we have

$$e^{i\theta} = \cos \theta + i \sin \theta.$$

Definition: Complex Exponential Form of a Complex Number The complex exponential form of a complex number with trigonometric form $r(\cos \theta + i \sin \theta)$ is $re^{i\theta}$.

Theorem: DeMoivre's Formula For a complex number $z = re^{i\theta} = r(\cos \theta + i \sin \theta)$ and positive integer n , we have

$$z^n = r^n e^{in\theta} = r^n [\cos(n\theta) + i \sin(n\theta)].$$

Theorem: n^{th} Root of a Complex Number For a complex number $z = re^{i\theta} = r(\cos \theta + i \sin \theta)$ and positive integer n , we have

$$z^{\frac{1}{n}} = r^{\frac{1}{n}} e^{i\frac{\theta+2\pi k}{n}} = r^{\frac{1}{n}} \left[\cos \left(\frac{\theta}{n} + \frac{2\pi k}{n} \right) + i \sin \left(\frac{\theta}{n} + \frac{2\pi k}{n} \right) \right]$$

for $k = 0, 1, \dots, n-1$.

The CORDIC Algorithm Our calculators and computers use this numerical algorithm based on complex number multiplication to calculate approximate sine and cosine values quickly and efficiently. We developed a unit-circle version of the algorithm. The steps of the algorithm are as follows:

- **Step 1 (Initialization):** Generate a table of cosine and sine values, starting at $\frac{\pi}{2}$, using the half-angle identities.
- **Step 2 (Framing the angle):** Place the angle θ in the range $-\pi < \theta \leq \pi$ by repeatedly adding or subtracting $\frac{\pi}{2}$, while keeping track of the number of times this is done.
- **Step 3 (Rotations):** Rotate the initial angle 0 to the given angle θ by adding and subtracting table angles ϕ , while simultaneously multiplying the initial complex number $1 + 0i$ by the complex numbers $\cos \phi \pm i \sin \phi$, respectively.
- **Step 4 (Unframe the angle):** Multiply by $\pm i$ according to the number of times $\frac{\pi}{2}$ was added or subtracted in Step 2.
- **Step 5 (Output the results):** The cosine of θ will be the real part of the resulting complex number, and the sine of θ will be the imaginary part.

The real CORDIC algorithm replaces the cosines and sines of the half-angles in our algorithm with the complex numbers $1 + 2^{-n}i$, $n = 0, 1, \dots$ and the corresponding angles to take advantage of efficiencies in binary calculations.

Review Exercises

1) Perform the following complex number operations.

a) $(4 - 2i) + (-1 + 8i)$

b) $(2 - 8i) - (3 - 7i)$

c) $(4 - i) \left(2 + \frac{1}{2}i \right)$

d) $\frac{1 + i}{-1 + 3i}$

2) Convert the following complex numbers from rectangular form to trigonometric form.

a) $1 - i$

b) $\sqrt{7} + \sqrt{21}i$

c) $-\frac{1}{8}i$

d) $-5 - 5i$

3) Convert the following complex numbers from trigonometric form to rectangular form.

a) $2\sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$ b) $2\sqrt{2}\left(\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right)$

4) Verify that the formula in equation (1) works by multiplying the following pairs of complex numbers two different ways: first, by multiplying the numbers in rectangular form, and second, by converting the numbers to trigonometric form, applying equation (1), and converting back to rectangular form.

a) $\sqrt{3} + 3i$ and $-i$ b) $\sqrt{2} - \sqrt{2}i$ and $\frac{\sqrt{2}}{4} + \frac{\sqrt{2}}{4}i$

5) Use successively higher degree approximations of e^x using its series representation, given in equation (2), to approximate e . For the n^{th} term approximation, $n = 0, 1, 2, 3, 4, 5, 6$, calculate the error in your approximation compared to your calculator's decimal value of e .

6) Use successively higher degree approximations of $\sin x$ using its series representation, given in equation (3), to approximate $\sin 1$. For the n^{th} term approximation, $n = 0, 1, 2, 3, 4, 5, 6$, calculate the error in your approximation compared to your calculator's decimal value of $\sin 1$.

7) Use successively higher degree approximations of $\cos x$ using its series approximation, given in equation (4), to approximate $\cos 1$. For the n^{th} term approximation, $n = 0, 1, 2, 3, 4, 5, 6$, calculate the error in your approximation compared to your calculator's decimal value of $\cos 1$.

8) Convert the following complex numbers from rectangular form to complex exponential form.

a) $1 - i$ b) $\sqrt{7} + \sqrt{21}i$
c) $-\frac{1}{8}i$ d) $-5 - 5i$

9) Convert the following complex numbers from complex exponential form to rectangular form.

a) $2\sqrt{2}e^{i\frac{5\pi}{4}}$ b) $2\sqrt{2}e^{-i\frac{3\pi}{4}}$

10) Verify that the exponent rules work with the complex exponential by multiplying the following pairs of complex numbers two different ways: first, by multiplying the numbers in rectangular form, and second, by converting the numbers to complex exponential form, applying the exponent rules, and converting back to rectangular form.

a) $\sqrt{3} + 3i$ and $-i$ b) $\sqrt{2} - \sqrt{2}i$ and $\frac{\sqrt{2}}{4} + \frac{\sqrt{2}}{4}i$

11) Calculate each of the following exactly using two methods: a) direct multiplication in rectangular form, and b) converting to trigonometric form, using DeMoivre's Formula, and converting back to rectangular form.

a) $(1 + i)^4$ b) $(1 - \sqrt{3}i)^3$

12) Find each solution to the following equations. In each case, verify that your solutions satisfy the equation.

a) $x^4 = -1$ b) $x^6 = 64$

13) Perform Step 2 of the CORDIC algorithm on the following angles to get an angle θ in the range $-\pi < \theta \leq \pi$. Give the value of **tmod** and **count** at the end of the step for each angle.

a) $\frac{13\pi}{6}$

b) -5π

14) Using exact values for the first three rows of Table 1 (given below), perform Step 3 of our unit-circle version of the CORDIC algorithm on the following angles, converted to decimals, to find the third-row CORDIC approximation of **tmod**. Then perform Step 4 to unframe the angle. Give the final value of **curr**, with the real and imaginary parts rounded to 8 decimal places. Compare to your calculator's values for the cosine and sine of the angle, respectively.

Radians	Cosine	Sine
$\frac{\pi}{2}$	0	1
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{8}$	$\frac{\sqrt{2 + \sqrt{2}}}{2}$	$\frac{\sqrt{2 - \sqrt{2}}}{2}$

a) $\frac{5\pi}{6}$

b) $-\frac{\pi}{3}$

Chapter 6: Trigonometry and Sound

MATH 117: Trigonometry

6.1 Introduction to Sound

Sound is a pressurized wave through a medium, that is interpreted by our ears as speech, music, noise, etc. The medium carries the wave – there is no sound in the vacuum of space, for example, despite what you may have seen in *Star Wars* and other science-fiction movies. The medium affects the sound that you hear. Voices sound different underwater, for example, than above water.

There are two types of waves that occur in nature. A **transversal** wave occurs in strings and surfaces, at a right angle to the surface. Examples of transversal waves in nature would include

- ripples on the surface of a pond after a rock is thrown in,
- a guitar string after it is plucked, and
- the surface of a drum after it is struck.

It should be no surprise that transversal waves are naturally modeled with sine and cosine functions of various amplitudes and periods. See the animations **transversal1.mov** and **transversal2.mov** on the class webpage to see transversal waves in motion.

The other type of naturally-occurring waves are **longitudinal** waves. Longitudinal waves occur through mediums, parallel to the movement of the wave through the medium. Imagine taking a Slinky™ and “slink-ing” it from one hand to the other. This is a longitudinal wave. Examples of longitudinal waves in nature would include

- waves at the beach and
- sound through any medium.

Although the oscillation is parallel to the direction of movement, we can still use sines and cosines to model longitudinal waves. We simply take the displacement of the medium, and place it at a right angle to time on a time-vs.-distance graph. See the animation **longitudinal.mov** to see a longitudinal wave and **together.mov** to see how it can be modeled with sines and cosines, both of which are on the class webpage.

When modeling sound waves with sines and cosines, we will have two quantities that we will measure. The **amplitude** of the sine or cosine will correspond to the loudness of the sound. The **frequency** is the number of oscillations of the sound wave over a period of time, typically measured in **hertz (Hz)**, or cycles per second. The frequency of a sound wave

corresponds to the pitch of the sound. The frequency F in hertz of a sound wave is inversely proportional to the period p of the sine or cosine with seconds as the independent variable:

$$F = \frac{1}{p}.$$

The number of cycles that a sound wave makes over a period of time T seconds long is the product of the frequency F in hertz times T , or

$$\text{number of oscillations} = FT = \left(\frac{1}{p}\right)(T) = \frac{T}{p}. \quad (1)$$

If the two curves shown in Figure 1 represent sounds, then the sound from the curve at left is louder than the one at right, but at a lower pitch.

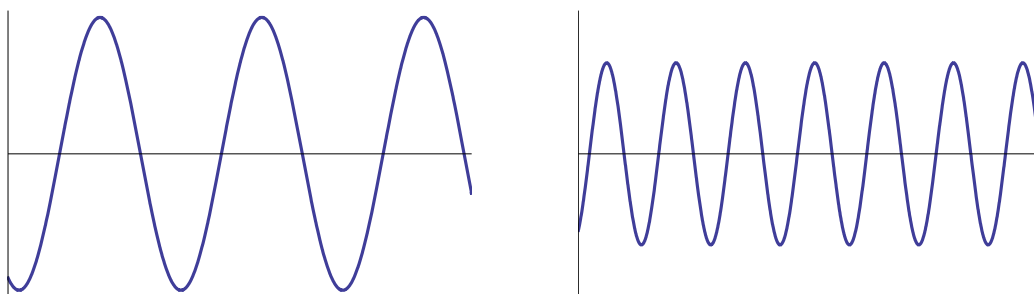


Figure 1

Example: Find the amplitude, period, and frequency of the sound wave represented by $\frac{1}{2} \cos(86\pi t)$, where t is measured in seconds. How many oscillations will the sound wave make over 8 seconds?

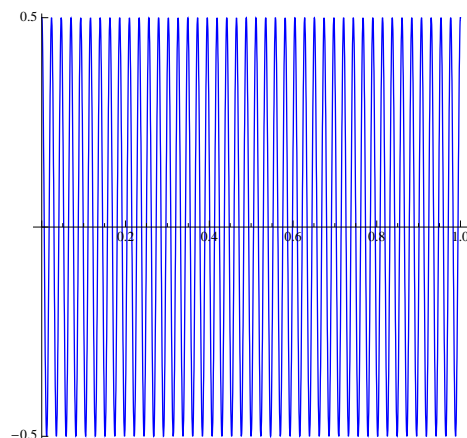
The amplitude is the leading coefficient $a = \frac{1}{2}$. The period is

$$p = \frac{2\pi}{86\pi} = \frac{1}{43} \text{ seconds.}$$

Therefore, the frequency is

$$F = \frac{1}{\frac{1}{43}} = 43 \text{ Hz.}$$

The number of oscillations over 8 seconds would be $43 \cdot 8 = 344$. The graph of the function is shown over 1 second. Notice that the graph oscillates 43 times over 1 second.



Sound is a natural tool to use when talking about modeling situations with sines and cosines, since it is so easy to generate and manipulate. The question that we have to answer is, how do we use trigonometry to analyze, generate, and alter sounds? We start to answer this question by noting some of the curious properties of regularly-sampled sine and cosine functions.

