

49. Zhang W, Hill RW (2000) A template-based and pattern-driven approach to situation awareness and assessment in virtual humans. In: Fourth International Conference on Autonomous Agents, Barcelona, pp 116–123

Soliton Perturbation

JI-HUAN HE

Modern Textile Institute, Donghua University,
Shanghai, China

Article Outline

[Glossary](#)

[Definition of the Subject](#)

[Introduction](#)

[Methods for Soliton Solutions](#)

[Future Directions](#)

[Bibliography](#)

Glossary

Soliton A soliton is a nonlinear pulse-like wave that can exist in some nonlinear systems. The isolated wave can propagate without dispersing its energy over a large region of space; collision of two solitons leads to unchanged forms, solitons also exhibit particlelike properties.

Soliton perturbation theory The soliton perturbation theory is used to study the solitons that are governed by the various nonlinear equations in presence of the perturbation terms.

Homotopy perturbation method The homotopy perturbation method is a useful tool to the search for solitons without the requirement of presence of small perturbations. In this method, a homotopy is constructed with a homotopy parameter, p . When $p = 0$, it becomes a nonlinear wave equation such as a KdV equation with a known soliton solution; when $p = 1$, it turns out to be the original nonlinear equation. To change p from zero to unity, one must only change from a trial soliton to the solved soliton.

Variational iteration method The variational iteration method is a new method for obtaining soliton-type solutions of various nonlinear wave equations. The method begins with a soliton-type solution with some unknown parameters which can be determined after few iterations. The iteration formulation is constructed by a general Lagrange multiplier which can be identified optimally via variational theory.

Exp-function method The exp-function method is a new method for searching for both soliton-type solutions and periodic solutions of nonlinear systems. The method assumes that the solutions can be expressed in arbitrary forms of the exp-function.

Definition of the Subject

The soliton is a kind of nonlinear wave. There are many equations of mathematical physics which have solutions of the soliton type. The first observation of this kind of wave was made in 1834 by John Scott Russell [1]. In 1895, the famous KdV equation, which possesses soliton solutions, was obtained by D. J. Korteweg and H. de Vries [2], who established a mathematical basis for the study of various solitary phenomena.

From a modern perspective, the soliton is used as a constructive element to formulate the complex dynamical behavior of wave systems throughout science: from hydrodynamics to nonlinear optics, from plasmas to shock waves, from tornados to the Great Red Spot of Jupiter, from traffic flow to the Internet, from Tsunamis to turbulence [3]. More recently, solitary waves are of key importance in the quantum fields: on extremely small scales and at very high observational resolution equivalent to a very high energy, space–time resembles a stormy ocean and particles and their interactions have soliton-type solutions [4].

Introduction

The soliton was first discovered in 1834 by John Scott Russell, who observed that a canal boat stopping suddenly gave rise to a solitary wave which traveled down the canal for several miles, without breaking up or losing strength. Russell named this phenomenon the ‘soliton’.

In a highly informative as well as entertaining article [1] J.S. Russell gave an engaging historical account of the important scientific observation:

I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped – not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at

a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation.

His ideas did not earn attention until 1965 when N.J. Zabusky and M.D. Kruskal began to use a finite difference approach to the study of KdV equation [5], and various analytical methods also led to a complete understanding of Solitons, especially the inverse scattering transform proposed by Gardner, Greene, Kruskal, and Miura [6] in 1967. The significance of Russell's discovery was then fully appreciated. It was discovered that many phenomena in physics, electronics and biology can be described by the mathematical and physical theory of the 'Soliton'.

The particle-like properties of solitons [7] also caught much attention, and were proposed as models for elementary particles [8]. More recently it has been realized that some of the quantum fields which are used to describe particles and their interactions also have solutions of the soliton type [9].

Methods for Soliton Solutions

The investigation of soliton solutions of nonlinear evolution equations plays an important role in the study of nonlinear physical phenomena. There are many analytical approaches to the search for soliton solutions, such as soliton perturbation, tanh-function method, projective approach, F-expansion method, and others [10,11,12,13,14].

Soliton Perturbation

We consider the following perturbed nonlinear evolution equation [15,16]

$$u_T + N(u) = \varepsilon R(u), \quad 0 < \varepsilon \ll 1. \quad (1)$$

When $\varepsilon = 0$, we have the un-perturbed equation

$$u_T + N(u) = 0, \quad (2)$$

which is assumed to have a soliton solution.

