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Notes on Displaced CES Functional Forms

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NOTES ON DISPLACED CES FUNCTIONAL FORMS

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The constant elasticity of substitution (CES) utility function can be "displaced" from the origin by the use of an extra b_i parameter for each commodity:

$$\text{maximize} \quad U = \left[\sum_{i=1}^{N} a_i^{\frac{1}{\sigma}} (X_i - b_i)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} \quad \text{subject to} \quad I = \sum_{i=1}^{N} P_i X_i \, .$$

This functional form is written slightly differently, with a_i to the power $1/\sigma$, so that a_i will be marginal expenditure shares, with $\sum a_i = 1$. Notice that it reduces to three special cases:

<u>CES</u>: where all $b_i = 0$.

<u>Stone-Geary</u>: (displaced Cobb-Douglas) where $\sigma = 1$ (in limit).

$$U = \prod_{i=1}^{N} (X_i - b_i)^{a_i}$$

<u>Cobb-Douglas</u>: where all $b_i = 0$ <u>and</u> $\sigma = 1$.

For the most general form, we differentiate the Lagrangean:

$$L = \left[\sum_{i=1}^{N} a_i^{\frac{1}{\sigma}} (X_i - b_i)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} + \lambda \left[I - \sum P_i X_i \right]$$

$$\frac{\partial L}{\partial X_j} = \frac{\sigma}{\sigma-1} \cdot \left[\quad\right]^{\frac{\sigma}{\sigma-1}-1} \cdot a_j^{\frac{1}{\sigma}} \cdot \frac{\sigma-1}{\sigma}(X_j-b_j)^{\frac{\sigma-1}{\sigma}-1} - \lambda P_j = 0,\ j = 1\ldots N$$

$$\left[\quad\right]^{\frac{\sigma}{\sigma-1}-1} a_j^{\frac{1}{\sigma}} (X_j - b_j)^{\frac{\sigma-1}{\sigma}-1} = \lambda P_j$$

$$* \qquad U a_j^{\frac{1}{\sigma}} (X_j - b_j)^{\frac{\sigma-1}{\sigma}} = \lambda\left[\quad\right] P_j (X_j - b_j)$$

Sum over N goods to get

$$U\left[\sum_{j=1}^{N} a_j^{\frac{1}{\sigma}} (X_j - b_j)^{\frac{\sigma-1}{\sigma}}\right] = \lambda\left[\quad\right]\sum_{j=1}^{N} P_j (X_j - b_j)$$

$$U = \lambda(I - \sum P_j b_j)$$

Let $I_D \equiv I - \sum P_j b_j$ be discretionary income. Then $U = \lambda I_D$, and:

$$\left[\quad\right]^{\frac{\sigma}{\sigma-1}} = \lambda I_D$$

$$\left[\quad\right] = (\lambda I_D)^{\frac{\sigma-1}{\sigma}}$$

From * above,

$$\left(\frac{U}{\lambda}\right) a_j^{\frac{1}{\sigma}} (X_j - b_j)^{\frac{\sigma-1}{\sigma}} = \left[\quad\right] P_j (X_j - b_j)$$

$$I_D a_j^{\frac{1}{\sigma}} (X_j - b_j)^{\frac{\sigma-1}{\sigma}} = \lambda^{\frac{\sigma-1}{\sigma}} I_D^{\frac{\sigma-1}{\sigma}} P_j (X_j - b_j)$$

$$I_D^{\frac{\sigma}{\sigma} - \frac{\sigma-1}{\sigma}} a_j^{\frac{1}{\sigma}} = \lambda^{\frac{\sigma-1}{\sigma}} P_j (X_j - b_j)^{\frac{\sigma}{\sigma} - \frac{\sigma-1}{\sigma}}$$

$$I_D^{\frac{1}{\sigma}} a_j^{\frac{1}{\sigma}} = \lambda^{\frac{\sigma-1}{\sigma}} P_j (X_j - b_j)^{\frac{1}{\sigma}}$$

Raise to σ power:

$$** \qquad I_D a_j = \lambda^{\sigma-1} P_j^{\sigma} (X_j - b_j)$$

$$I_D a_j P_j^{1-\sigma} = \lambda^{\sigma-1} P_j (X_j - b_j)$$

Sum over N goods to get:

$$I_D \sum_{j=1}^{N} a_j P_j^{1-\sigma} = \lambda^{\sigma-1} I_D$$

$$*** \qquad \left[\sum_{j=1}^{N} a_j P_j^{1-\sigma}\right]^{\frac{1}{\sigma-1}} = \lambda$$

