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Exp-function method to solve the nonlinear dispersive K(m,n) equations

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Abstract

Some new exact solutions are obtained for the nonlinear dispersive K(m, n) equations using the exp-function method. The results show that the method is straightforward and concise and its applications are promising.

Keywords: Exp-function method; Nonlinear dispersive K(m,n) equations; Higher order nonlinearity..

1 Introduction

In this paper we consider the following nonlinear dispersive K(m,n) equation:

$$u_t + a\left(u^m\right)_x + \left(u^n\right)_{xxx} = 0 \tag{1}$$

first proposed by Rosenau and Hyman [1]. For certain values of m and n, the K(m,n) equation has solitary waves which are compactly supported. It was shown in Refs.[1,2] that Eq.(1) presents compactly supported solutions with nonsmooth fronts. A large number of methods were suggested recently to study the nonlinear equations, such as the variational iteration method [3-8], the homotopy perturbation method [9-13], the variational method [14] and the parameter-expansion method[15], a complete review on recently developed analytical methods is available in Refs[16,17]. In this paper, we will apply the Exp-function method[18-20] to K(s+1,1) equation in the form [21].

$$u_t + au^s u_x + u_{xxx} = 0 \tag{2}$$

2 Exp-function method

The Exp-function method[18-20] is widely used to search for generalized solitary solutions and periodic solutions[22-26]. To illustrate the general solution procedure, we consider the following general partial differential equation:

$$F\left(u, u_t, u_x, u_{tt}, u_{xx}, \cdots\right) = 0\,. \tag{3}$$

We aim at its wave solution, so we introduce a complex variable, η, defined as

$$\eta = kx + \omega t\,. \tag{4}$$

We, therefore, can convert the partial differential equation, Eq.(3), into an ordinary differential equation :

$$G\left(u, \omega u', ku', \omega^2 u'', k^2 u'', \cdots\right) = 0\,. \tag{5}$$

The Exp-function method is to search for its solution in the form

$$u(\eta) = \frac{\sum_{j=-c}^{d} a_j \exp(j\eta)}{\sum_{i=-q}^{q} b_i \exp(i\eta)} \tag{6}$$

where c, d, p and q are positive integers unknown to be further determined , a_j and b_i are unknown constants.

3 Solution Procedure

Using the transformation (4), Eq. (2) becomes

$$\omega u' + kau^s u' + k^3 u''' = 0 \tag{7}$$

Integrating (7), and setting the constant of integration to be zero, we obtain

$$\omega u+\frac{aks}{s+1}u^{s+1}+k^3u''=0 \tag{8}$$

where the prime denotes the differential with respect to η.

Making the transformation

$$u=v^{1/s}, \tag{9}$$

Eq.(8) becomes

$$s^2(s+1)\omega v^2+aks^2v^3$$
$$+k^3\left(1-s^2\right)\left(v'\right)^2+k^3s(s+1)vv''=0 \tag{10}$$

We assume that the solution of Eq.(10) can be expressed in the form

$$v(\eta)=\frac{a_1\exp(\eta)+a_0+a_{-1}\exp(-\eta)}{\exp(\eta)+b_0+b_{-1}\exp(-\eta)} \tag{11}$$

Substituting Eq. (11) into Eq. (10), we have

$$\frac{1}{A}\{C_0+C_1\exp(\eta)+\cdots+C_8\exp(8\eta)\}=0 \tag{12}$$

Equating the coefficients of $\exp(n\eta)$ in Eq.(12) to be zero yields a set of algebraic equations:

$$C_0=0,C_1=0,C_2=0,C_3=0,$$
$$C_4=0,C_5=0,C_6=0,C_7=0,C_8=0$$

Solving the above algebraic system with the help of symbolic computation system, we obtain the following results

Case 1:

$$a_1=\frac{A_1+B_1}{6sab_{-1}},\ a_0\neq 0,\ a_{-1}=\frac{A_1+B_1}{6sa},$$
$$b_0=\frac{A_1-B_1}{saa_0},\ b_{-1}\neq 0,$$
$$k\neq 0,\ \omega=\frac{-k\left(A_1+B_1\right)}{6s\left(s+1\right)b_{-1}}; \tag{13}$$

Case 2:

$$a_1=\frac{A_1-B_1}{6sab_{-1}},\ a_0\neq 0,\ a_{-1}=\frac{A_1-B_1}{6sa},$$
$$b_0=\frac{A_1+B_1}{saa_0},\ \ b_{-1}\neq 0,$$
$$k\neq 0,\ \ \omega=\frac{-k\left(A_1-B_1\right)}{6s\left(s+1\right)b_{-1}}; \tag{14}$$

where

$A_1=4(s+1)k^2b_{-1}$ and $B_1=\sqrt{A_1^2-6s^4a^2a_0^2b_{-1}}$.