When $\varepsilon \neq 0$, but $0 < \varepsilon \ll 1$, we can use perturbation theory [15,16], and look for approximate solutions of Eq. (1), which are close to the soliton solutions of Eq. (2). Using multiple time scales (a slow time τ and a fast time t , such that $\partial_T = \partial_t + \varepsilon \partial_\tau$), we assume that the soliton solution can be expressed in the form

$$u(x, T) = u_0(\xi, \tau) + \varepsilon u_1(\xi, \tau, t) + \varepsilon^2 u_2(\xi, \tau, t) + \dots \quad (3)$$

where $\xi = x - ct$, and τ is a slow time and t is a fast time.

Substituting Eq. (3) into Eq. (1) and then equating like-powers of ε , we can obtain a series of linear equations for u_i ($i = 0, 1, 2, 3, \dots$).

In most cases the nonlinear term $R(u)$ in Eq. (1) plays an import role in understanding various solitary phenomena, and the coefficient ε is not limited to a "small parameter".

Variational Approach

Recently, variational theory and homotopy technology have been successfully applied to the search for soliton solutions [17,18] without requiring the small parameter assumption. Both variational and homotopy technologies can lead to an extremely simple and elementary, but rigorous, derivation of soliton solutions.

Considering the KdV equation

$$\frac{\partial u}{\partial t} - 6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0, \quad (4)$$

we seek its traveling wave solutions in the following frame

$$u(x, t) = U(\xi) \quad v(x, t) = V(\xi), \quad \xi = x - ct, \quad (5)$$

where c is angular frequency. Substituting Eq. (5) into Eq. (4) yields

$$-cu' - 6uu' + u''' = 0, \quad (6)$$

where a prime denotes the differential with respect to ξ .

Integrating Eq. (6) yields the result

$$-cu - 3u^2 + u'' = 0. \quad (7)$$

By the semi-inverse method [19], the following variational formulation is established

$$J = \int_0^\infty \left(\frac{1}{2} cu^2 + u^3 + \frac{1}{2} \left(\frac{du}{d\xi} \right)^2 \right) d\xi. \quad (8)$$

The semi-inverse method is a powerful mathematical tool to the search for variational formulae for real-life physical problems.

By the Ritz method, we search for a solitary wave solution in the form

$$u = p \operatorname{sech}^2(q\xi), \quad (9)$$

where p and q are constants to be further determined.

Substituting Eq. (9) into Eq. (8) results in

$$\begin{aligned} J &= \int_0^\infty \left[\frac{1}{2} c p^2 \operatorname{sech}^4(q\xi) + p^3 \operatorname{sech}^6(q\xi) \right. \\ &\quad \left. + \frac{1}{2} (4p^2 q^2 \operatorname{sech}^4(q\xi) \tanh^2(q\xi)) \right] d\xi \\ &= \frac{c p^2}{2q} \int_0^\infty \operatorname{sech}^4(z) dz + \frac{p^3}{q} \int_0^\infty \operatorname{sech}^6(z) dz \\ &\quad + 2p^2 q \int_0^\infty \{ \operatorname{sech}^4(z) \tanh^4(z) \} dz \\ &= \frac{c p^2}{3q} + \frac{8p^3}{15q} + \frac{4p^2 q}{15}. \end{aligned} \quad (10)$$

Making J stationary with respect to p and q results in

$$\frac{\partial J}{\partial p} = \frac{2cp}{3q} + \frac{24p^2}{15q} + \frac{8pq}{15} = 0, \quad (11)$$

$$\frac{\partial J}{\partial q} = -\frac{cp^2}{3q^2} - \frac{8p^3}{15q^2} + \frac{4p^2}{15} = 0, \quad (12)$$

or simplifying

$$5c + 12p + 4q^2 = 0, \quad (13)$$

$$-5c - 8p + 4q^2 = 0. \quad (14)$$

From Eqs. (13) and (14), we can easily obtain the following relations:

$$p = -\frac{1}{2}c, \quad q = \sqrt{\frac{c}{4}}. \quad (15)$$

So the solitary wave solution can be approximated as

$$u = -\frac{c}{2} \operatorname{sech}^2 \sqrt{\frac{c}{4}} (x - ct - \xi_0), \quad (16)$$

which is the exact solitary wave solution of KdV equation (4).