6.1 Exercises

1) Find the amplitude and period of the following sine and cosine functions. Assuming the curves represent sound waves and t is measured in seconds, find the frequency in hertz of each of the following, and find the number of oscillations over the given time span.

a) $\sin(3t)$; 5 seconds

b) $5 \cos(2\pi t)$; 2 seconds

c) $\frac{3 \cos(\frac{\pi}{4}t)}{2}$; 1.5 seconds

d) $\sin(800\pi t)$; 1 minute

e) $\cos\left(\frac{7\pi}{3}t\right)$; $\frac{1}{3}$ second

f) $4 \sin(24000\pi t)$; $\frac{1}{12000}$ second

2) Let sound A be represented by the function $f(t)$, and sound B be represented by the function $g(t)$. Determine which sound is louder, and which has the higher pitch. Explain your answer.

a) $f(t) = 2 \sin(3t)$ and $g(t) = \sin(3\pi t)$

b) $f(t) = 15 \cos(200\pi t)$ and $g(t) = 20 \sin(300\pi t)$

c) $f(t) = 5 \sin\left(\frac{231\pi}{4}t\right)$ and $g(t) = 4 \sin(181t) + 4 \cos(181t)$ (See 1.5 #9 and 1.6 #10.)

d) $f(t) = \sqrt{\frac{3}{2}} \sin\left(\pi t - \frac{\pi}{3}\right)$ and $g(t) = \sin(3.14t) - \cos(3.14t)$

3) Develop a formula for the frequency of a sound wave represented by either $a \sin(bt - c)$ or $a \cos(bt - c)$.

- 4) Missing from our conversation of amplitude and frequency is the constant function.
- Generally, we say that the period of a function is the smallest nonzero value p such that $f(t + p) = f(t)$ for all t . Given that definition, what is the period of the constant function $f(t) = 1$, if it exists?
 - Given our current definition of frequency based on the period of a function, what is the frequency of the constant function $f(t) = 1$, if it exists?
 - To find the actual frequency of $f(t) = 1$, graph the functions $y = \cos\left(\frac{2\pi}{p}t\right)$ for $p = 10^k$, $k = 1, 2, \dots, 8$, where t measures time in seconds, over the interval $[0, 1]$. What is the frequency of each graph? What do the graphs look like for very large values of p ?
 - Based on the results of part c), what is a more plausible answer for the frequency of the constant function $f(t) = 1$? Since $\cos(0t) = 1$, is this answer consistent with your formula for frequency found in problem 3)? Is this answer consistent with the fact that $f(t) = 1$ has no oscillations?
 - Generally, the amplitude, which measures the height of fluctuations, of $a \cos(bt - c)$ is a . What is the amplitude of $\cos(0t) = 1$? Do we need to restate the first sentence of this problem regarding amplitude?

6.2 Properties of Sampled Sine and Cosine Values

In order to better understand the curious phenomenon of regularly-sampled sines and cosines, we need to introduce the concept of vectors and dot products.

n -Vectors and Dot Products

A **vector** of length n , called a n -**vector**, is an ordered list of length n , denoted

$$\mathbf{u} = (u_1, u_2, \dots, u_n).$$

If $n = 2$ or 3 , we can interpret vectors graphically, as directed line segments starting at the origin and ending at the associated line segment. See Figure 2 for examples of vectors when $n = 2$ and $n = 3$.

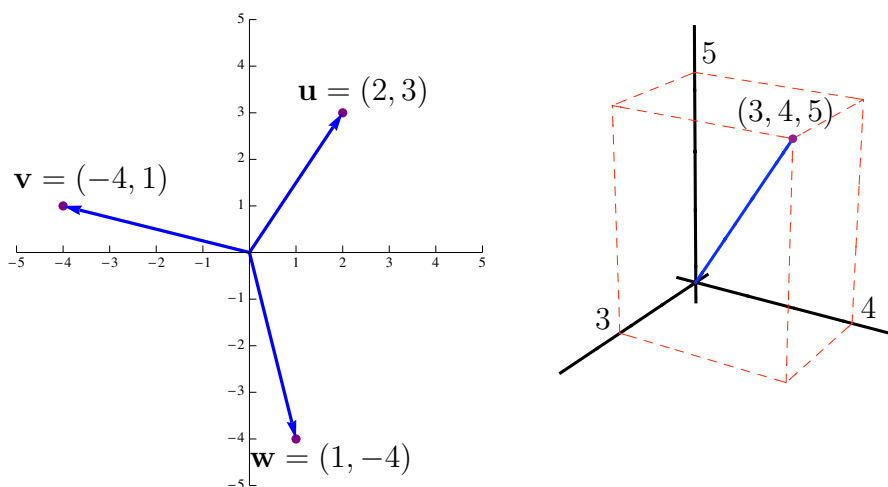


Figure 2

The **dot product** of two n -vectors $\mathbf{u} = (u_1, \dots, u_n)$ and $\mathbf{v} = (v_1, \dots, v_n)$, denoted $\mathbf{u} \cdot \mathbf{v}$, is given by

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + \dots + u_n v_n.$$

When $n = 2$, the square root of $\mathbf{u} \cdot \mathbf{u} = u_1^2 + u_2^2$ gives the length of the vector \mathbf{u} . We call this the **norm** of the vector, denoted $\|\mathbf{u}\|$, defined as

$$\|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}} = \sqrt{u_1^2 + u_2^2}.$$

We extend the concept of the norm as “length” of the vector for all values of n , with

$$\|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}} = \sqrt{u_1^2 + u_2^2 + \dots + u_n^2}.$$

The real question is, what does the dot product of \mathbf{u} and \mathbf{v} tell us about u and v ? To investigate this question, let’s look at some examples with $n = 2$.

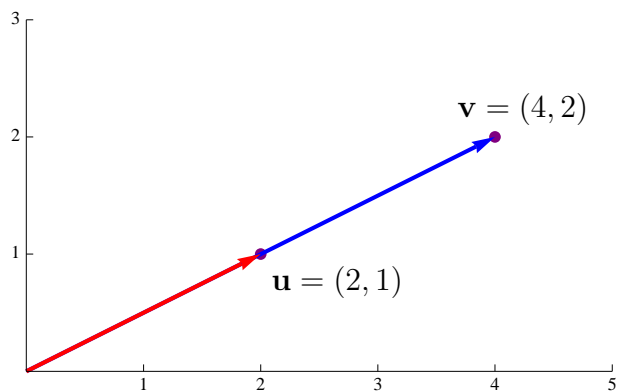


Figure 3

Example: Let $\mathbf{u} = (2, 1)$ and $\mathbf{v} = (4, 2)$, as in Figure 3. Find the norm of both \mathbf{u} and \mathbf{v} , and then $\mathbf{u} \cdot \mathbf{v}$.

$$\|\mathbf{u}\| = \sqrt{2^2 + 1^2} = \sqrt{5} \qquad \|\mathbf{v}\| = \sqrt{4^2 + 2^2} = \sqrt{20} = 2\sqrt{5}$$

$$\mathbf{u} \cdot \mathbf{v} = 2(4) + 1(2) = 8 + 2 = 10$$

Notice that, in this case, $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|$.

Example: Let $\mathbf{u} = (2, 1)$ and $\mathbf{v} = (-2, 4)$, as in Figure 4. Find the norm of both \mathbf{u} and \mathbf{v} ,

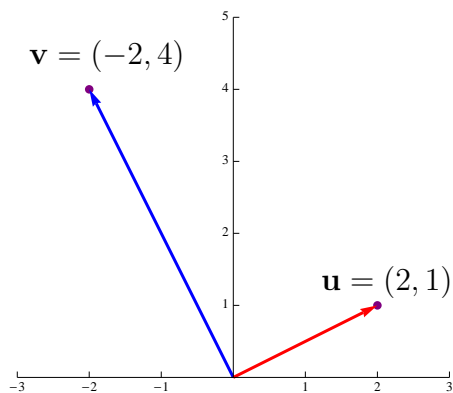


Figure 4

and then $\mathbf{u} \cdot \mathbf{v}$.

$$\|\mathbf{u}\| = \sqrt{2^2 + 1^2} = \sqrt{5} \qquad \|\mathbf{v}\| = \sqrt{(-2)^2 + 4^2} = \sqrt{20} = 2\sqrt{5}$$

$$\mathbf{u} \cdot \mathbf{v} = 2(-2) + 1(4) = -4 + 4 = 0$$

Notice that, in this case, $\mathbf{u} \cdot \mathbf{v} = 0$.

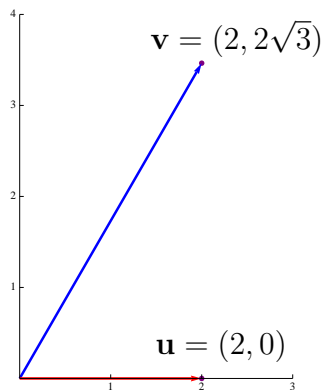


Figure 5

Example: Let $\mathbf{u} = (2, 0)$ and $\mathbf{v} = (2, 2\sqrt{3})$, as in Figure 5. Find the norm of both \mathbf{u} and \mathbf{v} , and then $\mathbf{u} \cdot \mathbf{v}$.

$$\|\mathbf{u}\| = \sqrt{2^2 + 0^2} = \sqrt{4} = 2 \quad \|\mathbf{v}\| = \sqrt{(2)^2 + (2\sqrt{3})^2} = \sqrt{4 + 12} = \sqrt{16} = 4$$

$$\mathbf{u} \cdot \mathbf{v} = 2(2) + 0(2\sqrt{3}) = 4$$

Notice that, in this case, $\mathbf{u} \cdot \mathbf{v} = \frac{1}{2}\|\mathbf{u}\|\|\mathbf{v}\|$.

The examples give us a clue as to what the dot product is giving us. Let θ be the angle between vectors \mathbf{u} and \mathbf{v} , and suppose that $\mathbf{u} \cdot \mathbf{v}$ gives us the product $\|\mathbf{u}\|\|\mathbf{v}\|$ times some function dependent upon θ ; that is,

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|f(\theta).$$

In the first example, when $\theta = 0$, we had

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|f(0) = \|\mathbf{u}\|\|\mathbf{v}\|$$

so $f(0) = 1$. In the second example, when $\theta = \frac{\pi}{2}$, we had

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|f\left(\frac{\pi}{2}\right) = 0,$$

so $f(\frac{\pi}{2}) = 0$. In the last example, when $\theta = \frac{\pi}{3}$, we had

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|f\left(\frac{\pi}{3}\right) = \frac{1}{2}\|\mathbf{u}\|\|\mathbf{v}\|,$$

so $f(\frac{\pi}{3}) = \frac{1}{2}$. Notice that the function $f(\theta)$ assumes the same values as the cosine function for these three values. In fact, it can be shown (in a higher-level course) that

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos \theta$$

where θ is the angle between \mathbf{u} and \mathbf{v} . In fact, we can extend this result for $n = 3$ and higher values, although it is hard to visualize for $n > 3$. Therefore, we can find the angle θ between the vectors \mathbf{u} and \mathbf{v} using the formula

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}.$$

This means that, if $\mathbf{u} \cdot \mathbf{v} = 0$ (and neither vector has length 0), then the vectors \mathbf{u} and \mathbf{v} are at right angles to each other (in some sense). In this case, we say that \mathbf{u} and \mathbf{v} are **orthogonal**.

Vectors of Sampled Sine and Cosine Values

Consider the trigonometric functions

$$\sin\left(\frac{2\pi n}{N}t\right) \quad \text{and} \quad \cos\left(\frac{2\pi n}{N}t\right). \quad (2)$$

The period of sine and cosine functions of the form $\sin(ct)$ and $\cos(ct)$ is $p = \frac{2\pi}{c}$, so the period of the functions in equations (2) is

$$p = \frac{2\pi}{\frac{2\pi n}{N}} = 2\pi \cdot \frac{N}{2\pi n} = \frac{N}{n}, \text{ for } n \neq 0.$$

If we look at the functions over the interval $[0, N]$, then using the formula for the number of oscillations given in equation (1), the oscillations over that interval will be

$$\frac{N}{\frac{N}{n}} = N \cdot \frac{n}{N} = n.$$

Figure 6 shows the functions in equation (2) with $N = 4$, and then $n = 0, 1, 2, 3$, each graphed over the interval $[0, 4]$.

Let \mathbf{u}_n be the vector formed by sampling $\cos\left(\frac{2\pi n}{4}t\right)$ at the values $t = 0, 1, 2, 3$, and let \mathbf{v}_n be the analogous vector formed by sampling $\sin\left(\frac{2\pi n}{4}t\right)$, for $n = 0, 1, 2, 3$. That means that

$$\begin{aligned} \mathbf{u}_0 &= (1, 1, 1, 1) & \mathbf{v}_0 &= (0, 0, 0, 0) \\ \mathbf{u}_1 &= (1, 0, -1, 0) & \mathbf{v}_1 &= (0, 1, 0, -1) \\ \mathbf{u}_2 &= (1, -1, 1, -1) & \mathbf{v}_2 &= (0, 0, 0, 0) \\ \mathbf{u}_3 &= (1, 0, -1, 0) & \mathbf{v}_3 &= (0, -1, 0, 1). \end{aligned}$$

The sampled values are shown in Figure 6. Notice that

- $\mathbf{v}_0 = \mathbf{v}_2 = 0$,
- $\mathbf{u}_1 = \mathbf{u}_3$, while $\mathbf{v}_1 = -\mathbf{v}_3$, and
- any two vectors from the set $B = \{\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1\}$ are orthogonal.