Cases 3 and 4:

$$a_1=\frac{(s+1)k^2A_2}{15saB_2},\ a_0=\pm\frac{\left(s+1\right)k^2}{s}\sqrt{\frac{B_2}{15a^2}},$$
$$a_{-1}=\frac{(s+1)b_{-1}k^2A_2}{15saB_2},$$
$$b_0=\pm 2\sqrt{\frac{3}{5}}\frac{b_{-1}\left(19B_2-192b_{-1}\right)}{\left(B_2\right)^{\frac{3}{2}}},\ b_{-1}=b_{-1},$$
$$\omega=k-\frac{4kb_{-1}^2C_1}{15sB_2^3}; \tag{15}$$

where

$$A_2=215b_{-1}+\sqrt{145b_{-1}^2},$$
$$B_2=23b_{-1}+\sqrt{145b_{-1}^2},$$
$$C_1=\sqrt{145b_{-1}^2}\left(2641k^2+6495s\right)$$
$$+5b_{-1}\left(7579k^2+16629s\right).$$

Cases 5 and 6

$$a_1=\frac{(s+1)k^2A_3}{15saB_3},\ a_0=\pm\frac{\left(s+1\right)k^2}{s}\sqrt{\frac{B_3}{15a^2}},$$
$$a_{-1}=\frac{(s+1)k^2b_{-1}A_3}{15saB_3},$$
$$b_0=\pm 2\sqrt{\frac{3}{5}}\frac{b_{-1}\left(19B_3-192b_{-1}\right)}{\left(B_3\right)^{\frac{3}{2}}},$$
$$\omega=k-\frac{4kb_{-1}^2C_2}{15sB_3^3} \tag{16}$$

where

$$A_3 = 215b_{-1} - \sqrt{145b_{-1}^2}\,,$$
$$B_3 = 23b_{-1} - \sqrt{145b_{-1}^2}\,,$$
$$C_2 = \sqrt{145b_{-1}^2}\left(2641k^2 + 6495s\right) - 5b_{-1}\left(7579k^2 + 16629s\right).$$

Cases 7 and 8

$$a_1 = \frac{A_4C_3}{30},\ a_0 = \pm\sqrt{\frac{B_4b_{-1}}{15}}C_3,\ a_{-1} = \frac{A_4C_3b_{-1}}{30},$$
$$b_0 = \pm\frac{\left(5\sqrt{5}a - \sqrt{29a^2}\right)\sqrt{B_4b_{-1}}}{16\sqrt{3}},\ b_{-1} = b_{-1},$$
$$\omega = -\frac{aA_4k^3}{30s}. \tag{17}$$

Cases 9 and 10:

$$a_1 = \frac{A_5C_3}{30},\ a_0 = \pm\sqrt{\frac{B_5b_{-1}}{15}}C_3,\ a_{-1} = \frac{A_5C_3b_{-1}}{30},$$
$$b_0 = \pm\frac{\left(5\sqrt{5}a + \sqrt{29a^2}\right)\sqrt{B_5b_{-1}}}{16\sqrt{3}},\ b_{-1} = b_{-1},$$
$$\omega = -\frac{aA_5k^3}{30s}. \tag{18}$$

where

$$A_4 = \frac{25a + \sqrt{145a^2}}{a^2},\ B_4 = \frac{23a - \sqrt{145a^2}}{a^3},$$
$$A_5 = \frac{25a - \sqrt{145a^2}}{a^2},\ B_5 = \frac{23a + \sqrt{145a^2}}{a^3},$$
$$C_3 = \frac{k^2(s+1)}{s}.$$

For each case, we obtain the corresponding solutions as follows:

$$u_1(x,t) = \left(\frac{a_0(A_1 + B_1)e^{kx+\omega_1 t} + 6saa_0^2b_{-1} + a_0b_{-1}(A_1 + B_1)e^{-(kx+\omega_1 t)}}{6saa_0b_{-1}e^{kx+\omega_1 t} + 6b_{-1}(A_1 - B_1) + 6saa_0b_{-1}^2e^{-(kx+\omega_1 t)}}\right)^{\frac{1}{s}}, \tag{19}$$

$$u_2(x,t) = \left(\frac{a_0(A_1 - B_1)e^{kx+\omega_2 t} + 6saa_0^2b_{-1} + a_0b_{-1}(A_1 - B_1)e^{-(kx+\omega_2 t)}}{6saa_0b_{-1}e^{kx+\omega_2 t} + 6b_{-1}(A_1 + B_1) + 6saa_0b_{-1}^2e^{-(kx+\omega_2 t)}}\right)^{\frac{1}{s}}, \tag{20}$$

where $A_1 = 4(s+1)k^2b_{-1}$, $B_1 = \sqrt{A_1^2 - 6s^4a^2a_0^2b_{-1}}$, $\omega_1 = \dfrac{-k(A_1 + B_1)}{6s(s+1)b_{-1}}$, and $\omega_2 = \dfrac{-k(A_1 - B_1)}{6s(s+1)b_{-1}}$.