The preceding analysis has the virtue of utter simplicity. The suggested variational approach can be readily applied to the search for solitary wave solutions of other nonlinear problems, and the present example can be used as paradigms for many other applications in searching for solitary wave solutions of real-life physics problems.

Variational Iteration Method

The variational iteration method [20] is an alternative approach to soliton solutions without the requirement of establishing a variational formulation for the discussed problems [17,21,22,23,24]. As an illustrating example, we consider the $K(3,1)$ equation in the form [17]:

$$u_t + u^2 u_x + u_{xxx} = 0. \quad (17)$$

According to the variational iteration method, its iteration formulation can be constructed as follows

$$\begin{aligned} u_{n+1}(x, t) \\ &= u_n(x, t) - \int_0^t \{ (u_n)_t + u_n^2 (u_n)_x + (u_n)_{xxx} \} dt. \end{aligned} \quad (18)$$

To search for its compacton-like solution, we assume the solution has the form

$$u_0(x, t) = \frac{a \sin^2(kx + wt)}{b + c \sin^2(kx + wt)}, \quad (19)$$

where a , b , k , and w are unknown constants further to be determined after few iterations [17].

Homotopy Perturbation Method

The homotopy perturbation method [25] provides a simple mathematical tool for searching for soliton solutions without any small perturbation [18,26]. Considering the following nonlinear equation

$$\frac{\partial u}{\partial t} + au \frac{\partial u}{\partial x} + b \frac{\partial^3 u}{\partial x^3} + N(u) = 0, \quad a > 0, b > 0, \quad (20)$$

we can construct a homotopy in the form

$$\begin{aligned} (1-p) \left\{ \frac{\partial u}{\partial t} + 6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} \right\} \\ + p \left\{ \frac{\partial u}{\partial t} + au \frac{\partial u}{\partial x} + b \frac{\partial^3 u}{\partial x^3} + N(u) \right\} = 0. \end{aligned} \quad (21)$$

When $p = 0$, we have

$$\frac{\partial u}{\partial t} + 6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0, \quad (22)$$

a well-known KdV equation whose soliton solution is known. When $p = 1$, Eq. (21) turns out to be the original equation. According to the homotopy perturbation method, we assume

$$u = u_0 + pu_1 + p^2 u_2 + \dots \quad (23)$$

Substituting Eq. (23) into Eq. (21), and proceeding with the same process as the traditional perturbation method does, we can easily solve u_0 , u_1 and other components. The solution can be expressed finally in the form

$$u = u_0 + u_1 + u_2 + \dots \quad (24)$$

The homotopy perturbation method always stops before the second iteration, so the solution can be expressed as

$$u = u_0 + u_1 \quad (25)$$

for most cases.

Parameter-Expansion Method

The parameter-expansion method [27,28,29,30] does not require one to construct a homotopy. To illustrate its solution procedure, we re-write Eq. (20) in the form

$$\frac{\partial u}{\partial t} + au \frac{\partial u}{\partial x} + b \frac{\partial^3 u}{\partial x^3} + 1 \cdot N(u) = 0. \quad (26)$$

Supposing that the parameters a , b , and 1 can be expressed in the forms

$$a = a_0 + pa_1 + p^2 a_2 + \dots \quad (27)$$

$$b = b_0 + pb_1 + p^2 b_2 + \dots \quad (28)$$

$$1 = pc_1 + p^2 c_2 + \dots \quad (29)$$

where p is a bookkeeping parameter, $p = 1$. Substituting Eqs. (23), (27), (28) and (29) into Eq. (26) and proceeding the same way as the perturbation method, we can easily obtain the needed solution.

Exp-function Method

The exp-function method [31,32,33] provides us with a straightforward and concise approach to obtaining generalized solitary solutions and periodic solutions and the solution procedure, with the help of Matlab or Mathematica, is utterly simple. Consider a general nonlinear partial differential equation of the form

$$F(u, u_x, u_y, u_z, u_t, u_{xx}, u_{yy}, u_{zz}, u_{tt}, u_{xy}, u_{xt}, u_{yt}, \dots) = 0. \quad (30)$$

Using a transformation

$$\eta = ax + by + cz + dt, \quad (31)$$

we can re-write Eq. (30) in the form of the following nonlinear ordinary differential equation:

$$G(u, u', u'', u''', \dots) = 0, \quad (32)$$

where a prime denotes a derivation with respect to η .

