How to Study Mathematics

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This essay describes a number of strategies for studying college level mathematics. It has sections entitled

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How is college mathematics different from high school math?

In high school mathematics much of your time was spent learning algorithms and manipulative techniques which you were expected to be able to apply in certain well-defined situations. This limitation of material and expectations for your performance has probably led you to develop study habits which were appropriate for high school mathematics but may be insufficient for college mathematics. This can be a source of much frustration for you and for your instructors. My object in writing this essay is to help ease this frustration by describing some study strategies which may help you channel your abilities and energies in a productive direction.

The first major difference between high school mathematics and college mathematics is the amount of emphasis on what the student would call theory---the precise statement of definitions and theorems and the logical processes by which those theorems are established. To the mathematician this material, together with examples showing why the definitions chosen are the correct ones and how the theorems can be put to practical use, is the essence of mathematics. A course description using the term ``rigorous'' indicates that considerable care will be taken in the statement of definitions and theorems and that proofs will be given for the theorems rather than just plausibility arguments. If your approach is to go straight to the problems with only cursory reading of the ``theory'' this aspect of college math will cause difficulties for you.

The second difference between college mathematics and high school mathematics comes in the approach to technique and application problems. In high school you studied one technique at a time---a problem set or unit might deal, for instance, with solution of quadratic equations by factoring or by use of the quadratic formula, but it wouldn't teach both and ask you to decide which was the better approach for particular problems. To be sure, you learn individual techniques well in this approach, but you are unlikely to learn how to attack a problem for which you are not told what technique to use or which is not exactly like other applications you have seen. College mathematics will offer many techniques which can be applied for a particular type of problem---individual problems may have many possible approaches, some of which work better than others. Part of the task of working such a problem lies in choosing the appropriate technique. This requires study habits which develop judgment as well as technical competence.

We will take up the problem of how to study mathematics by considering specific aspects individually. First we will consider definitions---first because they form the foundation for any part of mathematics and are essential for understanding theorems. Then we'll take up theorems, lemmas, propositions, and corollaries and how to study the way the subject fits together. The subject of proofs, how to decipher them and why we need them, comes next. And finally, we will discuss development of judgment in problem solving.
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What should you do with a definition?

A definition in mathematics is a precise statement delineating and naming a concept by relating it to previously defined concepts or such undefined concepts as ``number'' or ``set.'' Careful definitions are necessary so that we know exactly what we are talking about. Unfortunately, for many of the concepts in undergraduate mathematics the definition is rather difficult to understand, so often at low levels an intuitive feeling for the meaning of a term is all that is given or required. This intuitive feeling, while necessary, is not sufficient at the college level. This means that you need to grapple with and master the formal statement of definitions and their meanings. How do you do it?

Step 1. Make sure you understand what the definition says.

This sounds obvious, but it can cause some difficulties, particularly for definitions with complicated logical structure (like the definition of the limit of a function at a point in its domain). Definitions are not a good place to practice your speed reading. In general there are no wasted words or extraneous symbols in established definitions and the easily overlooked small words like *and, or, if ... then, for all,* and *there is* are your clues to the logical structure of the definition.

First determine what general class of things is being talked about: the definition of a polynomial describes a particular kind of algebraic expression; the definition of a continuous function specifies a kind of function; the definition of a basis for a vector space specifies a kind of set of vectors.

Next decipher the logical structure of the definition. What do you have to do to show that a member of your general class of things satisfies the definition: what do you have to do to show that an expression is a polynomial, or a function is continuous, or a set of vectors is a basis.

Step 2. Determine the scope of the definition with examples.

Most definitions have standard examples that go with them. While these are useful, they may lead you to expect that all examples look like the standard example. To understand a definition you should make up your own examples: find three examples that do satisfy the definition but which are as different as possible from each other; find two examples of items in the general class described by the definition which do not satisfy it. Prove that your five examples do what you think they do---such proofs are usually short, follow the structure of the definition quite closely, and help immensely in understanding the definition. These examples should be neatly written up so that you can refer to them later. Your own examples will have more meaning for you than mine or the book's when it comes time to review.

Step 3. Memorize the exact wording of the definition.