$$u_{3,4}(x,t) = \left(\frac{\sqrt{5}(s+1)}{15sa^2}\frac{ak^2A_2(B_2)^{\frac{1}{2}}e^{kx+\omega_1 t} \pm k^2B_2^2\sqrt{15a^2} + ab_{-1}k^2A_2(B_2)^{\frac{1}{2}}e^{-(kx+\omega_1 t)}}{\sqrt{5}(B_2)^{\frac{3}{2}}e^{kx+\omega_1 t} \pm 2\sqrt{3}b_{-1}(19B_2 - 192b_{-1}) + \sqrt{5}b_{-1}(B_2)^{\frac{3}{2}}e^{-(kx+\omega_1 t)}}\right)^{\frac{1}{s}}, \tag{21}$$

$$u_{5,6}(x,t) = \left(\frac{\sqrt{5}(s+1)}{15sa^2}\frac{ak^2A_3(B_3)^{\frac{1}{2}}e^{kx+\omega_2 t} \pm k^2aB_3^2\sqrt{15a^2} + ab_{-1}k^2A_3(B_3)^{\frac{1}{2}}e^{-(kx+\omega_2 t)}}{\sqrt{5}(B_3)^{\frac{3}{2}}e^{kx+\omega_2 t} \pm 2\sqrt{3}b_{-1}(19B_3 - 192b_{-1}) + \sqrt{5}b_{-1}(B_3)^{\frac{3}{2}}e^{-(kx+\omega_2 t)}}\right)^{\frac{1}{s}}, \tag{22}$$

where $A_2 = 215b_{-1} + \sqrt{145b_{-1}^2}$, $B_2 = 23b_{-1} + \sqrt{145b_{-1}^2}$, $A_3 = 215b_{-1} - \sqrt{145b_{-1}^2}$,
$B_3 = 23b_{-1} - \sqrt{145b_{-1}^2}$, $\omega_1 = k - \dfrac{4b_{-1}^2kC_1}{15sB_2^3}$, $\omega_2 = k - \dfrac{4b_{-1}^2kC_2}{15sB_3^3}$,

$C_1=\sqrt{145b_{-1}^2}\left(2641k^2+6495s\right)+5b_{-1}\left(7579k^2+16629s\right)$,

$C_2=\sqrt{145b_{-1}^2}\left(2641k^2+6495s\right)-5b_{-1}\left(7579k^2+16629s\right)$.

$$u_{7,8}\left(x,t\right)=\left(\frac{8\sqrt{3}A_4C_3e^{kx+\omega_1 t}\pm 48\sqrt{5B_4b_{-1}}C_3+8\sqrt{3}A_4C_3b_{-1}e^{-(kx+\omega_1 t)}}{240\sqrt{3}e^{kx+\omega_1 t}\pm 15\left(5\sqrt{5}a-\sqrt{29a^2}\right)\sqrt{B_4b_{-1}}+240\sqrt{3}b_{-1}e^{-(kx+\omega_1 t)}}\right)^{\frac{1}{s}}, \tag{23}$$

$$u_{9,10}\left(x,t\right)=\left(\frac{8\sqrt{3}A_5C_3e^{kx+\omega_2 t}\pm 48\sqrt{5B_5b_{-1}}C_3+8\sqrt{3}A_5C_3b_{-1}e^{-(kx+\omega_2 t)}}{240e^{kx+\omega_2 t}\pm 15\left(5\sqrt{5}a+\sqrt{29a^2}\right)\sqrt{B_5b_{-1}}+240b_{-1}e^{-(kx+\omega_2 t)}}\right)^{\frac{1}{s}}, \tag{24}$$

where $A_4=\dfrac{25a+\sqrt{145a^2}}{a^2}$, $B_4=\dfrac{23a-\sqrt{145a^2}}{a^3}$, $A_5=\dfrac{25a-\sqrt{145a^2}}{a^2}$, $B_5=\dfrac{23a+\sqrt{145a^2}}{a^3}$, $\omega_1=-\dfrac{aA_4k^3}{30s}$, $\omega_2=-\dfrac{aA_5k^3}{30s}$ and $C_3=\dfrac{k^2\left(s+1\right)}{s}$.