According to the exp-function method, the traveling wave solutions can be expressed in the form

$$u(\eta) = \frac{\sum_{n=-k}^l a_n \exp(n\eta)}{\sum_{m=-i}^j b_m \exp(m\eta)}, \quad (33)$$

where i , j , k , and l are positive integer which could be freely chosen, a_n and b_m are unknown constants to be determined. The solution procedure is illustrated in [32].

Future Directions

It is interesting to point out the connection of catastrophe theory to loop soliton chaos, and finally to chaotic Cantorian spacetime [34,35].

El Naschie [34,35] studied the Eguchi–Hanson gravitational instanton solution and its interpretation by 't Hooft in the context of a quantum gravitational Hilbert space, as an event and a possible solitonic “extended” particle. Transferring a certain solitonic solution of Einstein’s field equations in Euclidean “real” space–time to the mathematical infinitely-dimensional Hilbert space, it is possible to observe a new non-standard process by which a definite mass can be assigned to massless particles. Thus by invoking Einstein’s gravity in “solitonic” gauge theory and vice versa, an alternative explanation for how massless particles acquire mass is found, which is also in harmony with the basic structure of our standard model as it stands at present [34,35].

Bibliography

Primary Literature

1. Russell JS (1844) Report on Waves. Fourteenth Meeting of the British Association for the Advancement of Science, John Murray, London, pp 311–390
2. Korteweg DJ, De Vries G (1895) On the change of form of long waves advancing in a rectangular channel, and a new type of long stationary wave. *Phil Mag Ser* 539:422–443
3. Eilbeck C (2007) John Scott Russell and the solitary wave. Heriot-Watt University, Edinburgh http://www.ma.hw.ac.uk/~chris/scott_russell.html
4. El Naschie MS (2004) A review of E infinity theory and the mass spectrum of high energy particle physics. *Chaos Solit Fract* 19(1):209–236
5. Zabusky NJ, Kruskal MD (1965) Interaction of ‘Solitons’ in a collisionless plasma and the recurrence of initial states. *Phys Rev Lett* 15:240–243
6. Gardner CS, Greene JM, Kruskal MD, Miura RM (1967) Method for solving the KdV equation. *Phys Rev Lett* 19:1095–1097
7. Bode M, Liehr AW, Schenk CP et al (2002) Interaction of dissipative solitons: particle-like behavior of localized structures in a three-component reaction-diffusion system. *Physica D* (1–2):45–66
8. Braun HB, Kulda J, Roessli B et al (2005) Emergence of soliton chirality in a quantum antiferromagnet. *Nat Phys* 1(3):159–163
9. Ahufinger V, Mebrahtu A, Corbalan R et al (2007) Quantum switches and quantum memories for matter-wave lattice solitons. *New J Phys* 9:4
10. Ye JF, Zheng CL, Xie LS (2006) Exact solutions and localized excitations of general Nizhnik–Novikov–Veselov system in (2+1)-dimensions via a projective approach. *Int J Nonlinear Sci Num Simul* 7(2):203–208
11. Bogning JR, Tchakoutio-Nguetcho AS, Kofane TC (2005) Gap solitons coexisting with bright soliton in nonlinear fiber arrays.