This step may sound petty, but the use of definitions demands knowledge of exactly what they say. For this reason you can count on being asked for the statement of any definition on an exam. The importance of precise wording should have been made clear by your examples in step 2 and it certainly is essential in the proof of theorems.

Solid knowledge of definitions is more than a third of the battle. Time spent gaining such knowledge is not wasted.
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Theorems, Propositions, Lemmas, and Corollaries

Occasionally definitions are useful in and of themselves, but usually we need to relate them to each other and to general problems before they can be made to work for us. This is the role of theory.

The relative importance and the intended use of statements which are then proved is hinted at by the names they are given. Theorems are usually important results which show how to make concepts solve problems or give major insights into the workings of the subject. They often have involved and deep proofs. Propositions give smaller results, often relating different definitions to each other or giving alternate forms of the definition. Proofs of propositions are usually less complex than the proofs of theorems. Lemmas are technical results used in the proofs of theorems. Often it is found that the same trick is used several times in one proof or in the proof of several theorems. When this happens the trick is isolated in a lemma so that its proof will not have to be repeated every time it is used. This often makes the proofs of theorems shorter and, one hopes, more lucid. Corollaries are immediate consequences of theorems either giving special cases or highlighting the interest and importance of the theorem. If the author or instructor has been careful (not all authors and instructors are) with the use of these labels, they will help you figure out what is important in the subject.

The steps to understanding and mastering a theorem follow the same lines as the steps to understanding a definition.

Step 1. Make sure you understand what the theorem says.

Part of this is a vocabulary problem. Theorems use terms which have been given precise meanings by definitions. So you may need to review the definitions to understand the words in a theorem.

Next you need to understand the logical structure of the theorem: what are the hypotheses and what are the conclusions? If you have several hypotheses, must they all be satisfied (that is, do they have an ***and*** between them) or will it suffice to have only some of them (an ***or*** between them)? In most cases theorems require that all of their hypotheses be satisfied. A theorem tells you nothing about a situation which does not satisfy the hypothesis. The hypothesis tells you what you must show in order for the theorem to apply to a particular case. The conclusions tell you what the theorem tells you about each case.

Step 2. Determine how the theorem is used.

This involves finding examples of problems for which the theorem gives a technique for finding the answer. Make up your own problems and show how the theorem helps with them. Again writing this down will help solidify the theorem in your mind and make it easier to review.

Step 3. Find out what the hypotheses are doing there.

This is a little tricky and is probably more important in advanced courses than in beginning courses. What you do is find examples (either your own or someone else's) to show that if individual hypotheses are omitted the conclusion can be false. For instance, in calculus many theorems have a hypothesis that the functions involved be continuous; why does the theorem fail if this hypothesis is left out? Usually an example will make this clearer than an examination of how the hypothesis was used in the proof will. A catalog of such examples can be very useful. See for instance the books Counterexamples in Analysis and Counterexamples in Topology.

In some cases a hypothesis is included just because it makes an otherwise complicated proof easy. This means that you may not be able to find examples which illustrate that each hypothesis is essential.

Step 4. Memorize the statement of the theorem.

If you are going to use a theorem you need to know exactly what it says. Pay particular attention to hypotheses. We will take up proofs later, but for now let me note that it is not a good idea to try to memorize the proof of a theorem. What you want to do is understand the proof well enough that you can prove the theorem yourself.
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Fitting the subject together

Mathematics is not a collection of miscellaneous techniques but rather a way of thinking---a unified subject. Part of the task of studying mathematics is getting the various definitions and theorems properly related to each other. This is particularly important at the end of a course, but it will help you make sense of the content and organization of a subject if you keep the overall organization in mind as you go along. There are two techniques I know of which help with this process: working backwards and definition-theorem outlines.