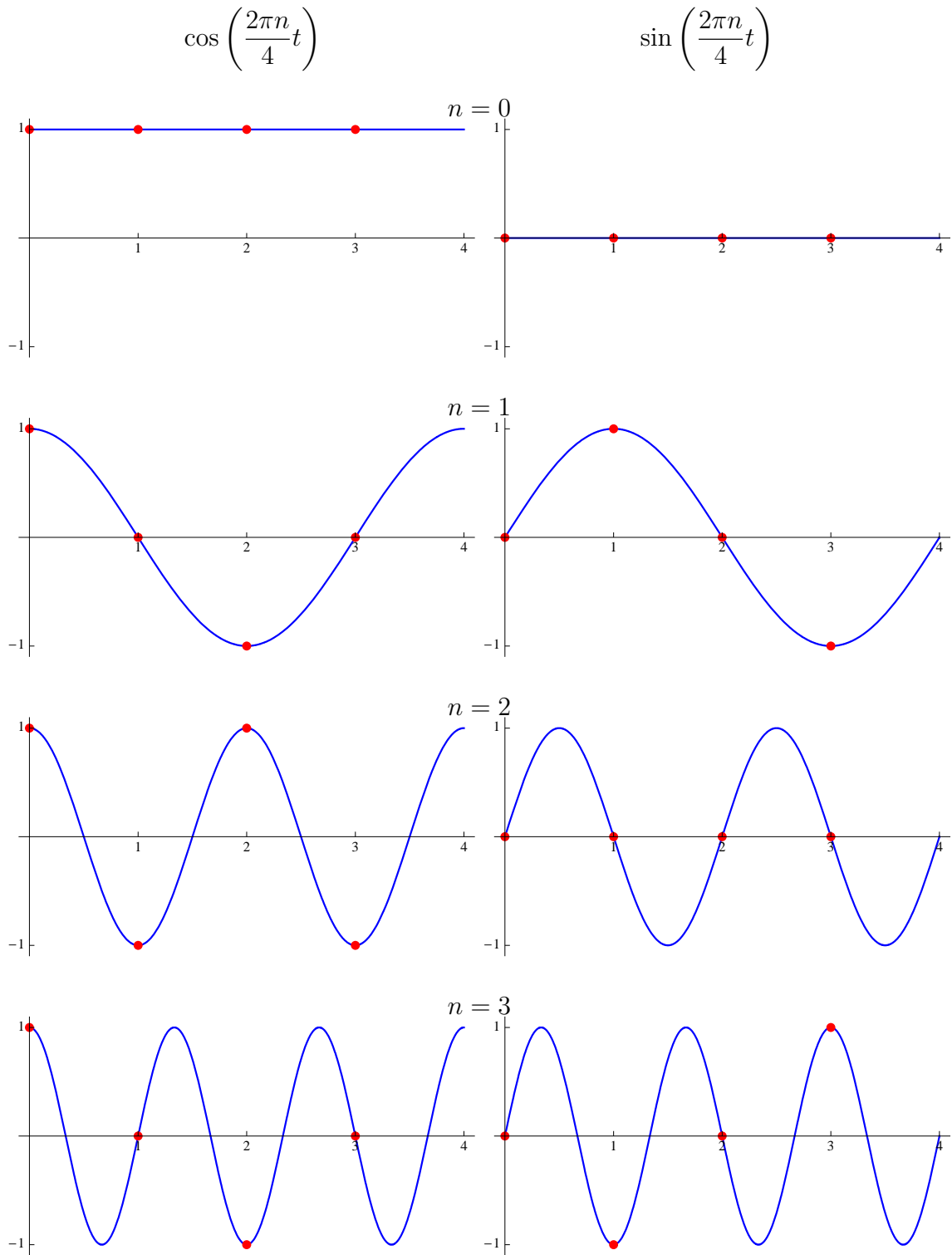


Figure 6

Furthermore, I maintain that we can build any 4-vector with the vectors in B , and I can prove it.

Consider the 4-vector (a, b, c, d) , where a, b, c , and d are real numbers. Then

$$\begin{aligned} & \frac{a+b+c+d}{4}\mathbf{u}_0 + \frac{a-c}{2}\mathbf{u}_1 + \frac{a-b+c-d}{4}\mathbf{u}_2 + \frac{b-d}{2}\mathbf{v}_1 \\ &= \frac{a+b+c+d}{4}(1, 1, 1, 1) + \frac{a-c}{2}(1, 0, -1, 0) \\ & \quad + \frac{a-b+c-d}{4}(1, -1, 1, -1) + \frac{b-d}{2}(0, 1, 0, -1). \end{aligned}$$

The first component is

$$\frac{a+b+c+d}{4} + \frac{a-c}{2} + \frac{a-b+c-d}{4} = \frac{a+2a+a}{4} + \frac{b-b}{4} + \frac{c-2c+c}{4} + \frac{d-d}{4} = a.$$

The second component is

$$\frac{a+b+c+d}{4} - \frac{a-b+c-d}{4} + \frac{b-d}{2} = \frac{a-a}{4} + \frac{b+b+2b}{4} + \frac{c-c}{4} + \frac{d+d-2d}{4} = b.$$

The third component is

$$\frac{a+b+c+d}{4} - \frac{a-c}{2} + \frac{a-b+c-d}{4} = \frac{a-2a+a}{4} + \frac{b-b}{4} + \frac{c+2c+c}{4} + \frac{d-d}{4} = c.$$

The fourth and last component is

$$\frac{a+b+c+d}{4} - \frac{a-b+c-d}{4} - \frac{b-d}{2} = \frac{a-a}{4} + \frac{b+b-2b}{4} + \frac{c-c}{4} + \frac{d+d+2d}{4} = d.$$

There is nothing unique about the number “4”. The properties we have demonstrated above for $N = 4$ hold for all positive integers N . You will investigate other cases in the exercises.

6.2 Exercises

1) Find the length of the following vectors.

a) $\left(3, \frac{9}{\sqrt{5}}\right)$

b) $(1, -2, 3)$

c) $(\sqrt{3}, \sqrt{5}, \sqrt{7})$

d) $(1, -1, 1, -1, 1, -1)$

2) Graph each of the following pairs of vectors and determine visually whether they are orthogonal (at right angles). Make sure that your aspect ratio is set so that units are the same length in both the horizontal and vertical directions. Then calculate the dot product of the two vectors to whether or not they are orthogonal.

a) $\left(3, \frac{9}{\sqrt{5}}\right)$ and $(3, -\sqrt{5})$

b) $(1, -1.1)$ and $(11, 3\pi)$

c) $(\sqrt{3}, \sqrt{5}, \sqrt{7})$ and $(-1, -1, 1.5)$

d) $(1, -1, 1)$ and $\left(\frac{1}{2}, 1, \frac{1}{2}\right)$

3) Find the measure in radians (rounded to two decimal places) of the angle between the following pairs of vectors.

- a) $(1, 0)$ and $(1, 1)$ b) $(1, 1)$ and $(1, 2)$
c) $(1, 2, 3)$ and $(-1, 2, -1)$ d) $(2, 0, 0)$ and $(1, 1, 1)$

4) Verify the claim from the $N = 4$ example in the chapter – that any pair of vectors from the set $B = \{\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1\}$, where

$$\begin{array}{ll} \mathbf{u}_0 = (1, 1, 1, 1), & \mathbf{u}_1 = (1, 0, -1, 0), \\ \mathbf{u}_2 = (1, -1, 1, -1), & \text{and} \quad \mathbf{v}_1 = (0, 1, 0, -1), \end{array}$$

are orthogonal.

5) Let \mathbf{u}_n be the vector formed by sampling $\cos\left(\frac{2\pi n}{3}t\right)$ at the values $t = 0, 1, 2$, and let \mathbf{v}_n be the analogous vector formed by sampling $\sin\left(\frac{2\pi n}{3}t\right)$, for $n = 0, 1, 2$.

- a) Find the exact values of the vectors \mathbf{u}_n and \mathbf{v}_n for $n = 0, 1, 2$. Find a set B of three of these vectors so that any pair of vectors in the set will be orthogonal.

- b) Find coefficients for each of the vectors in B so that their sum equals (a, b, c) .

6) Let \mathbf{u}_n be the vector formed by sampling $\cos\left(\frac{2\pi n}{6}t\right)$ at the values $t = 0, 1, 2, 3, 4, 5$, and let \mathbf{v}_n be the analogous vector formed by sampling $\sin\left(\frac{2\pi n}{6}t\right)$, for $n = 0, 1, 2, 3, 4, 5$.

- Find the exact values of the vectors \mathbf{u}_n and \mathbf{v}_n for $n = 0, 1, 2, 3, 4, 5$.
- Show that any pair of vectors from the set $B = \{\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{v}_1, \mathbf{v}_2\}$ are orthogonal.
- Let $\mathbf{w} = (a, b, c, d, e, f)$, and let

$$\begin{aligned} k_0 &= \frac{\mathbf{w} \cdot \mathbf{u}_0}{\mathbf{u}_0 \cdot \mathbf{u}_0}, & k_1 &= \frac{\mathbf{w} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}, & k_2 &= \frac{\mathbf{w} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}, \\ k_3 &= \frac{\mathbf{w} \cdot \mathbf{u}_3}{\mathbf{u}_3 \cdot \mathbf{u}_3}, & k_4 &= \frac{\mathbf{w} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}, & k_5 &= \frac{\mathbf{w} \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}. \end{aligned}$$

Calculate each of these coefficients exactly.

- d) Verify component by component that

$$k_0 \mathbf{u}_0 + k_1 \mathbf{u}_1 + k_2 \mathbf{u}_2 + k_3 \mathbf{u}_3 + k_4 \mathbf{v}_1 + k_5 \mathbf{v}_2 = \mathbf{w}.$$

6.3 Discrete Fourier Transform

In 1807, French mathematician Baron Jean Baptiste Joseph Fourier (1768–1830) speculated that any function $f(x)$ defined over the interval $(c, c+p)$ could be written as an infinite series of sine and cosine functions of different frequencies; that is,

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos \left(\frac{2\pi nx}{p} \right) + b_n \sin \left(\frac{2\pi nx}{p} \right) \right],$$

for some constants a_0 , a_n , and b_n for $n = 1, 2, \dots$. We call this type of infinite trigonometric series a **Fourier** series. Figure 7 shows finite sum approximations of the Fourier series for a particular function, and how it converges to the function as we add more terms. For a full illustration of this process, see the animation **Fourierseries.mov** on the class webpage. The constants a_0 , a_n , and b_n are easily defined, but can be difficult to calculate in prac-