- Int J Nonlinear Sci Num Simul 6(4):371–385; Abdusalam HA (2005) On an improved complex tanh-function method. Int J Nonlinear Sci Num Simul 6(2):99–106
12. El-Sabbagh MF, Ali AT (2005) New exact solutions for (3+1)-dimensional Kadomtsev–Petviashvili equation and generalized (2+1)-dimensional Boussinesq equation. Int J Nonlinear Sci Num Simul 6(2):151–162
 13. Shen JW, Xu W (2004) Bifurcations of smooth and non-smooth travelling wave solutions of the Degasperis-Procesi equation. Int J Nonlinear Sci Num Simul 5(4):397–402
 14. Sheng Z (2007) Further improved F-expansion method and new exact solutions of Kadomtsev–Petviashvili equation. Chaos Solit. Fract 32(4):1375–1383
 15. Yu H, Yan J (2006) Direct approach of perturbation theory for kink solitons. Phys Lett A 351(1–2):97–100
 16. Herman RL (2005) Exploring the connection between quasi-stationary and squared eigenfunction expansion techniques in soliton perturbation theory. Nonlinear Anal 63(5–7): e2473–e2482
 17. He JH, Wu XH (2006) Construction of solitary solution and compacton-like solution by variational iteration method. Chaos Solit Fract 29(1):108–113
 18. He JH (2005) Application of homotopy perturbation method to nonlinear wave equations. Chaos Solit Fract 26(3): 695–700
 19. He JH (2004) Variational principles for some nonlinear partial differential equations with variable coefficients. Chaos Solit Fract 19(4):847–851
 20. He JH (1999) Variational iteration method – a kind of non-linear analytical technique: Some examples. Int J Non-Linear Mech 34(4):699–708
 21. Abulwafa EM, Abdou MA, Mahmoud AA (2007) Nonlinear fluid flows in pipe-like domain problem using variational-iteration method. Chaos Solit Fract 32(4):1384–1397
 22. Inc M (2007) Exact and numerical solitons with compact support for nonlinear dispersive $K(m,p)$ equations by the variational iteration method. Phys A 375(2):447–456
 23. Soliman AA (2006) A numerical simulation and explicit solutions of KdV-Burgers' and Lax's seventh-order KdV equations. Chaos Solit Fract 29(2):294–302
 24. Abdou MA, Soliman AA (2005) Variational iteration method for solving Burger's and coupled Burger's equations. J Comput Appl Math 181(2):245–251
 25. He JH (2000) A coupling method of a homotopy technique and a perturbation technique for non-linear problems. Int J Non-Linear Mech 35(1):37–43
 26. Ganji DD, Rafei M (2006) Solitary wave solutions for a generalized Hirota-Satsuma coupled KdV equation by homotopy perturbation method. Phys Lett A 356(2):131–137
 27. Shou DH, He JH (2007) Application of Parameter-expanding Method to Strongly Nonlinear Oscillators. Int J Nonlinear Sci Numer Simul 8:113–116
 28. He JH (2001) Bookkeeping parameter in perturbation methods. Int J Nonlinear Sci Numer Simul 2:257–264
 29. He JH (2002) Modified Lindstedt-Poincare methods for some strongly non-linear oscillations. Part I: expansion of a constant. Int J Non-Linear Mech 37:309–314
 30. Xu L (2007) He's parameter-expanding methods for strongly nonlinear oscillators. J Comput Appl Math 207(1):148–157
 31. He J-H, Wu X-H (2006) Exp-function method for nonlinear wave equations. Chaos Solit Fract 30(3):700–708
 32. He J-H, Abdou MA (2007) New periodic solutions for nonlinear evolution equations using Exp-function method. Chaos Solit Fract 34(5):1421–1429
 33. Wu X-H, He J-H (2008) EXP-function method and its application to nonlinear equations. Chaos Solit Fract 38(3):903–910
 34. El Naschie MS (2004) Gravitational instanton in Hilbert space and the mass of high energy elementary particles. Chaos Solit Fract 20(5):917–923
 35. El Naschie MS (2004) How gravitational instanton could solve the mass problem of the standard model of high energy particle physics. Chaos Solit Fract 21(1):249–260

Books and Reviews

- He JH (2006) Some Asymptotic Methods for Strongly Nonlinear Equations. Int J Mod Phys B 20(10):1141–1199; 20(18): 2561–2568
- He JH (2006) Non-perturbative methods for strongly nonlinear problems. dissertation.de-Verlag im Internet, Berlin
- Drazin PG, Johnson RS (1989) Solitons: An Introduction. Cambridge University Press, Cambridge

Solitons and Compactons

Ji-HUAN HE¹, SHUN-DONG ZHU²

¹ Modern Textile Institute, Donghua University, Shanghai, China

² Department of Science, Zhejiang Lishui University, Lishui, China

Article Outline

[Glossary](#)
[Definition of the Subject](#)
[Introduction](#)
[Solitons](#)
[Compactons](#)
[Generalized Solitons and Compacton-like Solutions](#)
[Future Directions](#)
[Cross References](#)
[Bibliography](#)

Glossary

Soliton A soliton is a stable pulse-like wave that can exist in some nonlinear systems. The soliton, after a collision with another soliton, eventually emerges unscathed.

Compacton A compacton is a special solitary traveling wave that, unlike a soliton, does not have exponential tails.

Generalized soliton A generalized soliton is a soliton with some free parameters. Generally a generalized soliton can be expressed by exponential functions.