From ** above:

$$I_D a_j = \left[\sum^{N} a_i P_i^{1-\sigma}\right] P_j^{\sigma}(X_j - b_j)$$

Solution for X_j yields the demand function. With the other special cases, we have:

Demand Functions

<u>Displaced</u> <u>CES</u>: $$X_j = b_j + \frac{a_j\left(I - \sum_{i=1}^{N} P_i b_i\right)}{P_j^{\sigma}\left[\sum_{i=1}^{N} a_i P_i^{1-\sigma}\right]}$$

<u>CES</u>: $$X_j = \frac{a_j I}{P_j^{\sigma}\left[\sum^{N} a_i P_i^{1-\sigma}\right]}$$

<u>Linear</u> <u>Expenditure</u> <u>System</u> <u>(LES)</u> <u>from</u> <u>Stone-Geary</u>: $$X_j = b_j + \frac{a_j I_D}{P_j}$$

<u>Cobb-Douglas</u>: $$X_j = \frac{a_j I}{P_j}$$

PARAMETERS

Since we started $a_j^{\frac{1}{\sigma}}$ in utility, a_j is the marginal expenditure share out of discretionary income, I_D. The b_j can be considered a "minimum required purchase," but it might be either positive or negative. It is possible to estimate these parameters using econometrics, but suppose we just have one set of balance sheets that describe the economy (the consumer's budget) in one year. Then we resort to backwards solution. That is, suppose we have:

1. Observations on expenditures $(X_j P_j)$ and income I in one year,
2. Extraneous estimate of σ (or assume Stone-Geary or Cobb-Douglas),
3. Units convention, so that every net price $P_j = 1$ in the benchmark,
4. Tax rates, so that gross-of-tax prices $P_j(1+t_j) = (1+t_j)$,
5. Extraneous estimates of b_j .

Then solve for a_j (for each consumer) from ** (or demand) above:

$$I_D a_j = \left[\sum^{N} a_j P_j^{1-\sigma} \right] P_j^{\sigma}(X_j - b_j)$$

and sum

$$I_D \sum a_j = \left[\sum^{N} a_j P_j^{1-\sigma} \right] \sum_{j=1}^{N} P_j^{\sigma}(X_j - b_j) \ .$$

Taking the ratio of one to the other, where $\sum a_j = 1$, we have:

<u>Backward</u> <u>Solution</u>

<u>Displaced</u> <u>CES</u>: $$a_j = \frac{P_j^{\sigma}(X_j - b_j)}{\sum_{j=1}^{N} P_j^{\sigma}(X_j - b_j)}$$

<u>CES</u>: $$a_j = \frac{P_j^{\sigma} X_j}{\sum_{j=1}^{N} P_j^{\sigma} X_j}$$

<u>Stone-Geary</u>: $$a_j = \frac{P_j(X_j - b_j)}{\sum P_j(X_j - b_j)}$$

<u>Cobb-Douglas</u>: $$a_j = \frac{P_j X_j}{\sum P_j X_j}$$

The system is now completely determined. Note that if we had econometric estimates or extraneous sources for both a_j and b_j, then the system would be overdetermined. It would generate expenditure amounts that are not equal to expenditures observed in any particular year. We could use the generated expenditures as our benchmark equilibrium, but then other data in the system would have to be made consistent with it. In terms of "degrees of freedom," we have one for each good. For example, if we used extraneous a_j and minimum required expenditure $\sum^{N} P_j b_j$, then we could instead use the units convention ($P_j = 1$) to let the data on $P_j X_j$ and I determine appropriate intercepts:

$$b_j = X_j - \frac{a_j(I - \sum P_j b_j)}{P_j^{\sigma}\left[\sum a_i P_i^{1-\sigma}\right]} \quad .$$

To derive the income elasticity, differentiate the demand:

$$X_j = b_j + \frac{a_j\left(I - \sum_{j=1}^{N} P_j b_j\right)}{P_j^{\sigma}\left[\sum^{N} a_j P_j^{1-\sigma}\right]}$$

with respect to income I, and define:

$$\eta_{Ij} = \frac{\partial X_j}{\partial I} \cdot \frac{I}{X_j} = \frac{a_j}{P_j^{\sigma}[\sum a_j P_j^{1-\sigma}]} \cdot \frac{I}{X_j}$$

$$= \frac{a_j I_D}{P_j^{\sigma}[\sum a_j P_j^{1-\sigma}]} \cdot \frac{I}{I_D} \cdot \frac{1}{X_j} = \frac{(X_j - b_j)}{X_j} \cdot \frac{I}{I_D}$$

If we have an extraneous estimate of the income elasticity, we can "force" it upon this demand system. Solve for b_j as:

$$\eta_{Ij} \cdot \frac{I_D}{I} \cdot X_j = X_j - b_j$$

$$b_j = X_j - X_j \eta_{Ij} \frac{I_D}{I} = X_j \left(1 - \frac{\eta_{Ij} I_D}{I} \right) .$$

That is, we can use available η_{Ij}, with choice of I_D, to set b_j.