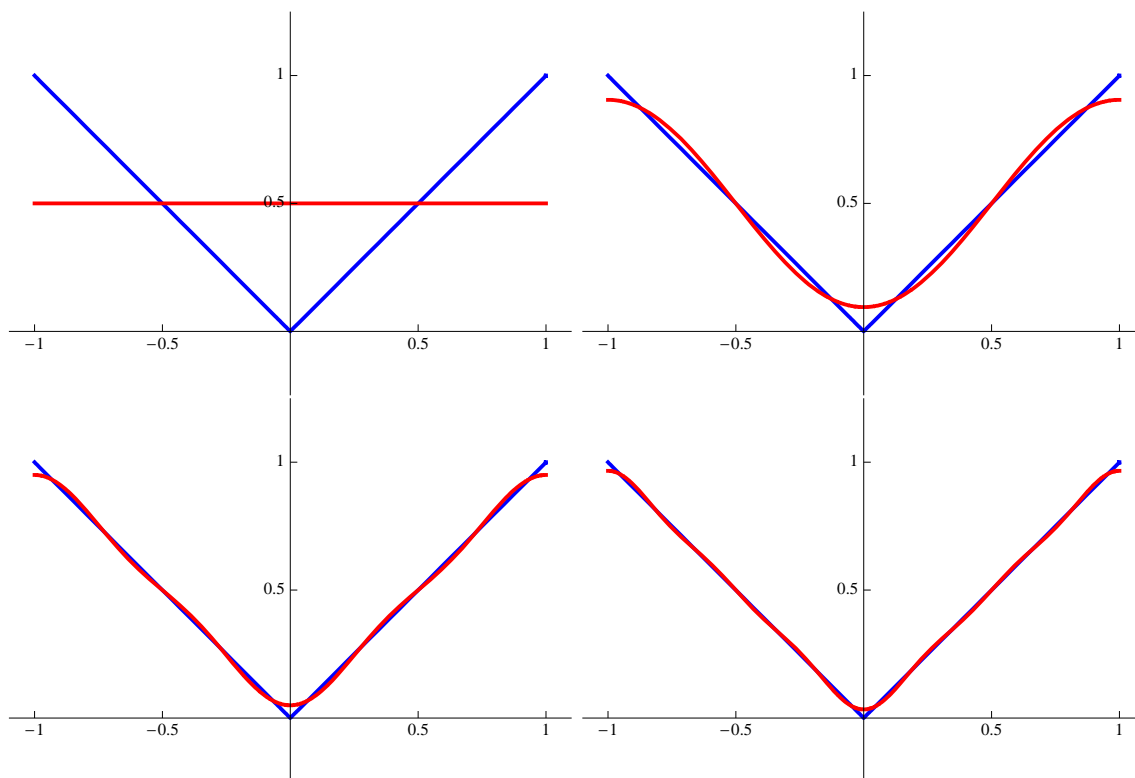


Figure 7: Partial Fourier series of $f(x) = |x|$ on $[-1, 1]$, with 1 term (top left), 2 terms (top right), 3 terms (bottom left), and 4 terms (bottom right).

tice. Calculus is required, and sometimes the calculus problems generated are not explicitly solvable. The process of finding these coefficients for the trigonometry functions is called the **Fourier transform**. We also have the issue (illustrated in **Fourierseries.mov**) that in order to exactly reproduce the function $f(x)$, many times we need the entire infinite number of terms.

However, there is a discrete version of the Fourier transform that uses only a finite number of points, sampled at regular intervals from the different period (hence, different frequency) sine and cosine functions. Let $\{y_k\}_{k=0}^{N-1}$ be a sequence of regularly-spaced y -values. Then

$$y_k = a_0 + \sum_{n=1}^{N-1} \left[a_n \cos\left(\frac{2\pi n}{N}k\right) + b_n \sin\left(\frac{2\pi n}{N}k\right) \right], \quad (3)$$

where

$$a_n = \frac{1}{N} \sum_{k=0}^{N-1} y_k \cos\left(\frac{2\pi k}{N}n\right) \quad \text{and} \quad b_n = \frac{1}{N} \sum_{k=0}^{N-1} y_k \sin\left(\frac{2\pi k}{N}n\right) \quad (4)$$

for $n = 0, 1, \dots, N-1$. (Notice that, by definition, $b_0 = 0$.) If we think of \mathbf{y} as a N -vector of the y_k values, and \mathbf{u}_n and \mathbf{v}_n as N -vectors where the k^{th} component is $\cos\left(\frac{2\pi k}{N}n\right)$ and $\sin\left(\frac{2\pi k}{N}n\right)$, respectively, then we can think of a_n and b_n defined in equation (4) in terms of dot products:

$$a_n = \frac{1}{N}(\mathbf{y} \cdot \mathbf{u}_n) \quad \text{and} \quad b_n = \frac{1}{N}(\mathbf{y} \cdot \mathbf{v}_n). \quad (5)$$

In the next few examples, we will let $N = 8$, meaning that

$$\begin{aligned} \mathbf{u}_0 &= (1, 1, 1, 1, 1, 1, 1, 1), & \mathbf{v}_0 &= (0, 0, 0, 0, 0, 0, 0, 0), \\ \mathbf{u}_1 &= \left(1, \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, -1, -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), & \mathbf{v}_1 &= \left(0, \frac{1}{\sqrt{2}}, 1, \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, -1, -\frac{1}{\sqrt{2}}\right), \\ \mathbf{u}_2 &= (1, 0, -1, 0, 1, 0, -1, 0), & \mathbf{v}_2 &= (0, 1, 0, -1, 0, 1, 0, -1), \\ \mathbf{u}_3 &= \left(1, -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, -1, \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right), & \mathbf{v}_3 &= \left(0, \frac{1}{\sqrt{2}}, -1, \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, 1, -\frac{1}{\sqrt{2}}\right), \\ \mathbf{u}_4 &= (1, -1, 1, -1, 1, -1, 1, -1), & \mathbf{v}_4 &= (0, 0, 0, 0, 0, 0, 0, 0), \\ \mathbf{u}_5 &= \left(1, -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, -1, \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right), & \mathbf{v}_5 &= \left(0, -\frac{1}{\sqrt{2}}, 1, -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, -1, \frac{1}{\sqrt{2}}\right), \\ \mathbf{u}_6 &= (1, 0, -1, 0, 1, 0, -1, 0), & \mathbf{v}_6 &= (0, -1, 0, 1, 0, -1, 0, 1), \\ \mathbf{u}_7 &= \left(1, \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, -1, -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), & \mathbf{v}_7 &= \left(0, -\frac{1}{\sqrt{2}}, -1, -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 1, \frac{1}{\sqrt{2}}\right). \end{aligned}$$

Notice that

- $\mathbf{v}_0 = \mathbf{v}_4 = \mathbf{0}$,
- $\mathbf{u}_1 = \mathbf{u}_7$, $\mathbf{u}_2 = \mathbf{u}_6$, and $\mathbf{u}_3 = \mathbf{u}_5$, and
- $\mathbf{v}_1 = -\mathbf{v}_7$, $\mathbf{v}_2 = -\mathbf{v}_6$, and $\mathbf{v}_3 = -\mathbf{v}_5$.

It can be verified (see the exercises) that any two distinct elements of the set

$$B = \{\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$$

are orthogonal.

Example: Let \mathbf{y} be the vector where the k^{th} entry is $y_k = \frac{1}{4}k(8 - k)$; that is,

$$\mathbf{y} = \left(0, \frac{7}{4}, 3, \frac{15}{4}, 4, \frac{15}{4}, 3, \frac{7}{4}\right),$$

as shown in Figure 8. Calculate the Fourier coefficients a_0 , a_n , and b_n , $n = 1, \dots, 7$.

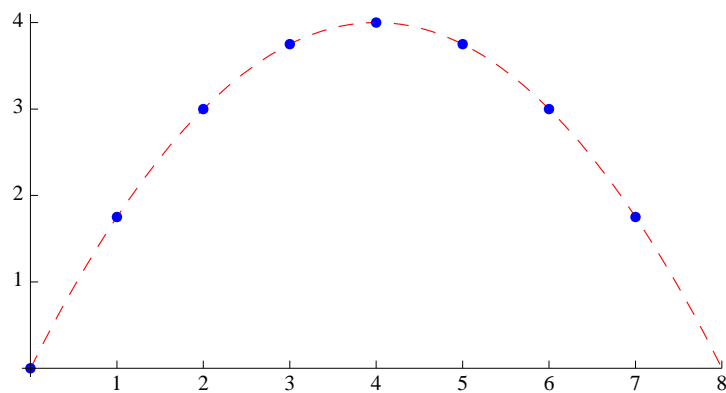


Figure 8: The original data.

$$a_0 = \frac{1}{8}(\mathbf{y} \cdot \mathbf{u}_0) = \frac{1}{8} \left(0, \frac{7}{4}, 3, \frac{15}{4}, 4, \frac{15}{4}, 3, \frac{7}{4}\right) \cdot (1, 1, 1, 1, 1, 1, 1, 1) = \frac{21}{8}$$

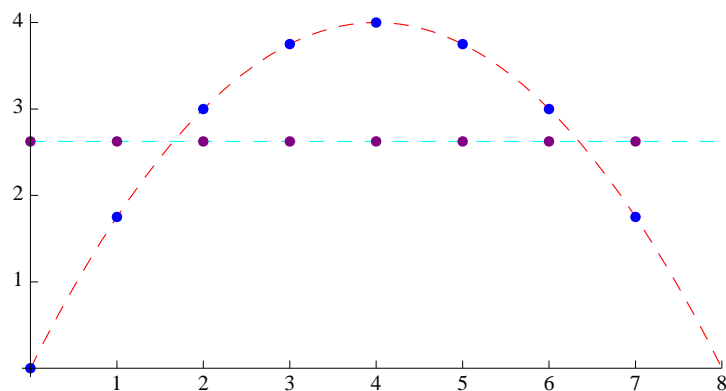


Figure 9: Partial reconstruction with $n = 0$.

$$a_1 = \frac{1}{8}(\mathbf{y} \cdot \mathbf{u}_1) = \frac{1}{8} \left(0, \frac{7}{4}, 3, \frac{15}{4}, 4, \frac{15}{4}, 3, \frac{7}{4}\right) \cdot \left(1, \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, -1, -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) = -\frac{2 + \sqrt{2}}{4}$$

$$a_7 = \frac{1}{8}(\mathbf{y} \cdot \mathbf{u}_7) = -\frac{2 + \sqrt{2}}{4}$$

$$b_1 = \frac{1}{8}(\mathbf{y} \cdot \mathbf{v}_1) = \frac{1}{8} \left(0, \frac{7}{4}, 3, \frac{15}{4}, 4, \frac{15}{4}, 3, \frac{7}{4} \right) \cdot \left(0, \frac{1}{\sqrt{2}}, 1, \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, -1, -\frac{1}{\sqrt{2}} \right) = 0$$

$$b_7 = \frac{1}{8}(\mathbf{y} \cdot \mathbf{v}_7) = 0$$

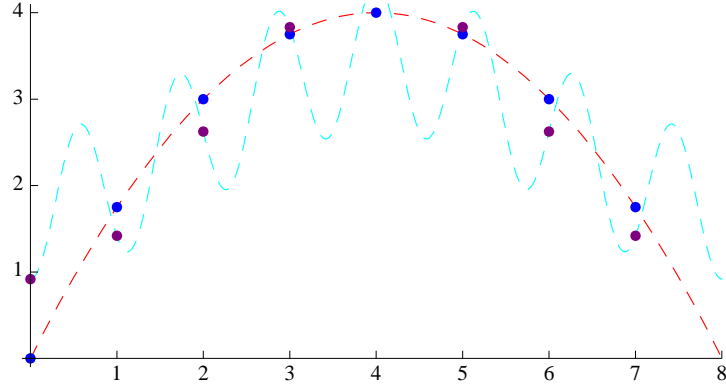


Figure 10: Partial reconstruction with $n = 0, 1$, and $n = 7$.

$$a_2 = \frac{1}{8}(\mathbf{y} \cdot \mathbf{u}_2) = \frac{1}{8} \left(0, \frac{7}{4}, 3, \frac{15}{4}, 4, \frac{15}{4}, 3, \frac{7}{4} \right) \cdot (1, 0, -1, 0, 1, 0, -1, 0) = -\frac{1}{4}$$

$$a_6 = \frac{1}{8}(\mathbf{y} \cdot \mathbf{u}_6) = -\frac{1}{4}$$

$$b_2 = \frac{1}{8}(\mathbf{y} \cdot \mathbf{v}_2) = \frac{1}{8} \left(0, \frac{7}{4}, 3, \frac{15}{4}, 4, \frac{15}{4}, 3, \frac{7}{4} \right) \cdot (0, 1, 0, -1, 0, 1, 0, -1) = 0$$

$$b_6 = \frac{1}{8}(\mathbf{y} \cdot \mathbf{v}_6) = 0$$

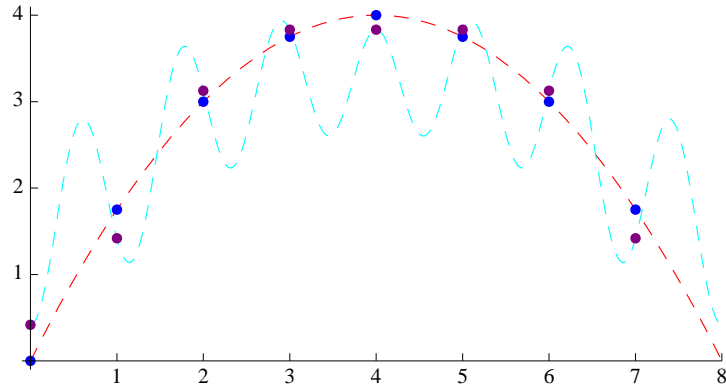


Figure 11: Partial reconstruction with $n = 0, 1, 2$, and $n = 6, 7$.

