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Upper Bounds on the Number of Channels to Ensure Collision-free Communications in Multi-Channel Wireless Networks Using Directional Antennas
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Abstract—Recent studies have found that using multiple channels can separate concurrent transmissions and significantly improve network throughput. However, these studies have only considered wireless nodes that are equipped with omni-directional antennas, which have high interference. On the other hand, other researchers have found that using directional antennas in wireless networks can reduce interference and improve the network performance. But their studies have only considered single channel to be used in wireless networks with directional antennas. Thus, integrating the two technologies of multiple channels and directional antennas together can potentially bring more benefits. Some previous works have studied the capacity on the multi-channel wireless networks using directional antennas. However, the channel assignment problem of such networks has not been well studied.

In this paper, we study the channel assignment problem in the multi-channel wireless networks using directional antennas. In particular, we consider the specific problem: how many channels are needed to ensure collision-free communication in a multi-channel wireless network? Given a set of wireless nodes randomly placed in a plane, it is difficult to derive the exact number of channels. In this paper, we try to derive upper bounds on the number of channels to ensure the collision-free communication.

The primary research contributions of our paper can be summarized as follows.

• We formulate the channel assignment problem as a classical graph coloring problem. By using the results of the vertex coloring for a graph, we derive the upper bounds on the number of channels under different network settings (i.e., the interference ratio).
• We also investigate the tightness of our derived upper bounds by constructing several scenarios.
• We compare our derived upper bounds with the results derived from the wireless networks using omni-directional antennas. It is shown that the upper bounds on the number of channels heavily depend on the node density and are also related to the interference ratio.

The remainder of the paper is organized as follows. In Section II, we describe the models and give the definitions of some fundamental concepts, which will be used in our analysis. Section III presents the derived upper bounds on the spatial reuse compared with omni-directional antennas. But, they only focused on a single-channel network which only allows finite concurrent transmissions. Thus, combining the two technologies of multiple channels and directional antennas together potentially brings more benefits.

Several studies including [17] and [18] have derived the capacity bounds of a Multi-Channel wireless network using Multiple Directional Antennas (MC-MDA). However, there are few studies on the channel assignment of such MC-MDA networks. In this paper, we study the channel assignment problem in such MC-MDA networks.

In this paper, we study the channel assignment problem in the multi-channel wireless networks using directional antennas. In particular, we study the problem: given a set of wireless nodes equipped with directional antennas, how many channels are needed to ensure collision-free transmission? We derive the upper bounds on the number of channels, which depend on the node density. We also construct several scenarios to examine the tightness of the derived bounds. Our result can be used to estimate the number of channels required for a practical wireless network. Besides, our results can also be used to provide a suggestion on the proper node density in the node deployment when the number of channels is given for a wireless network.

I. INTRODUCTION

The capacity of wireless networks is affected by two key factors: the interference among concurrent transmissions and the number of simultaneous transmissions on a single interface. Recent studies [1]–[8] found that using multiple channels can separate concurrent transmissions and greatly improve network throughput. However, those studies only consider that wireless nodes are equipped with omni-directional antennas, which cause high collisions. On the other hand, some researchers [9]–[16] found that directional antennas bring more benefits such as reduced interference and increased spatial reuse compared with omni-directional antennas. But, they only focused on a single-channel network which only allows finite concurrent transmissions. Thus, combining the two technologies of multiple channels and directional antennas together potentially brings more benefits.

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Two nodes $X_i$ and $X_j$ can establish a link denoted by $l_{ij}$ if and only if the following conditions are satisfied.

1. $X_j$ is within the transmission range of $X_i$ and $X_i$ is within the transmission range of $X_j$.
2. $X_j$ is covered by the antenna beam of $X_i$. Similarly, $X_i$ is also covered by the antenna beam of $X_j$.
3. No other node within the interference range$^1$ is simultaneously transmitting over the same channel and in the same direction toward $X_j$.

We call two nodes in conflict with each other if they are located within the interference range of each other and their antenna beams are pointed toward each other. For example, in Fig. 2, node $X_k$ within the interference range of node $X_j$ may conflict with $X_j$. Link $l_{ij}$ conflicts with link $l_{kl}$ if either node of one link conflicts with either node of the other link.