Step 1. Working backwards

In general there is very little difficulty recognizing a major result when you get to it. A good way of seeing how a subject works is to examine the proof of a major result and see what previous results were used in it. Then trace these results back to earlier results used to prove them. Eventually you will work your way back to definitions (unless there are theorems given without proof---in calculus, for instance, the proof of the intermediate value theorem is often omitted because it requires a deeper understanding of the real numbers than is usually available at the beginning of calculus 1). This information can be put into a sort of genealogy chart for results which helps you see at a glance how the results fit together. It helps to have descriptive names for your theorems and lemmas. Such a chart might look sort of like this:

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Mean Value Theorem
      Rolle's Theorem
           Candidate Lemma
                Meaning of the sign of the derivative
                     Definition of derivative
                Definition of max and min
      Existence  of  max  and  min   for  continuous  functions  on [a, b]
                Definition of max and min
                Definition of closed interval
                Least upper bound axiom
                Definition of continuity
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With such a road map through the theory you should be able to tell how you got where you are, if not where you are headed.

Step 2. Make a definition-theorem outline.

After you have worked backwards to the definitions for each of your major theorems in a section you should have a good idea of which results are needed before others can be proved. Some definitions will not make sense until certain theorems are proved (dimension of a vector space is an example: you can't give a number a name until you know you are talking about a unique number, and that requires a theorem). A definition theorem outline is an arrangement of the results in an order so that each result is introduced before it is needed in a proof. It should contain the precise statements of all definitions and theorems and a sketch of the proof of each theorem. A sketch of a proof will show which earlier results were used and how they were combined. It will usually omit calculations simplifying forms of expressions and routine checks that hypotheses are satisfied. This outline is both a good way to start a review and a useful thing to have to refer to.
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How to make sense of a proof

College level mathematics demands that the student work through (or at least sit through) many proofs. This is often unpopular, I think largely because it is hard work to follow a proof, hard work of an unfamiliar kind. Proof is largely absent or at most optional in high school math; it is neither absent nor optional in college mathematics.

Step 1. Make sure you know what the theorem says.

If you have the hypotheses mixed up with the conclusions you will not know what assumptions may be made nor will you know what conclusion you are trying to reach.

Step 2. Make a general outline of the proof.

This is what you would do in a definition theorem outline. See what the previous results used are and find out what the basic strategy of the proof is. On this pass through omit the details, else you miss the direction of the road by too close examination of the bricks in the pavement.

Most theorems have the form of implications: if the hypotheses are true, then the conclusion follows. The easiest structure for a proof to use is to assume the hypotheses and combine them, using previous results, to reach the conclusion through a chain of implications. Some proofs use other strategies: contrapositive argument, reductio ad absurdum, mathematical induction, perhaps even Zorn's lemma (a form of the axiom of choice). The more complicated kinds of proofs will need to be discussed in class.

Step 3. Fill in all of the details.

Once you understand the strategy of the proof concentrate on its tactics. Almost all expositions of proofs in undergraduate mathematics textbooks (and all expositions at higher levels) leave out many routine steps. An expression will be simplified without showing exactly how to get from one line to the next. Fill in these details. A theorem will be quoted and applied without explicitly checking all of its hypotheses. Check them. Some parts of the proof will be outlined with the details left to the reader. Put in those details. When you finish you should know why each step follows from what came before. You may not see how anyone could have thought to do the proof that way, but you should be able to see that it is correct.

Why bother with proofs at all?

For the mathematics major this question is easy to answer---a large portion of mathematics consists of proofs. The mathematician enjoys the logical puzzle which must be solved to find a proof and obtains aesthetic satisfaction from elegance in proofs. The student who wants to major in mathematics should do so because of ability in deciphering and producing proofs and enjoyment derived from proof well done. The major should also have skill in solving problems and finding applications as well.

But many of you will say ``I'm not a math major; I want applications so that I can use tools from mathematics in my field'' or ``I'm just taking this course because it's a requirement in my major and I sort of liked math in high school.'' Why should you learn about proof?

The applications you meet in other fields are not likely to look exactly like the math textbook applications, which are chosen for their appeal to a traditional audience (largely engineers) and for their representative character. Other applications work similarly, though not exactly the same way.