$$a_3 = \frac{1}{8}(\mathbf{y} \cdot \mathbf{u}_3) = \frac{1}{8} \left(0, \frac{7}{4}, 3, \frac{15}{4}, 4, \frac{15}{4}, 3, \frac{7}{4} \right) \cdot \left(1, -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, -1, \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}} \right) = \frac{-2 + \sqrt{2}}{4}$$

$$a_5 = \frac{1}{8}(\mathbf{y} \cdot \mathbf{u}_5) = \frac{-2 + \sqrt{2}}{4}$$

$$b_3 = \frac{1}{8}(\mathbf{y} \cdot \mathbf{v}_2) = \frac{1}{8} \left(0, \frac{7}{4}, 3, \frac{15}{4}, 4, \frac{15}{4}, 3, \frac{7}{4} \right) \cdot \left(0, \frac{1}{\sqrt{2}}, -1, \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, 1, -\frac{1}{\sqrt{2}} \right) = 0$$

$$b_5 = \frac{1}{8}(\mathbf{y} \cdot \mathbf{v}_5) = 0$$

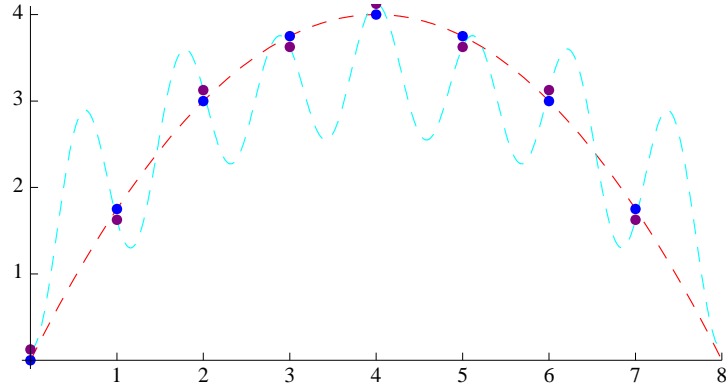


Figure 12: Partial reconstruction with $n = 0, 1, 2, 3$, and $n = 5, 6, 7$.

$$a_4 = \frac{1}{8}(\mathbf{y} \cdot \mathbf{u}_3) = \frac{1}{8} \left(0, \frac{7}{4}, 3, \frac{15}{4}, 4, \frac{15}{4}, 3, \frac{7}{4} \right) \cdot (1, -1, 1, -1, 1, -1, 1, -1) = -\frac{1}{8}$$

$$b_4 = \frac{1}{8}(\mathbf{y} \cdot \mathbf{v}_2) = \frac{1}{8} \left(0, \frac{7}{4}, 3, \frac{15}{4}, 4, \frac{15}{4}, 3, \frac{7}{4} \right) \cdot (0, 0, 0, 0, 0, 0, 0, 0) = 0$$

Figure 13 shows the difference between a continuous match to a curve and a discrete match to the data.

Example: Let \mathbf{y} be the vector where the k^{th} entry is $y_k = 2\sqrt{2} \sin\left(\frac{3\pi}{4}k - \frac{\pi}{6}\right)$; that is,

$$\mathbf{y} = \left(-\sqrt{2}, 1 + \sqrt{3}, -\sqrt{6}, -1 + \sqrt{3}, \sqrt{2}, -1 - \sqrt{3}, \sqrt{6}, 1 - \sqrt{3} \right),$$

as shown in Figure 14. Calculate the Fourier coefficients a_0 , a_n , and b_n , $n = 1, \dots, 7$.

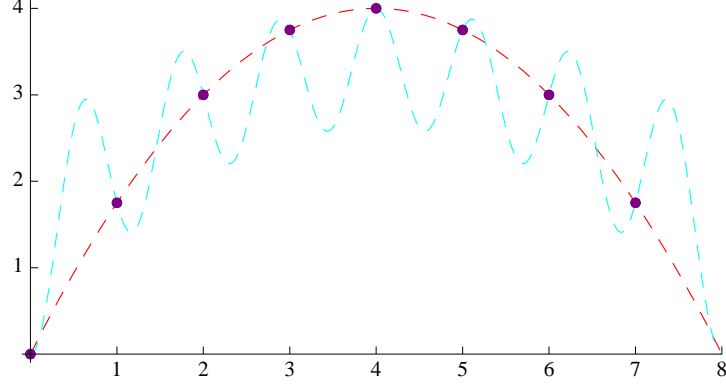


Figure 13: Full reconstruction.

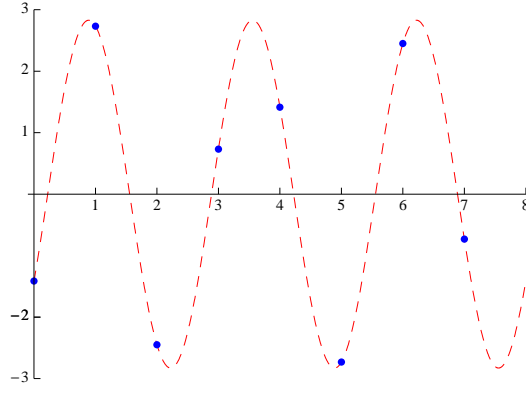


Figure 14: The original data.

$$a_0 = \frac{1}{8}(\mathbf{y} \cdot \mathbf{u}_0) = \frac{1}{8}\mathbf{y} \cdot (1, 1, 1, 1, 1, 1, 1, 1) = 0$$

$$a_1 = \frac{1}{8}(\mathbf{y} \cdot \mathbf{u}_1) = \frac{1}{8}\mathbf{y} \cdot \left(1, \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, -1, -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) = 0$$

$$a_7 = \frac{1}{8}(\mathbf{y} \cdot \mathbf{u}_7) = 0$$

$$b_1 = \frac{1}{8}(\mathbf{y} \cdot \mathbf{v}_1) = \frac{1}{8}\mathbf{y} \cdot \left(0, \frac{1}{\sqrt{2}}, 1, \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, -1, -\frac{1}{\sqrt{2}}\right) = 0$$

$$b_7 = \frac{1}{8}(\mathbf{y} \cdot \mathbf{v}_7) = 0$$

$$a_2 = \frac{1}{8}(\mathbf{y} \cdot \mathbf{u}_2) = \frac{1}{8}\mathbf{y} \cdot (1, 0, -1, 0, 1, 0, -1, 0) = 0$$

$$a_6 = \frac{1}{8}(\mathbf{y} \cdot \mathbf{u}_6) = 0$$

$$b_2 = \frac{1}{8}(\mathbf{y} \cdot \mathbf{v}_2) = \frac{1}{8}\mathbf{y} \cdot (0, 1, 0, -1, 0, 1, 0, -1) = 0$$

$$\begin{aligned}
b_6 &= \frac{1}{8}(\mathbf{y} \cdot \mathbf{v}_6) = 0 \\
a_3 &= \frac{1}{8}(\mathbf{y} \cdot \mathbf{u}_3) = \frac{1}{8}\mathbf{y} \cdot \left(1, -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, -1, \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right) = -\frac{1}{\sqrt{2}} \\
a_5 &= \frac{1}{8}(\mathbf{y} \cdot \mathbf{u}_5) = -\frac{1}{\sqrt{2}} \\
b_3 &= \frac{1}{8}(\mathbf{y} \cdot \mathbf{v}_2) = \frac{1}{8}\mathbf{y} \cdot \left(0, \frac{1}{\sqrt{2}}, -1, \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, 1, -\frac{1}{\sqrt{2}}\right) = \sqrt{\frac{3}{2}} \\
b_5 &= \frac{1}{8}(\mathbf{y} \cdot \mathbf{v}_5) = -\sqrt{\frac{3}{2}} \\
a_4 &= \frac{1}{8}(\mathbf{y} \cdot \mathbf{u}_3) = \frac{1}{8}\mathbf{y} \cdot (1, -1, 1, -1, 1, -1, 1, -1) = 0 \\
b_4 &= \frac{1}{8}(\mathbf{y} \cdot \mathbf{v}_2) = \frac{1}{8}\mathbf{y} \cdot (0, 0, 0, 0, 0, 0, 0, 0) = 0
\end{aligned}$$

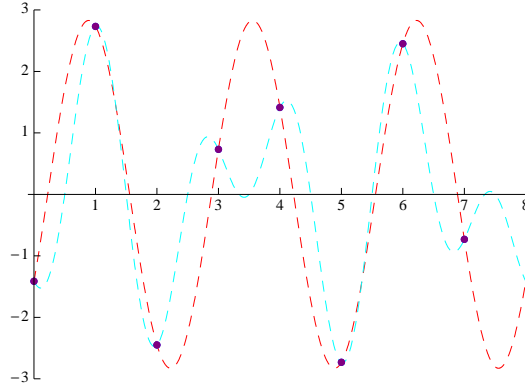


Figure 15: Full reconstruction.

This is a very interesting case, because, without the dashed line in Figure 14, the data would look random. However, if we take our Fourier coefficients, consider them as points (a_n, b_n) , $n = 0, 1, \dots, 7$, and then look at the distances from the origin $c_n = \sqrt{a_n^2 + b_n^2}$, $n = 0, 1, \dots, 7$, we get a data set with only two nonzero values:

$$\{c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7\} = \{0, 0, 0, \sqrt{2}, 0, \sqrt{2}, 0, 0\}.$$

If we graph this data relative to n , we get the graph shown in Figure 16, called the **power spectrum** of the original data. Each value of $n \leq \frac{N}{2}$ corresponds to a number of oscillations that is represented in the data. Since $a_n = a_{N-n}$ and $b_n = -b_{N-n}$ for $n = 1, \dots, N-1$, then the power spectrum (with c_0 omitted) is symmetric about $n = \frac{N}{2}$. Also, since $\mathbf{u}_n = \mathbf{u}_{N-n}$ and $\mathbf{v}_n = -\mathbf{v}_{N-n}$, the values at $n > \frac{N}{2}$ actually correspond to the reflected number of

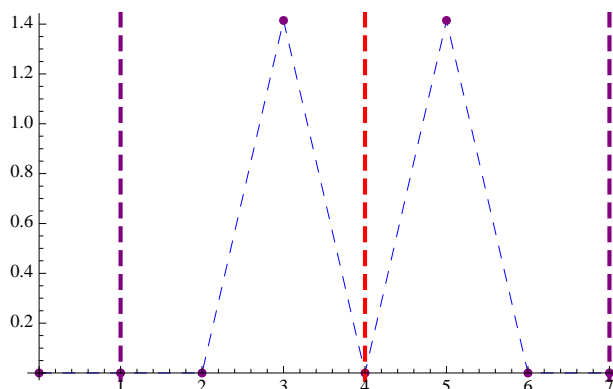


Figure 16: Power spectrum of the data in the second example.

oscillations below $n < \frac{N}{2}$. In our example, we have $\sqrt{2} + \sqrt{2} = 2\sqrt{2}$ in amplitude of sines and cosines present in the signal that oscillate 3 times over the interval, which corresponds to the function $y_k = 2\sqrt{2} \sin\left(\frac{3\pi}{4}k - \frac{\pi}{6}\right)$ that generated the data.

6.3 Exercises

1) Verify that the set $B = \{\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ from the $N = 8$ example in this section is orthogonal. (This involves 28 separate dot products, so you may wish to automate the process in some way. Either show all of your work if done by hand, or give a print-out if completed by computer.)

Using the vectors $\mathbf{u}_0, \dots, \mathbf{u}_7$ and $\mathbf{v}_0, \dots, \mathbf{v}_7$ from the $N = 8$ example in this section, perform the following operations in problems 2) through 5) for the given data vector \mathbf{y} .

a) Find the Fourier coefficients a_0, \dots, a_7 and b_0, \dots, b_7 . (Remember, there are some redundancies here, so there are less than 16 dot products to calculate.)

b) Generate the power spectrum $c_n = \sqrt{a_n^2 + b_n^2}$, $n = 0, \dots, 7$, and graph the values.