We assume that each node is equipped with an identical antenna with the same beamwidth $\theta$. Each node also has the same transmission range, denoted by $R_i$ and the same interference range, denoted by $R_t$. Typically, $R_t$ is no less than $R_i$, i.e., $R_t \geq R_i$. The number of interfering nodes around a node heavily depends on the interference ratio, which is defined as the ratio of the interference range to the transmission range, i.e., $r = R_t/R_i$. It is obvious that the interference ratio $r \geq 1$.

B. Definitions and Problem Formulation

In this paper, we assume that there are $n$ nodes in a plane and each node has only one antenna (interface), i.e., it can only transmit or receive with at most one neighboring node at one time. Basic definitions are stated as follows [20].

**Definition 1:** Link Set. A link set is defined as a set of links, among which no two links in this set share common nodes. Such a link set is denoted as LS. A link set is used to describe a set of links that needs to act simultaneously.

**Definition 2:** Valid Assignment. A valid assignment to a link set is an assignment of channels, such that no two conflicting links are assigned an identical channel. A link set is called a Schedulable Link Set if and only if there exists a valid assignment for the link set.

**Definition 3:** Node Density. There are $n$ nodes randomly located in the plane. Let $S$ denote the (infinite) set of sectors on the plane with radius $R_i$ and angle $\theta$. The number of nodes within sector $s$ is denoted as $N(s)$. The density of nodes is defined as $D = \max_{s \in S} N(s)$.

In order to compare our derived results to those with omnidirectional antennas, we re-state the definition in [20] here.

**Definition 4:** Node Density with Omni-directional Antennas. There are $n$ nodes uniformly located in the plane. Let $C$ denote the (infinite) set of circles on the plane with radius $R_i$. The number of nodes within circle $c$ is denoted as $N(c)$. The density of nodes is defined as $D_c = \max_{c \in C} N(c)$.

Then we give the definition of the upper bound on the number of channels to ensure collision-free transmission with directional antennas.

\(^1\) the interference range is used to denote the maximum distance within which a node can be interfered by an interfering signal.
Definition 5: Upper Bound on the number of channels. There exist possibly many valid link sets, which represent different combination of communication pairs among the nodes. The problem is to find a number, denoted as $U$, such that any link set $LS$ derived from $n$ nodes is schedulable using $U$ channels. In other word, $U$ is the upper bound of channels needed to ensure a collision-free link assignment.

C. Conflict Graph

The link assignment problem can be converted to a conflict graph problem which is first addressed in [21]. A conflict graph is used to model the effects of interference.

Definition 6: Conflict Graph. We define a graph in which every link from a link set $LS$ can be represented by a vertex. Two vertices in the graph are connected by an edge if and only if the two links conflict. Such a graph is called a conflict graph. The conflict graph $G$ constructed from link set $LS$ is denoted as $G(LS)$.

By constructing the conflict graph for a link set, and representing each channel by a different color, we have found that the requirement that no two conflicting links share the same channel is equivalent to the constraint that no two adjacent vertices share the same color in graph coloring. Therefore, the problem of channel assignment on a link set can be converted to the classical graph coloring problem on the conflict graph.

We enumerate several useful results as follows, which will be used to derive our results.

D. Graph Coloring

A graph coloring problem, as one of the most fundamental problems in graph theory, is known to be NP-hard even in the very restricted classes of planar graphs. A coloring is regarded as valid if no two adjacent vertices use the same color. We denote the minimum number for a valid coloring of vertices in a graph $G$ by a chromatic number, $\chi(G)$.

There are two well-known results on the upper bound of $\chi(G)$ [22], [23].

Proposition 1: If $\Delta(G)$ denotes the largest degree among $G$’s vertices, i.e., $\Delta(G) = \max_{v \in G} \text{Degree}(v)$, then we have

$$\chi(G) \leq \Delta(G) + 1$$

Proposition 2: If $G$ contains a subgraph $H$ in which each node has a degree at least $d > 0$, we define such degree as $LD(H) = \min_{v \in H} \text{Degree}(v)$. We denote the maximum one among these degrees by $\delta(G) = \max_{H \subseteq G} LD(H)$. We have

$$\chi(G) \leq \delta(G) + 1$$

In the following section, we will derive the upper bounds on the number of channels to ensure the collision-free transmission in a link set.

