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An elementary introduction to the homotopy perturbation method

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ABSTRACT

This paper is an elementary introduction to the concepts of the homotopy perturbation method. Particular attention is paid to giving an intuitive grasp for the solution procedure throughout the paper.

This short note is actually a comment on Ariel's publication [1] and a reply to El-Mistikawy's comment [2].

Ariel [1] considered the following coupled equations

$$F''' + (F+G)F'' - F'^2 = 0 \tag{1}$$

$$G''' + (F+G)G'' - G'^2 = 0 \tag{2}$$

$$F'(0) = 1, \quad G'(0) = \beta, \quad F'(\infty) = 0, \quad G'(\infty) = 0, \quad \text{and} \quad F(0) + G(0) = 0 \tag{3}$$

where prime denotes the derivative with respect to the similarity coordinate η.

Ariel constructed the following homotopy equations

$$F''' - b^2F' + P\left\{(F+G)F'' - F'^2 - b^2F'\right\} = 0 \tag{4}$$

$$G''' - b^2G' + p\left\{(F+G)G'' - G'^2 - b^2G'\right\} = 0. \tag{5}$$

The obtained results are of high accuracy even for the first-order approximate solution, showing that the solution procedure is acceptable.

El-Mistikawy [2] considers that the homotopy perturbation cannot obtain higher-order approximation except first-order approximation. Actually El-Mistikawy might misapprehend the method, which always tends to solve first-order approximations but the solution procedure can continue without any difficulty to obtain higher-order approximations, see Refs. [3–7].

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Let us outline the basic solution procedure of the homotopy perturbation method:

1. Qualitative sketch/trial function solution

From physical understandings and boundary conditions, we choose such initial guesses:

$$F'(\eta) = e^{-a\eta} \tag{6}$$

and

$$G'(\eta) = \beta e^{-b\eta} \tag{7}$$

where a and b are unknown constants to be further identified. It is obvious that Eqs. (6) and (7) satisfy the boundary condition $F'(0) = 1, F'(\infty) = 0$ and $G'(0) = \beta, G'(\infty) = 0$ respectively.

2. Construction of homotopy equations

According to the initial guesses, Eqs. (6) and (7), homotopy equations can be constructed as follows

$$F''' - a^2F' + P\left\{(F+G)F'' - F'^2 - a^2F'\right\} = 0 \tag{8}$$

$$G''' - b^2G' + p\left\{(F+G)G'' - G'^2 - b^2G'\right\} = 0. \tag{9}$$

When $p = 0$, we can obtain the initial guesses; When $p = 1$, Eqs. (8) and (9) turn out to be the original ones.

Our homotopy system turns out to be Ariel's system when we choose $a = b$ for simplicity.

3. Solution procedure similar to that of classical perturbation methods

Now we can use the homotopy parameter, p, as an expanding parameter used in classic perturbation methods. The simplest way is the method of straightforward expansion:

$$F = F_0 + pF_1 + p^2F_2 + \cdots \tag{10}$$

$$G = G_0 + pG_1 + p^2G_2 + \cdots. \tag{11}$$

Generally we stop before second iteration. Setting $p = 1$, we obtain the first-order approximation which reads

$$F(\eta) = F_0(\eta) + F_1(\eta) \tag{12}$$

$$G(\eta) = G_0(\eta) + G_1(\eta). \tag{13}$$

4. Optimal identification of the unknown parameter in the trial functions

There are many approaches to identify the unknown parameters in the obtained approximation, for example, the method of weighted residuals, especially the least squares method.

For the present problem, we set

$$\int_0^{+\infty} e^{-a\eta}\left\{F''' + (F+G)F'' - F'^2\right\} dt = 0 \tag{14}$$

and

$$F(0) + G(0) = 0 \tag{15}$$

to identify the unknown constants a and b. Ariel's identification of the unknown constant is equivalent to Eq. (14).

5. Higher-order approximations

If first-order approximation cannot meet the requirement of engineering applications, higher-order approximations should be solved.

As it is well known the straightforward expansion leads to secular terms [8] and various methods, such as Lindstedt–Poincare method [9–11], are suggested to eliminate the secular terms. One of the advantages of the homotopy perturbation method is that it can take full advantage of various known perturbation methods.

El-Mistikawy [2] actually uses the Lindstedt–Poincare method, of course, El-Mistikawy's solution procedure is valid for higher-order approximations.

In Ref. [9], the present author suggested a modified Lindstedt–Poincare method, which is also very useful and effective to solve higher-order approximations. We re-write Eqs. (1) and (2) in the forms

$$F''' + 0 \cdot F' + p\left\{(F+G)F'' - F'^2\right\} = 0 \tag{16}$$

$$G''' + 0 \cdot G' + p\left\{(F+G)G'' - G'^2\right\} = 0. \tag{17}$$

The zero coefficient in Eq. (16) can be expressed in [6,9,12]

$$0 = -a^2 + a_1 p + a_2 p^2 + \cdots. \tag{18}$$

The zero coefficient in Eq. (17) can be expanded in a similar way

$$0 = -b^2 + b_1 p + b_2 p^2 + \cdots. \tag{19}$$

The constants a_i and b_i can be identified in view of no η terms, i.e., no $\eta e^{-a\eta}$ term in F and no $\eta e^{-b\eta}$ term in G.

Why such expansion is valid is explained in Ref. [12].

The above solution procedure can be achieved by the parameter-expansion method [6]. We re-write Eqs. (1) and (2) in the forms:

$$F''' + 0 \cdot F' + 1 \cdot \left\{(F+G)F'' - F'^2\right\} = 0 \tag{20}$$

$$G''' + 0 \cdot G' + 1 \cdot \left\{(F+G)G'' - G'^2\right\} = 0. \tag{21}$$

We expand F, G and zero as before, and expand unit in Eqs. (20) and (21), respectively, as

$$1 = c + c_1 p + c_2 p^2 + \cdots \tag{22}$$

and

$$1 = d + d_1 p + d_2 p^2 + \cdots. \tag{23}$$

For detailed solution procedure for parameter-expansion method, please refer to Refs. [6,13–18].

For detailed description of the homotopy perturbation method, please refer to Refs. [6,19].

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