2) $\mathbf{y} = (5, 5, 5, 5, 5, 5, 5, 5)$

3) $\mathbf{y} = \left(0, \frac{\sqrt{2} - \sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{2} + \sqrt{2}}{2}, 1, \frac{\sqrt{2} + \sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{2} - \sqrt{2}}{2}\right)$

4) $\mathbf{y} = (0, 7, 8, 5, 0, -5, -8, -7)$

5) $\mathbf{y} = (0, 1, 2, 3, 4, 5, 6, 7)$

6) Let \mathbf{u}_n be the vector formed by sampling $\cos\left(\frac{2\pi n}{6}t\right)$ at the values $t = 0, 1, 2, 3, 4, 5$, and let \mathbf{v}_n be the analogous vector formed by sampling $\sin\left(\frac{2\pi n}{6}t\right)$, for $n = 0, 1, 2, 3, 4, 5$. (See problem 6) in section 5.2.) Use the vectors to find the Fourier coefficients and power spectrum for the data vector

$$\mathbf{y} = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{6} - \sqrt{2}}{4}, -\frac{\sqrt{6} + \sqrt{2}}{4}, \frac{\sqrt{2}}{2}, \frac{\sqrt{6} - \sqrt{2}}{4}, -\frac{\sqrt{6} + \sqrt{2}}{4} \right).$$

7) Use the same \mathbf{u}_n and \mathbf{v}_n as in problem 6) to find the Fourier coefficients and power spectrum for the data vector

$$\mathbf{y} = \left(\frac{1}{2}, \frac{1}{2}, -1, \frac{1}{2}, \frac{1}{2}, -1 \right).$$

How does the power spectrum for this set of data compare to the power spectrum to the data in the previous problem?

6.4 Finding Frequencies

From the last section, we saw that the presence of nonzero n and $N - n$ Fourier coefficients from a set of data means that the data, or **signal**, has components that can be generated by sampling a sine and/or cosine curves that oscillate n cycles over the length of the sample. We can use this technology to help us analyze, alter, and filter sound signals.

Let's demonstrate how this works. On the class webpage, you will find a sound file **dchord.wav** and a *Mathematica*® file **frequencies.nb**. Download both files to the desktop of your computer. After entering the initial command to set the correct directory, we can import the sound file data into *Mathematica*® using the command

```
data = Import["dchord.wav", "Data"][[1]];
```

This is a recording of a D-chord that was created in *Mathematica*®, by sampling the function

$$f(t) = \cos(146.8 * 2\pi t) + \cos(220 * 2\pi t) + \cos(293.7 * 2\pi t) + \cos(370 * 2\pi t)$$

at $t = 0, \frac{1}{44100}, \dots, \frac{44099}{44100}$. To hear the chord, use the command

```
ListPlay[data, SampleRate → 44100, PlayRange → All]
```

and hit the “play” button. To see the data over the first 0.02 seconds, use the command

```
ListPlot[data, Joined → True, PlotStyle → {Blue,Thick},  
PlotRange → {{1,882},All}, AxesOrigin → {1,0}] .
```

(Do not put the period – it only marks the end of the sentence.) The graph is shown on the left in Figure 17.

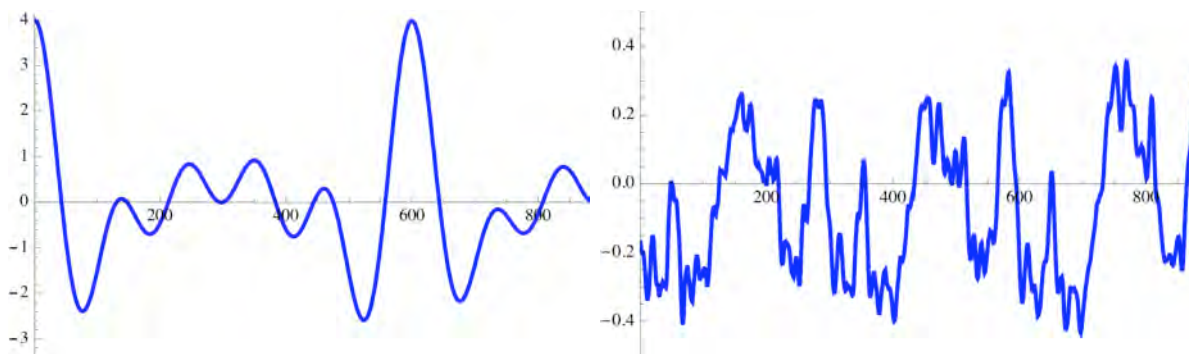


Figure 17: D-chords (synthesizer on the left, guitar on the right)

Mathematica® has a built-in command that will generate the Fourier coefficients for a signal:

```
fcoef = Fourier[data, FourierParameters → {-1,1}]; .
```

Then, to view the power spectrum (which is shown on the left in Figure 18), use the commands

```
pts=Table[{k-1,Abs[fcoef[[k]]]}, {k,44100}];
ListPlot[pts, Joined → True, PlotStyle → {Blue,Thickness[.002]},
PlotRange → {{0, 800},All},
Ticks → {{146.8, 220, 293.7, 370}, Automatic}] .
```

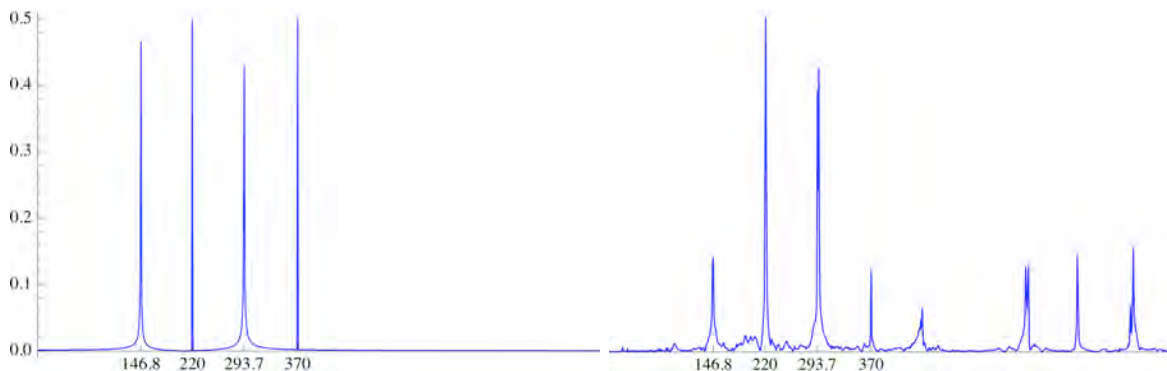


Figure 18

Notice that the power spectrum of the D-chord signal has peaks located at (or near) 146.8, 220, 293.7, and 370. These are the frequencies (in Hz) of the notes in the chord.

Now, let's repeat this exercise with the sound file **dchord_guitar.wav**. The raw data over the first 0.02 seconds is shown on the right in Figure 17, and looks nothing like the synthesizer D-chord on the left in that same figure. However, when we look at the power spectrum (shown on the right in Figure 18), we can still see peaks at the same locations, with some clutter between those peaks, plus some other regularly-spaced peaks of higher frequencies. These are called **overtones**, and are frequently (no pun intended) found in tones produced by stringed instruments. The difference between the two sounds is analogous to the difference in cartoon images and photographs. The synthesized sound has no overtones or texture that we find in the sound generated by the guitar.

It is important to point out that, in both of the examples shown above, the peaks occurred at the frequencies of the notes of the chord because the signal was exactly one second in duration. Run the same analysis on the sound file **dchord_twosec.wav** and note that the peaks are now located at 293.6, 440, 587.4, and 740. We have to remember that the discrete Fourier transform is measuring the number of oscillations over the length of the signal. If we know the duration of the signal, then we can find the frequency of the notes in the signal in hertz by dividing the peak locations by the length in seconds of the signal.

There is some additional *Mathematica*[®] code provided in the file **frequencies.nb** to help find the location of the peaks in the power spectrum. The command **hertz[low,high,time]** will locate the greatest value of the power spectrum on the interval [low, high], and will divide the location of that peak by the length of the sound duration in seconds. The command **tone[htz]** will play a 3-second tone at the frequency **htz**, so you can listen to see if the pitch is present in the sound file.

To show how these sound signals differ from just standard noise, let's run the same example with the sound file **noise.wav**, which was collected in a noisy classroom. The data

over a 0.02 second interval in this signal is shown on the left in Figure 19. Notice that the data really does not look that different from the data for the D-chords shown in Figure 17. However, if we take the Fourier transform of the data and look at the power spectrum, then we get the graph shown at right in Figure 19, which has no prominent peaks like the ones we saw in Figure 18.

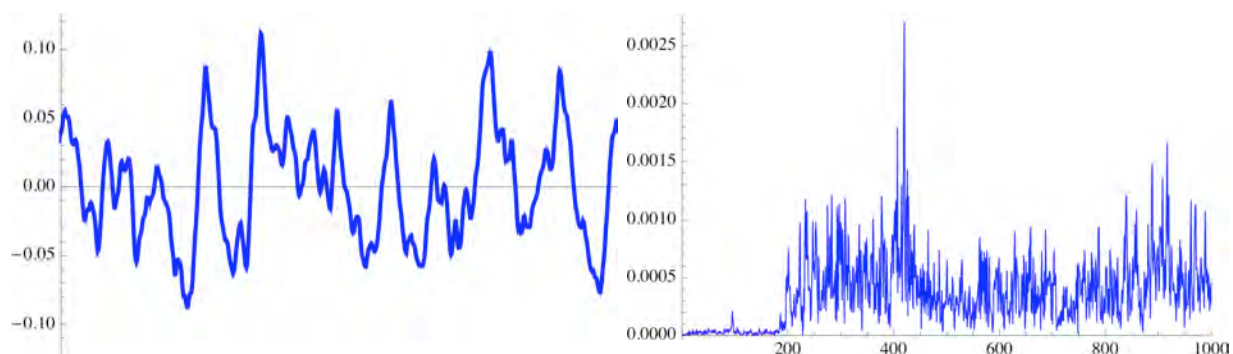


Figure 19

Although the raw data for the music and the noise look almost indistinguishable, we can now look at the power spectrum, and tell immediately which is melodic and which is not. The homework for this section will ask you to analyze several different sound files using *Mathematica*[®] to determine if the sound is musical in nature, and, if so, which frequencies are represented in the signal.

6.4 Exercises

Use the code in the file **frequencies.nb** to analyze the sound samples provided on the class webpage. Determine if the sample has any dominant frequencies, and if so, what they are.

- | | |
|-------------------|---------------------|
| 1) ex1.wav | 2) ex2.wav |
| 3) ex3.wav | 4) ex4.wav |
| 5) ex5.wav | 6) ex6.wav |
| 7) ex7.wav | 8) ex8.wav |
| 9) ex9.wav | 10) ex10.wav |

6.5 Changing Frequencies

Let's consider a sampling of length 16 from the function

$$f(x) = 2^{-\frac{x}{4}} \cos\left(\frac{2\pi}{16}4x\right),$$

with $x = 0, \dots, 15$, shown on the left in Figure 20. The data reflects the decay that you

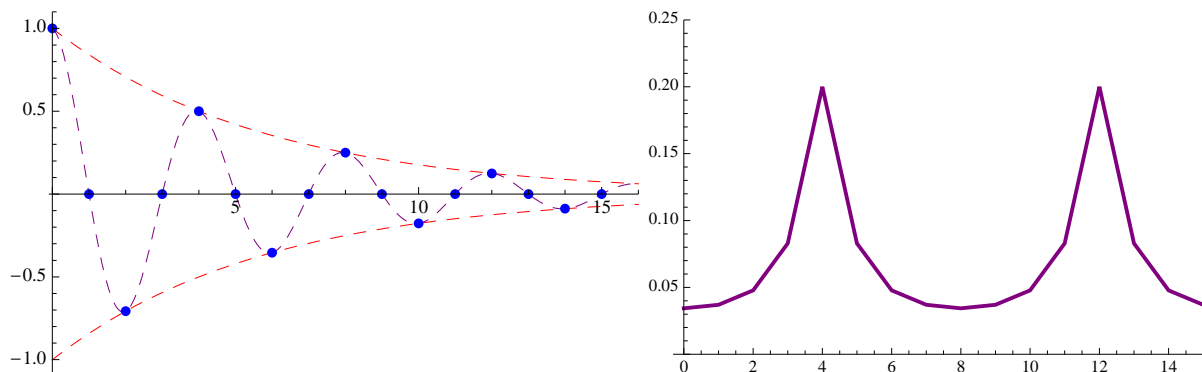


Figure 20

would normally see in the loudness of a tone when played on a guitar or a standard piano, for example. The power spectrum of the signal is shown on the right in Figure 20, and shows that the signal has a predominant frequency of 4, as indicated by the symmetric (about $n = 8$) peaks at $n = 4$ and $n = 16 - 4 = 12$.