III. UPPER BOUNDS ON THE NUMBER OF CHANNELS

We derive several upper bounds under different network settings in terms of the interference ratio $r$. We have also found that the upper bound is monotonically decreasing when the interference ratio increases.

Theorem 1: If there are $n$ nodes in a planar area with the density $D$ and each node is equipped with an antenna with the identical beamwidth $\theta$, for any valid link set $LS$ derived from the $n$ nodes, the corresponding conflict graph $G(LS)$ can be colored by using $2D - 1$ colors.

Proof. Given link $l_{ij}$ which consists of nodes $X_i$ and $X_j$, as shown in Fig. 3. The interference region is denoted as two sectors with radius $R_l$ and angle $\theta$ (the gray area in Fig. 3). From the definition of node density, each sector has at most $D$ nodes. Other than nodes $X_i$ and $X_j$, there are at most $D - 1$ nodes in either sector. After we combine the nodes in the two sectors, the gray area contains at most $2D - 2$ nodes excluding nodes $X_i$ and $X_j$.

Suppose link $l_{kl}$ is one of the links that conflicts with $l_{ij}$. It is obvious that at least one node of that link, e.g., $X_k$, should be in $X_j$’s interference region, the gray sector centered at $X_j$ in Fig. 3. At the same time, the antenna of $X_k$ should be pointed to $X_j$ if it can interfere with $X_j$. Thus, $X_k$’s interference region must also cover $X_j$. So, $|X_k - X_j| \leq R_l$. Since the antenna beam of the other node $X_l$ should be turned toward $X_k$, it must also fall in the interference region of $X_j$, as shown in Fig. 3. Hence, $|X_l - X_j| \leq R_l$.

It seems that any link that conflicts with link $l_{ij}$ must totally fall in the gray area representing the interference regions of nodes $X_i$ and $X_j$. However, consider the case that $X'_k$ and $X'_l$ forms a link $l'_{kl}$ in Fig. 3. $X'_l$ is outside the gray region of $l_{ij}$, but $X'_k$ can interfere with $X_i$ since its beam covers $X_i$. So, a link conflicting with link $l_{ij}$ must contain at least one node falling in the gray area.

Therefore, there are at most $2D - 2$ links that conflict with $l_{ij}$. Hence, the maximum degree of the vertices of $G$ is $\Delta(G) \leq 2D - 2$. From Proposition 1, the conflict graph can be colored by using $2D - 1$ colors.

Theorem 1 can be applied to any settings of the interference ratio $r$. When $r$ is greater than 1, we can get tighter upper bounds. Particularly, we have the result when $r = 2$.

Theorem 2: When $r = 2$ and $n$ nodes are distributed in a planar area with density $D$, and each node is equipped with
an antenna with the identical beamwidth $\theta$, for any valid link set $LS$ derived from those $n$ nodes, the corresponding conflict graph $G(LS)$ can be colored by using $\frac{1}{2}D$ colors.

**Proof.** Without loss of generality, assume $R_i = 1$ so $R_i = 2$. Since the number of nodes $n$ is a finite number, the number of links derived from $n$ is also a finite number. Given a finite number of links on the plane, we can always find a line, such that at least one node is on the line, and all the other nodes are on the right hand side of the plane (as shown in Fig. 4). We denote the node on the line as $X_i$, and the other node on the corresponding link $l_{ij}$ is $X_j$. Then we will calculate the number of links that may conflict with link $l_{ij}$.