When k is an imaginary number, the obtained solitary solutions can be converted into periodic solutions [18]. Setting $k=iK$ and $\omega=i\Omega$, we obtain the following transformations:

$$e^{kx+\omega t}=\cos[Kx+\Omega t]+i\sin[Kx+\Omega t], \tag{25}$$

and

$$e^{-(kx+\omega t)}=\cos[Kx+\Omega t]-i\sin[Kx+\Omega t],. \tag{26}$$

Substituting (25) and (26) into Eqs. (19)-(24), respectively, leads to the following periodic solutions:

$$u_{11}\left(x,t\right)=\left(\frac{a_0\left(\delta_1+\delta_2\right)\cos[Kx+\Omega_1 t]+3saa_0^2}{6saa_0\cos[Kx+\Omega_1 t]+3\left(\delta_1+\delta_2\right)}\right)^{\frac{1}{s}}, \tag{27}$$

$$u_{12}\left(x,t\right)=\left(\frac{a_0\left(\delta_1-\delta_2\right)\cos[Kx+\Omega_2 t]+3saa_0^2}{6saa_0\cos[Kx+\Omega_2 t]+3\left(\delta_1-\delta_2\right)}\right)^{\frac{1}{s}}, \tag{28}$$

where $\delta_1=4k^2(s+1)$, $\delta_2=\sqrt{\delta_1^2-6a^2a_0^2s^4}$, $\Omega_1=\dfrac{-K\left(\delta_1+\delta_2\right)}{6s\left(s+1\right)}$ and $\Omega_2=\dfrac{-K\left(\delta_1-\delta_2\right)}{6s\left(s+1\right)}$.

$$u_{13,14}\left(x,t\right)=\left(\frac{2\sqrt{5}(s+1)ak^2\delta_3\left(\delta_4\right)^{\frac{1}{2}}\cos[Kx+\Omega_3 t]\pm 5\sqrt{3}\left(s+1\right)\sqrt{a^2}k^2\delta_4^2}{30\sqrt{5}sa^2\left(\delta_4\right)^{\frac{3}{2}}\cos[Kx+\Omega_3 t]\pm 30\sqrt{3}sa^2\left(19\delta_4-192\right)}\right)^{\frac{1}{s}}, \tag{29}$$

$$u_{15,16}(x,t)=\left(\frac{2\sqrt{5}(s+1)ak^2\delta_5(\delta_6)^{\frac{1}{2}}\cos[Kx+\Omega_4 t]\pm 5\sqrt{3}(s+1)\sqrt{a^2}k^2\delta_6^2}{30\sqrt{5}sa^2(\delta_6)^{\frac{3}{2}}\cos[Kx+\Omega_4 t]\pm 30\sqrt{3}sa^2(19\delta_6-192)}\right)^{\frac{1}{s}}, \tag{30}$$

where $\delta_3=215+\sqrt{145}$, $\delta_4=23+\sqrt{145}$, $\delta_5=215-\sqrt{145}$, $\delta_6=23-\sqrt{145}$, $\Omega_3=K-\frac{4K\alpha_1}{15s\delta_4^3}$, $\Omega_4=K-\frac{4K\alpha_2}{15s\delta_6^3}$, $\alpha_1=\sqrt{145}\left(2641k^2+6495s\right)+5\left(7579k^2+16629s\right)$ and $\alpha_2=\sqrt{145}\left(2641k^2+6495s\right)-5\left(7579k^2+16629s\right)$.

$$u_{17,18}(x,t)=\left(\frac{16\sqrt{3}\delta_7\delta_{11}\cos[Kx+\Omega_5 t]\pm 48\sqrt{5\delta_8}\delta_{11}}{480\sqrt{3}\cos[Kx+\Omega_5 t]\pm 15\left(5\sqrt{5}a-\sqrt{29a^2}\right)\sqrt{\delta_8}}\right)^{\frac{1}{s}}, \tag{31}$$

$$u_{19,20}(x,t)=\left(\frac{16\sqrt{3}\delta_9\delta_{11}\cos[Kx+\Omega_6 t]\pm 48\sqrt{5\delta_{10}}\delta_{11}}{480\sqrt{3}\cos[Kx+\Omega_6 t]\pm 15\left(5\sqrt{5}a+\sqrt{29a^2}\right)\sqrt{\delta_{10}}}\right)^{\frac{1}{s}}, \tag{32}$$

where

$\delta_7=\frac{25a+\sqrt{145a^2}}{a^2}$, $\delta_8=\frac{23a-\sqrt{145a^2}}{a^3}$, $\delta_9=\frac{25a-\sqrt{145a^2}}{a^2}$, $\delta_{10}=\frac{23a+\sqrt{145a^2}}{a^3}$, $\delta_{11}=\frac{k^2(s+1)}{s}$, $\Omega_5=\frac{a\delta_7K^3}{30s}$ and $\Omega_6=\frac{a\delta_9K^3}{30s}$.

4. Conclusion

The exp-function method itself is of utter simplicity. Using the method we can obtain a series of exact solutions with some free parameters which can be determined according to the boundary/initial conditions.

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