Now, let's repeat this exercise with a sampling of length 16 from the function

$$f(x) = 2^{-\frac{x}{4}} \cos\left(\frac{2\pi}{16}2x\right),$$

with $x = 0, \dots, 15$, shown on the left in Figure 21. The power spectrum of this signal is

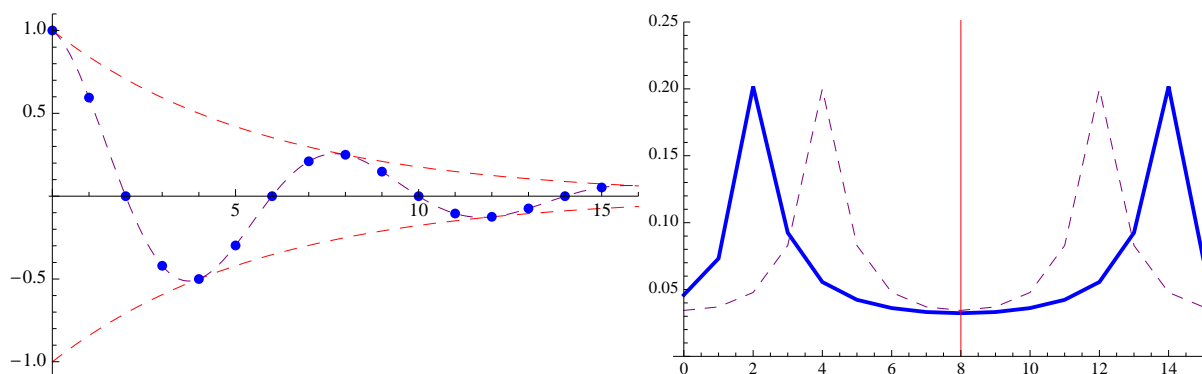


Figure 21

shown on the right in Figure 21, and shows that the signal has a predominant frequency of 2, as indicated by the symmetric peaks at $n = 2$ and $n = 16 - 2 = 14$.

Notice that the second signal was at roughly half the frequency of first one, and this halving is demonstrated by halving the n values in the first spectrum on the left of $n = 8$, and then producing the reflection across $n = 8$, to get the second spectrum. If the signals were sounds, the second signal would be one octave lower than the first one. This gives us a hint as to how we could take a sound signal and lower the tone by one octave.

Let $\{a_n\}_{n=0}^{N-1}$ and $\{b_n\}_{n=1}^{N-1}$ be the Fourier coefficients for the higher frequency signal of length N , where N is divisible by 4. Let's try to construct Fourier coefficients \tilde{a}_n and \tilde{b}_n for a one-lower-octave signal by doing the following:

1. let $\tilde{a}_0 = a_0$ (leaves the constant term alone);

2. if

$$\sqrt{a_{2n-1}^2 + b_{2n-1}^2} > \sqrt{a_{2n}^2 + b_{2n}^2}$$

then we let $\tilde{a}_n = a_{2n-1}$ and $\tilde{b}_n = b_{2n-1}$, else let $\tilde{a}_n = a_{2n}$ and $\tilde{b}_n = b_{2n}$, for $n = 1, \dots, \frac{N}{4}$ (moves the higher of each pair of the non-zero frequency terms over halfway to the left);

3. let $\tilde{a}_n = a_{\frac{N}{2}}$ and $\tilde{b}_n = b_{\frac{N}{2}}$ for $n = \frac{N}{4} + 1, \dots, \frac{N}{2}$ (fills in with the middle value); and

4. let $\tilde{a}_n = \tilde{a}_{N-n}$ and $\tilde{b}_n = -\tilde{b}_{N-n}$ for $n = \frac{N}{2} + 1, \dots, N - 1$ (creates a symmetric right side of the graph).

We have a sound sample **lalala.wav** on the class webpage of someone with a high-pitched voice saying “la” nine times. Using code found in the *Mathematica*® file **lowerfrequency.nb**, we can import this sound and perform a Fourier transform on the sound data. The first 5000 components of the power spectrum for this sample are shown on the left in Figure 22. The

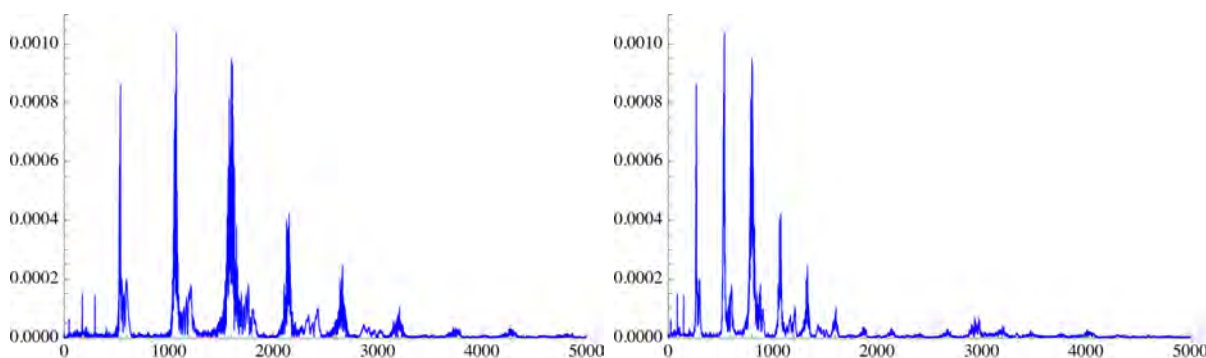


Figure 22

right graph is of the first 5000 components of the new power spectrum created by running the above algorithm. We can turn the Fourier coefficients back into a sound signal by using the *Mathematica*® command

recon = Re[InverseFourier[altered, FourierParameters \rightarrow $\{-1, 1\}$]

When we listen to this reconstructed signal, we have lowered the tone by one octave. However, we have also lowered the frequency of the “la’s” by one-half as well.

What we have neglected to consider is that words and other intermittent sounds have a frequency in the signal as well. In order to say exactly the same thing with a lowered octave, we need to break the signal up into disjoint “windows”. The idea is that over a small window, there are no words being conveyed, just a tone. Your digital camera uses basically the same trick, splitting your digital pictures into “pixels”. We can try the same algorithm, just applying it separately over each window. The *Mathematica*[®] code for this repeated looping through the data, 2000 data values at a time, is included in **lowerfrequency.nb**. The first 5000 components of the power spectrum for this new signal are shown in Figure 23. The sound file **lowlalala.wav** showing the results of applying the algorithm to small windows is on the class webpage.

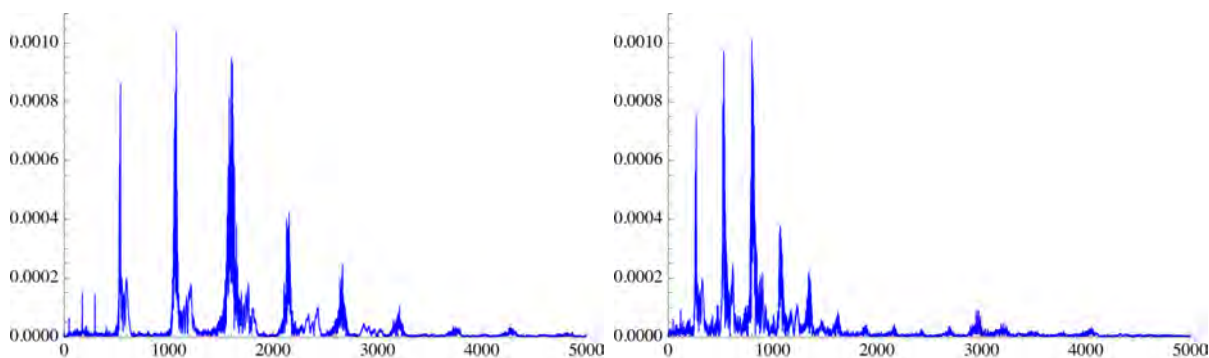


Figure 23: Original power spectrum on the left, altered on the right.

6.5 Exercises

Use the code in the file **lowerfrequency.nb** to lower the sound samples provided on the class webpage by one octave. Run both the first and second blocks of code and note the effectiveness of each method.

- | | |
|-------------------|--------------------|
| 1) ex1.wav | 2) ex3.wav |
| 3) ex6.wav | 4) ex7.wav |
| 5) ex8.wav | 6) ex11.wav |

6.6 Removing Frequencies

One of the most common uses of the discrete Fourier transform is filtering unwanted noise out of sound signals, by removing nonzero coefficients that correspond to higher frequencies. The *Mathematica*[®] file **filtering.nb** on the class webpage contains code that will allow us to filter sound samples.

Let's start with the sound file **singing.wav**, also on the class webpage. We may import that signal into *Mathematica*[®], and generate the power spectrum, which is shown in Figure 24.

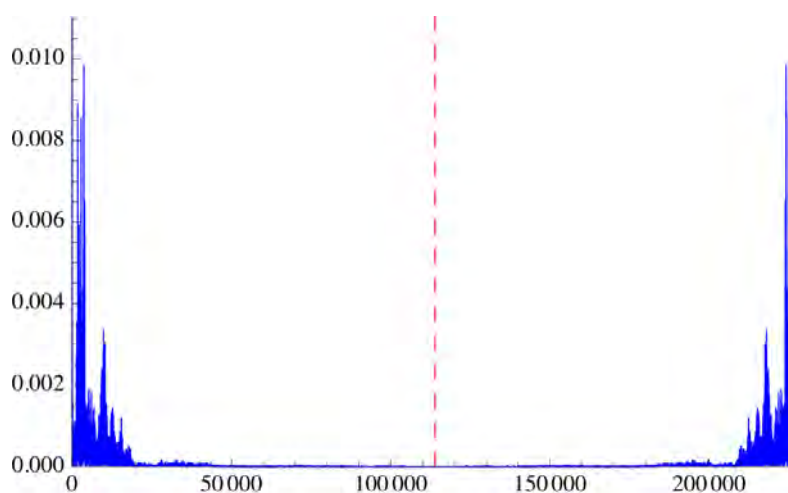


Figure 24

Now let's load in the sound file **singing_static.wav**, also on the class webpage, which is identical to the previous file except that a large amount of static has been added. The power spectrum is shown in Figure 25. Notice the difference in this graph compared to Figure 24. The static shows up in the spectrum as high frequencies, near the symmetry line of the graph, lying over the top of the original signal.

Suppose that we did not have the original signal to compare to, and we wanted to remove as much of the static from the signal as possible. Let's zero-out all of the Fourier coefficients from 13001 to its symmetric location on the right of the graph by setting **cutoff = 13000**; that is, let

$$a_n = b_n = 0 \text{ for } n = 13001, \dots, N - 13001.$$

The new power spectrum is shown in Figure 26. It is clear that we have not removed all of the effects of the static, but hopefully we will have removed a great deal of it. Execute the code in **filtering.nb**, and hear the effects yourself.

The static in the signal was generated by adding Gaussian white noise to the original data. Gaussian noise is among the toughest noise to remove from a signal. Let's try an easier example. The sound file **singing_whistle.wav** has the same song being sung, but this time there is a whistle blowing in the background. The power spectrum is shown in Figure 27.