Let us consider link $l_{ij}$ consisting of two nodes $X_i$ and $X_j$ (as shown in Fig. 5). For any link $l_{ij}$ that interferes with node $X_j$, at least one node of that link must fall in the interference range of $X_i$. Thus, any interfering link must have an acute angle with link $l_{ij}$. So, Therefore, we draw a line parallel to the line segment $X_iX_j$ and a line parallel to the upper border of the interference region of $X_j$ to bound those interfering nodes. Similarly, we draw two other lines parallel to $X_iX_j$ and the lower border of the interference region of $X_i$. Those lines and the arc of the interference region of $X_j$ form the region $ABCDEFG$ with the bold border, as shown in Fig. 5 (Note that the length of $CD$ is equal to the length of $AG$, which is equal to the length of $X_kX_j$). Thus, those interfering nodes should totally fall in this region.

Then we illustrate that this region can be covered by three identical sectors with radius $R_i = 2$ and angle $\theta$. We place these three sectors as follows. First, we put a sector erectly. By calculating the coordinates of point $A$ and point $B$, we can prove that $R_i$ is greater than segment $AB$. Then we place the second sector next to the first one as shown in Fig. 5. Similarly, we can prove that points $C$ and $D$ fall in the second sector by calculating the coordinates of $C$ and $D$. Then we put the third sector next to the second one. Point $E$ falls in the third sector. Points $F$ and $G$ also fall in the first sector. So, the region $ABCDEFG$ can be covered by the three sectors.

Since the region $ABCDEFG$ can be covered by three identical sectors with radius $R_i$ and angle $\theta$, by definition of the node density, the number of nodes in region $ABCDEFG$ is at most $3D$. Those $3D$ nodes can form at most $\frac{3}{2}D$ links in this area. Other than link $l_{ij}$, there are at most $\frac{3}{2}D - 1$ links can interfere with link $l_{ij}$. Therefore, every vertex in subgraph $H$ (the gray area in Fig. 5) of $G$ has a vertex with degree at most $\frac{3}{2}D - 1$. From Proposition 2, the conflict graph can be colored by using $\frac{1}{2}D$ colors.

Note that the result of Theorem 2 also holds for any $r > 2$. More specifically, we have the following result.

**Theorem 3:** If an upper bound $U$ is valid for the interference ratio $r = r_1$, $r_1 \geq 2$ and $r_2 > r_1$, then the upper bound $U$ is also valid for $r = r_2$.

A similar approach to the omni-directional case [20] can be used to prove Theorem 3. Thus, the proof of Theorem 3 is omitted here.

From Theorem 2, the upper bound is monotonically non-increasing as interference ratio $r$ increases. Intuitively, the larger the interference ratio $r$ is, the further reduced the upper bound $U$ can be. When $r = 4$, we have the following result.

**Theorem 4:** When $r = 4$ and $n$ nodes are distributed in a planar area with density $D$, and each node is equipped with an antenna with beamwidth $\theta$, for any valid link set $LS$ derived from those $n$ nodes, the corresponding conflict graph $G(LS)$ can be colored by using $D$ colors.

The similar proof techniques can be used to prove Theorem 4, as shown in Fig. 6. Due to the space limitation, we omit the proof details of Theorem 4 here.

**IV. LOWER CONSTRAINTS ON THE UPPER BOUND**

The upper bound on the number of channels to ensure a collision-free transmission can be further reduced. In this section, we construct several scenarios to illustrate how far the upper bounds can be reduced.

**Theorem 5:** When $r = 1$, the upper bound cannot be reduced to be lower than $D - 1$.

**Proof.** When $r = 1$, $R_i = R_i$. We construct a scenario, as shown in Fig. 7. The density $D$ is 14 in Fig. 7. We first draw a sector of radius $R_i/2$ and angle $\theta$. Then we place $D - 1$ (13) nodes equally on the arc of the sector with radius $R_i/2$. For each node on the circle, we establish a link with length $R_i = R_i$ toward the center of the sector, as shown in Fig. 7.

It is obvious that the node set we have just constructed is of density $D$ since there are $D$ nodes within the sector of radius $R_i$ and angle $\theta$. For the link set from the constructed

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**Fig. 5.** The proof of Theorem 2 **Fig. 6.** The proof of Theorem 4

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**Fig. 7.** The proof of theorem 5 **Fig. 8.** The proof of theorem 6
node set, the corresponding conflict graph is a \((D - 1)\)-clique (i.e., each link interferes with each other), which needs exactly \(D - 1\) colors to color. So the upper bound cannot be lower than \(D - 1\).