This means that you need to learn how to apply the concepts in your math courses to situations not discussed in those courses. (There is no way that a course could discuss every possible known application: about 500 papers appear every two weeks with applications, and those are just the applications published in the ``mathematical'' literature!) To do so you need the best possible understanding of the mathematics you want to apply. Certainly this means that you need to know the hypotheses of theorems so that you don't apply them where they won't work. It is helpful to know the proof so that you can see how to circumvent the failed hypothesis if necessary. One of the major pitfalls of applied mathematics, particularly as practiced by nonmathematicians, is the danger of conveniently overlooking the assumptions of a mathematical model. (Mathematicians trying to do applied mathematics are more likely to fall into the trap of making models which have no relationship to reality.)

Many applications consist of recognizing the definition of a mathematical concept phrased in the terms of another discipline---the more familiar you are with the definition, the more likely you are to be able to recognize the disguised version elsewhere. The nuances of definitions are made most clear in the proofs of propositions relating definitions and pointing out unexpected equivalent variants, some of which may look more like a situation in another discipline than the precise form used in your math class.

Arguments for theory as an aid to application rest on an obvious premise: it is much easier to apply something you understand thoroughly. This is, however, a better argument for care in learning the statements of theorems than it is for spending time understanding proofs. The best justification for the inclusion of proof in math classes is more philosophical:

Proof is the ultimate test of validity in mathematics.

Once one accepts the logical processes involved in a proof no further observation or change in fashion will change the validity of a mathematical result. No other discipline has such an immutable criterion for validity.

The major benefit derived from an education is the ability to think clearly and make considered judgment. Each discipline should teach a body of material, appropriate modes of thought in dealing with that material, and a means for determining the validity of the conclusions reached. A chemistry curriculum with no lab work would be seriously deficient since experiment is the test of validity in science. Similarly mathematics without proof is severely deficient, indeed it is not mathematics.
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Developing technique

About a third to a half of any math course deals with technique---the process of making theorems work for you in specific situations rather than in the general setting in which they are usually stated. Sometimes this is fairly easy: many proofs give explicit constructions which you follow for the special case. In these situations the only problems are with algebraic and trigonometric manipulations and keeping track of where you are in the process. In other situations (technique of integration is a good example) there are lots of approaches which might apply to a given problem and several tricks which might be used to make the problem more tractable. For these you need to develop judgment.

Step 1. Read through the theorems and examples.

Some students make the whole process of learning how to do problems more difficult by acting like it had no connection with the other material in the course. Often problems follow a pattern which is given explicitly in the proof of the major theorem they follow. Knowing the general pattern in advance is easier than trying to find it by trial and error.

Step 2. Work enough problems to master the technique.

At this stage you should work enough problems so that the single technique which the problems illustrate is firmly in your mind. Since you have ultimate responsibility for your education, you should take the initiative to work enough problems for your own practice needs. This may well be more problems than are assigned to be turned in.

Step 3. Work a few problems in as many different ways as possible.

Too often the practice obtained in step 2 leads the student to think that there is only one approach to each problem. Sometimes one approach is easy and another is complicated, but often several different attacks will work equally well. Complicated approaches give the student practice in solving problems which take more than one step and more than one technique.

Step 4. Make yourself a set of randomly chosen problems.

One difficulty with learning many techniques to solve a particular kind of problem is that you have to figure out which technique to use before you can get to work on a solution. This is exacerbated by the tendency for problems to be grouped so that the appropriate technique to use is the one which immediately preceded the problem set. Putting two or three problems from each of the problem sets in a chapter on technique on 3 by 5 cards and then shuffling the cards will give you a set of problems on which to practice deciding which technique to use.
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A few final suggestions

Mathematical prose has a very low redundancy rate and mathematics is a very cumulative subject. Pay close attention as you read---once introduced, a concept is rarely repeated and it will be assumed later. Allow yourself adequate time to read the book before starting the problems.

Few students write fast enough to get complete and readable notes in class. For this reason it is useful to go back over your class notes shortly after each class and make a complete, clean copy with all of the definitions and theorems clearly stated. This practice will also help you identify parts you don't understand so you can ask your professor about them in a timely fashion.

Do not let yourself fall behind. Mathematics requires precision, habits of clear thought, and practice. Cramming for an exam will not only fail to produce the desired result on the exam, it will also reinforce a bad habit---that of trying to do mathematics by memorization rather than understanding. A

good night's sleep and a clear head will serve you better than last minute memorization.
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