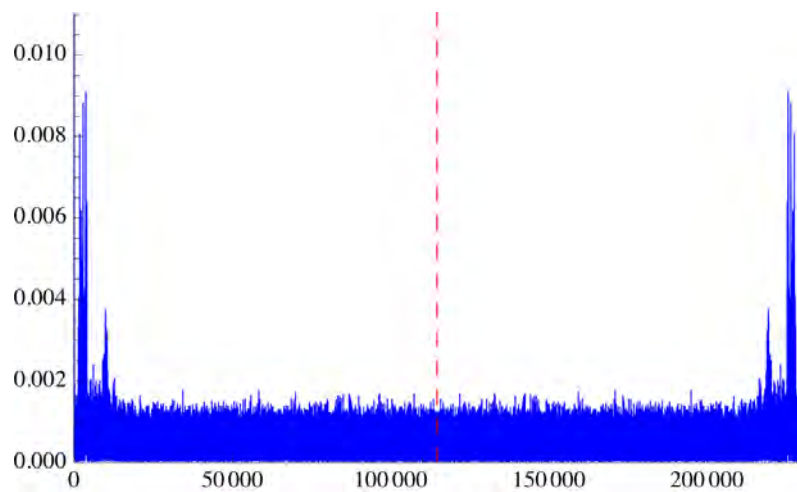


Figure 25

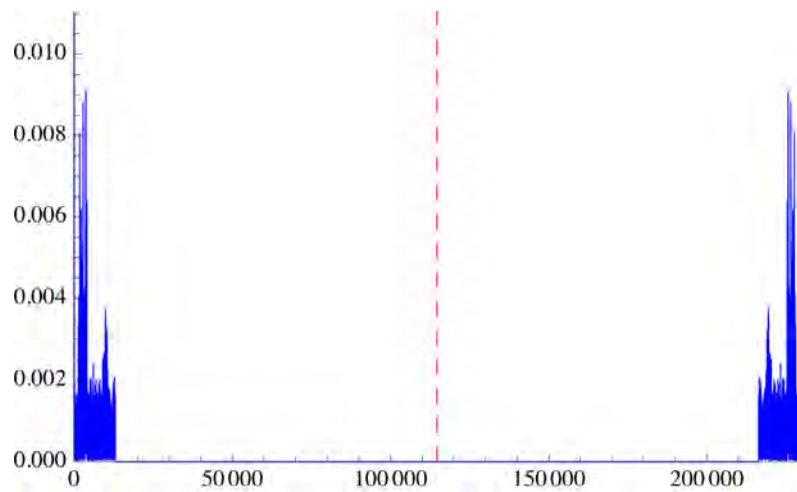


Figure 26

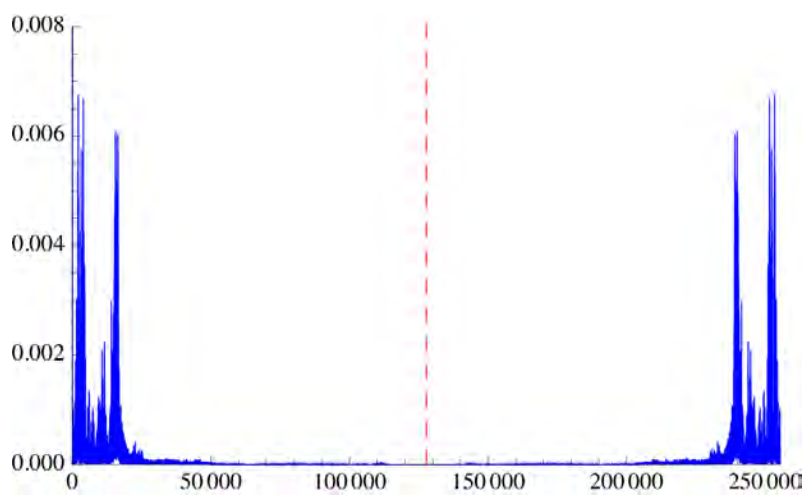


Figure 27

The innermost large peak (around $n = 17000$) and its twin on the right are caused by the high frequency of the whistle. If we again zero out the high-frequency Fourier coefficients, we stand a chance of being able to remove the sound of the whistle from the reconstructed signal. Figure 28 shows the new spectrum with the coefficients zeroed-out from $n = 13000$ on to its symmetric value on the right side of the graph. Reconstruct the altered Fourier

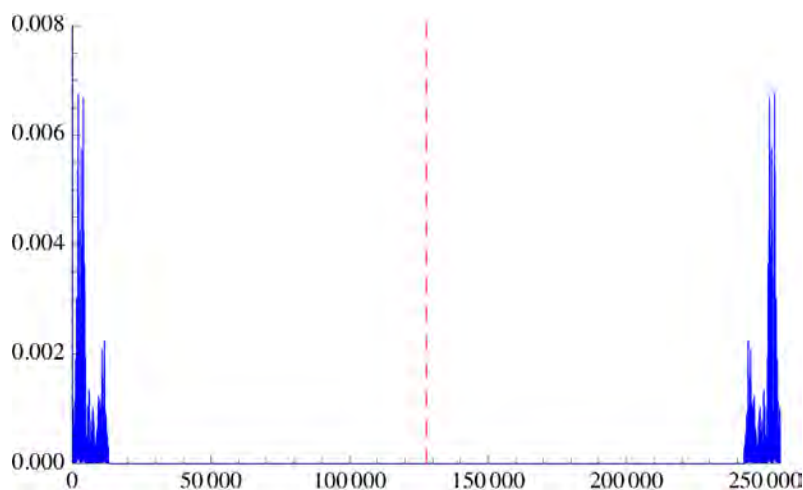


Figure 28

coefficients to hear the results.

6.6 Exercises

Use the code in the file **filtering.nb** to analyze the sound samples provided on the class webpage. Determine the best cutoff level to produce a filtered sound file without noise, whistles, etc.

- | | |
|--------------------|--------------------|
| 1) ex12.wav | 2) ex13.wav |
| 3) ex14.wav | 4) ex15.wav |
| 5) ex16.wav | 6) ex17.wav |
| 7) ex18.wav | 8) ex19.wav |

6.7 Review: What have we learned (or relearned)?

Definition: Sound *Sound is a pressurized wave through a medium, that is interpreted by our ears as speech, music, noise, etc.*

Definition: Transversal Waves *A transversal wave occurs in strings and surfaces, at a right angle to the surface.*

Definition: Longitudinal Waves *Longitudinal waves occur through mediums, parallel to the movement of the wave through the medium.*

Definition: Sound Volume *The amplitude of the sine or cosine will correspond to the loudness of the sound.*

Definition: Sound Frequency *The frequency is the number of oscillations of the sound wave over a period of time, typically measured in hertz (Hz), or cycles per second. The frequency of a sound wave corresponds to the pitch of the sound. The frequency F of a sound wave is inversely proportional to the period p of the sine or cosine; that is*

$$F = \frac{1}{p}.$$

The number of oscillations of a sound wave of frequency F over a time interval of T seconds is FT .

Definition: n -Vectors *A vector of length n , called a n -vector, is an ordered list of length n , denoted*

$$\mathbf{u} = (u_1, u_2, \dots, u_n).$$

Definition: Dot Products *The dot product of two n -vectors $\mathbf{u} = (u_1, \dots, u_n)$ and $\mathbf{v} = (v_1, \dots, v_n)$, denoted $\mathbf{u} \cdot \mathbf{v}$, is given by*

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + \dots + u_nv_n.$$

Definition: Norm of a Vector *The norm of the vector, denoted $\|\mathbf{u}\|$, is defined as*

$$\|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}} = \sqrt{u_1^2 + u_2^2 + \dots + u_n^2}.$$

Theorem: Angle Between Two Nonzero Vectors *The angle θ between the nonzero vectors \mathbf{u} and \mathbf{v} is given by*

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}.$$

Definition: Orthogonal Vectors *Two nonzero vectors \mathbf{u} and \mathbf{v} are said to be orthogonal if $\mathbf{u} \cdot \mathbf{v} = 0$.*

Theorem: Discrete Fourier Transform Let $\{y_k\}_{k=0}^{N-1}$ be a sequence of regularly-spaced y -values. Then

$$y_k = a_0 + \sum_{n=1}^{N-1} \left[a_n \cos\left(\frac{2\pi n}{N}k\right) + b_n \sin\left(\frac{2\pi n}{N}k\right) \right],$$

where

$$a_n = \frac{1}{N} \sum_{k=0}^{N-1} y_k \cos\left(\frac{2\pi k}{N}n\right) \quad \text{and} \quad b_n = \frac{1}{N} \sum_{k=0}^{N-1} y_k \sin\left(\frac{2\pi k}{N}n\right)$$

for $n = 0, 1, \dots, N-1$. (Notice that, by definition, $b_0 = 0$.) If we think of \mathbf{y} as a N -vector of the y_k values, and \mathbf{u}_n and \mathbf{v}_n as N -vectors where the k^{th} component is $\cos\left(\frac{2\pi k}{N}n\right)$ and $\sin\left(\frac{2\pi k}{N}n\right)$, respectively, then we can think of a_n and b_n defined in equation (4) in terms of dot products:

$$a_n = \frac{1}{N}(\mathbf{y} \cdot \mathbf{u}_n) \quad \text{and} \quad b_n = \frac{1}{N}(\mathbf{y} \cdot \mathbf{v}_n).$$

The Fourier coefficients are symmetric about $n = \frac{N}{2}$. Converting the a_n and b_n back into y_k is called the inverse Fourier transform.

Definition: Power Spectrum of a Signal After taking the discrete Fourier transform of a signal $\{y_k\}_{k=0}^{N-1}$ (generating the Fourier coefficients a_n and b_n for $n = 0, \dots, N-1$, let

$$c_n = \sqrt{a_n^2 + b_n^2} \text{ for } n = 0, \dots, N-1.$$

Then the sequence $\{c_n\}_{n=0}^{N-1}$ is called the power spectrum of the signal. The graph of the power spectrum is symmetric about $n = \frac{N}{2}$, and the n^{th} component gives a measure of how much of the signal is of sines and cosines oscillating n times over the length of the signal.

Review Exercises

1) Find the amplitude and period of the following sine and cosine functions. Assuming the curves represent sound waves and t is measured in seconds, find the frequency in hertz of each of the following, and find the number of oscillations over the given time span.

- | | |
|-----------------------------------|------------------------------------|
| a) $3 \sin(6\pi t)$; 2 seconds | b) $\cos(20000t)$; 12 seconds |
| c) $2 \sin(1000\pi t)$; 1 second | d) $5 \cos(5000\pi t)$; 4 seconds |

2) Find the measure in radians (rounded to two decimal places) of the angle between the following pairs of vectors. Determine whether the vectors are orthogonal.

- | | |
|---|--|
| a) $\left(2, \frac{1}{2}\right)$ and $\left(-2, \frac{1}{2}\right)$ | b) $(5, 2)$ and $\left(-\frac{1}{2}, \frac{5}{4}\right)$ |
| c) $(2, 1, 3)$ and $(-2, 1, 1)$ | d) $(1, 1, 1)$ and $(-1, -1, -1)$ |

- 3) Let \mathbf{u}_n be the vector formed by sampling $\cos\left(\frac{2\pi n}{6}t\right)$ at the values $t = 0, 1, 2, 3, 4, 5$, and let \mathbf{v}_n be the analogous vector formed by sampling $\sin\left(\frac{2\pi n}{6}t\right)$, for $n = 0, 1, 2, 3, 4, 5$.
- Find the exact values of the vectors \mathbf{u}_n and \mathbf{v}_n for $n = 0, 1, 2, 3, 4, 5$.
 - Find the Fourier coefficients a_0, \dots, a_5 and b_0, \dots, b_5 for the signal $y = (0, 1, 2, 3, 2, 1)$. (Remember, there are some redundancies here, so there are less than 12 dot products to calculate.)
 - Generate the power spectrum $c_n = \sqrt{a_n^2 + b_n^2}$, $n = 0, \dots, 5$, and graph the values.
- 4) Use the code in the file **frequencies.nb** to analyze the sound samples provided on the class webpage. Determine if the sample has any dominant frequencies, and if so, what they are.
- ex20.wav**
 - ex21.wav**
- 5) Use the code in the file **lowerfrequency.nb** to lower the sound samples provided on the class webpage by one octave. Run both the first and second blocks of code and note the effectiveness of each method.
- ex22.wav**
 - ex23.wav**
- 6) Use the code in the file **filtering.nb** to analyze the sound samples provided on the class webpage. Determine the best cutoff level to produce a filtered sound file without noise, whistles, etc.
- ex24.wav**
 - ex25.wav**