When the interference ratio \(r\) is increased, the upper bound can also be reduced. More specifically, when \(r \geq 2\), we have the following result.

**Theorem 6:** When \(r \geq 2\), the upper bound cannot be reduced to lower than \(\frac{1}{D} \cdot \frac{D}{2} + 1\), where 
\[
\beta = 2 \arctan\left(\frac{\tan \frac{\theta}{2} \cdot \sqrt{(R_i^2 - d^2) \tan^2 \frac{\theta}{2} + R_i^2 - d^2}}{d \tan^2 \frac{\theta}{2} + \sqrt{R_i^2 - d^2} \tan^2 \frac{\theta}{2} + R_i^2}ight).
\]

Since \(R_i = r \cdot R_t\) and \(d = R_i - R_t = (r - 1)R_t\), we substitute the corresponding parts in Eq. (3). Then we have
\[
\beta = 2 \arctan\left(\frac{\tan \frac{\theta}{2} \cdot \sqrt{(2r - 1) \tan^2 \frac{\theta}{2} + r^2 - (r - 1)}\right)
\]

From Eq. (4), the coverage angle \(\beta\) is less than the beamwidth \(\theta\). It only depends on \(\theta\) and the interference ratio \(r\). This angle monotonously increases with \(\theta\) when \(0 < \theta \leq \frac{\pi}{2}\). Furthermore, it monotonously decreases with the increased interference ratio \(r\). There are nearly \(\lfloor \frac{\beta}{\frac{\theta}{2}} \rfloor + 1\) links falling in the interference region of node \(X_j\). Thus, in order to separate those links, we need at least \(\lfloor \frac{\beta}{\frac{\theta}{2}} \rfloor + 1\) colors.

It is shown in Theorem 6 that the number of required channels can be reduced when \(r\) is increased.

**Theorem 7:** The upper bound cannot be reduced to lower than \(\frac{1}{D} \cdot \frac{D}{2}\), for any \(r\) and any \(\theta\).

**Proof:** Suppose that there are \(D\) nodes that are closely located. The distance between any two of them is \(\epsilon\), where \(\epsilon\) is a quite small number and \(\epsilon > 0\). Any link is constructed from any two of the \(D\) nodes. When the distance is quite narrow, the collisions among links are quite high and any link can almost conflict with other links. So, there are \(\frac{1}{D} D\) that conflict with each other. Therefore, the number of channels cannot be reduced to \(D/2\).

V. DISCUSSIONS AND IMPLICATIONS

We summarize our results in Table I. We also compare our results with omni-directional cases [20]. Note that we assume \(R_i = 1\) is fixed and \(R_t\) is adjustable. The coverage angle \(\beta\) is given in Eq. 3, which decreases with the increased interference ratio \(r\). When \(r = 2\), the angle is denoted by \(\beta_1\). When \(r = 4\), the angle is denoted by \(\beta_2\). Thus, we have \(\beta_2 < \beta_1\).

When interference ratio \(r = 1\), the upper bound for the network with directional antennas is \(2D - 1\) and the upper bound for the network with omni-directional antennas is \(2D_o - 3\). Different from the node density \(D\) with directional antennas, \(D_o\) is defined as the maximum number of nodes within a circle of interference range. Generally, we have \(D \neq D_o\).

<table>
<thead>
<tr>
<th>(r)</th>
<th>Omni-directional antennas [20]</th>
<th>Directional antennas</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(2D_o - 3) (D_o - 1) (2D - 1) (D - 1)</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>(\frac{1}{2} D_o) (\frac{1}{2} (D_o - 1)) (\frac{1}{2} D) (\frac{1}{2} D_o + 1)</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>(\frac{1}{2} D_o) (\frac{1}{2} D_o) (D) (\frac{1}{2} D_o + 1)</td>
<td></td>
</tr>
<tr>
<td>(\infty)</td>
<td>(\frac{1}{2} D_o) (\frac{1}{2} D_o) (\frac{1}{2} D) (1)</td>
<td></td>
</tr>
</tbody>
</table>

When \(r = 2\), the upper bound on the number of channels is \(\frac{1}{2} D\) for directional antennas and \(\frac{1}{2} D_o\) for omni-directional antennas. Similarly, when \(r = 4\), the upper bound is reduced to \(D\) for directional antennas and \(D_o\) for omni-directional antennas. From those results, we have found that the number of channels needed for a collision-free transmission scales linearly with the node density \(D\), and is non-increasing as the interference ratio \(r\) increases.

When the interference ratio \(r\) approximates the infinity, then \(R_t\) approximates 0. This means all links have length 0. So, any link will conflict with at most \(\frac{1}{2} D - 1\) links at the directional case and \(\frac{1}{2} D_o - 1\) links at the omni-directional case.

From Table I, we also observe that upper bounds derived from the omni-directional antennas have almost the same coefficients as those derived from the directional antennas except for the case when \(r = 1\) (although they have different node densities, i.e., \(D\) and \(D_o\)). Both the upper bounds derived from the omni-directional case and those derived from the directional case heavily depend on the node density and the interference ratio. An interesting question is whether the upper bounds are independent of the actual radiation patterns of antennas.

Our derived theoretical results can be applied to solve many practical problems. For example, given a wireless network with a number of wireless nodes, our derived bounds can be used to estimate the number of channels required to ensure a collision-free communication. For another example, when the number of channels is given (e.g., there are 14 channels but only three orthogonal channels available in IEEE 802.11), our results can be used to offer suggestions on the node density in the node deployment or suggestions on the channel assignment for a given network.
VI. CONCLUSION

Previous studies are focused on using multiple channels to improve the network performance. However, those studies only consider that wireless nodes are mounted with only omni-directional antennas, which have high interference. There are other studies considering using directional antennas in wireless networks, which can cause low interference. But, they only consider using a single channel at the networks. Thus, integrating the two technologies together can potentially improve the network performance further.

This paper might be the first study on deriving upper bounds on the number of channels to ensure the collision-free communication in multi-channel wireless networks using directional antennas. In particular, we derive the upper bounds on the number of channels to ensure a collision-free communication. The upper bounds heavily depend on the node density and are also related to the interference ratio. Our results also offer some useful suggestions on deploying wireless nodes (e.g., the node density).

REFERENCES


APPENDIX

Calculation of the Coverage Angle $\beta$

To calculate the coverage angle $\beta$, we need to obtain coordinates $x_1$ and $y_1$ of the intersection point $A$ first, as shown in Fig. 9. Since the circle is denoted by the equation

$$x^2 + y^2 = R^2$$

The line $l_1$ is denoted by the equation

$$y = \tan \frac{\theta}{2} \cdot (x - d)$$

After joining Eq. (5) and Eq. (6), we have the coordinates $x$ and $y$ of the point $A$.

$$x_1 = \frac{d \cdot \tan 2 \theta + \sqrt{(R_1^2 - d^2) \tan^2 \frac{\theta}{2} + R_1^2}}{1 + \tan^2 \frac{\theta}{2}}$$

$$y_1 = \frac{\theta}{2} \cdot \frac{\sqrt{(R_1^2 - d^2) \tan^2 \frac{\theta}{2} + R_1^2 - d}}{1 + \tan^2 \frac{\theta}{2}}$$

On the other hand, we have $\tan \frac{\theta}{2} = \frac{y_1}{x_1} = \frac{\tan \frac{\theta}{2} \cdot (\sqrt{(R_1^2 - d^2) \tan^2 \frac{\theta}{2} + R_1^2 - d})}{d \tan^2 \frac{\theta}{2} + \sqrt{(R_1^2 - d^2) \tan^2 \frac{\theta}{2} + R_1^2}}$. Thus, we have

$$\beta = 2 \arctan \left( \frac{\sqrt{(R_1^2 - d^2) \tan^2 \frac{\theta}{2} + R_1^2 - d}}{d \tan^2 \frac{\theta}{2} + \sqrt{(R_1^2 - d^2) \tan^2 \frac{\theta}{2} + R_1^2}} \right)$$