For the uncompensated price elasticity, differentiate demand with respect to price, and calculate:

$$\eta_{P_j} = \frac{\partial X_j}{\partial P_j} \cdot \frac{P_j}{X_j} = ?$$

The strategy is similar. If we have an extraneous estimate of the price elasticity to force on the system, then rearrange this most recent derivation to express σ as a function of η_{P_j} and other parameters.

<u>Two methods</u>

1. Differentiate the demand function, where P_j appears in three places. Things may drop out nicely, but some models have complications.

2. Simulate, or, use numerical differentiation. That is, set up the model with demand functions, etc. Set the elasticities σ and solve for an equilibrium (benchmark). Then change prices slightly, calculate new demands

and $\frac{\Delta X}{\Delta P} \cdot \frac{P}{X}$. Then we change σ and repeat until we obtain the desired elasticity.

The point is just to use available information and the one degree of freedom appropriately. Parameters (a_j, b_j, σ) imply a (price elasticity, income elasticity, expenditure outcome). The attempt to match an observed outcome or a measured elasticity will have implications for what parameters must be used.

INDIRECT UTILITY AND EXPENDITURE FUNCTIONS

Now that all parameters (a_j, b_j, σ) are available, we can substitute demand into utility to get the indirect utility function:

$$U = \left[\sum_{j=1}^{N} a_j^{\frac{1}{\sigma}} \left(\left[b_j + \frac{a_j(I_D)}{P_j^{\sigma}[\sum a_i P_i^{1-\sigma}]}\right] - b_j\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}$$

$$U^{\frac{\sigma-1}{\sigma}} = \sum_{j=1}^{N} a_j^{\frac{1}{\sigma}} a_j^{\frac{\sigma-1}{\sigma}} P_j^{-\sigma(\frac{\sigma-1}{\sigma})} I_D^{\frac{\sigma-1}{\sigma}} \left[\sum a_i P_i^{1-\sigma}\right]^{\frac{1-\sigma}{\sigma}}$$

$$= I_D^{\frac{\sigma-1}{\sigma}} \left[\sum a_i P_i^{1-\sigma}\right]^{\frac{1-\sigma}{\sigma}} \cdot \left[\sum_{j=1}^{N} a_j P_j^{1-\sigma}\right]$$

$$= I_D^{\frac{\sigma-1}{\sigma}} \left[\sum a_i P_i^{1-\sigma}\right]^{\frac{1}{\sigma}}$$

$$U = I_D \cdot \left[\sum a_i P_i^{1-\sigma}\right]^{\frac{1}{\sigma-1}} = I_D \Big/ \left[\sum a_i P_i^{1-\sigma}\right]^{\frac{1}{1-\sigma}}$$

$$U = I_D \,/\, \overline{P}$$

It is extremely useful to define $\overline{P}$ as the "ideal" price index, a composite of the N commodity prices. Note from *** above that:

$$\overline{P} \equiv \frac{1}{\lambda} = \left[\sum^{N} a_j P_j^{1-\sigma}\right]^{\frac{1}{1-\sigma}}$$

Since the b_j do not appear, this composite price is identical for the CES form. The CES price index is not defined where $\sigma = 1$. Take limit as $\sigma \rightarrow 1$. Or, the same maximization procedure for the Stone-Geary or Cobb-Douglas utility function would yield:

$$\overline{P} = \frac{1}{\lambda} = \prod_{j=1}^{N} \left(\frac{P_j}{a_j}\right)^{a_j} .$$

Thus we have:

$$\begin{array}{l} \text{Indirect} \\ \text{Utility} \\ \text{Functions} \end{array} \left\{ \begin{array}{ll} \text{Displaced CES:} & U = I_D/\bar{P} \\[2ex] \text{CES:} & U = I/\bar{P} \\[2ex] \text{Stone-Geary:} & U = I_D/\bar{P} \\[2ex] \text{Cobb-Douglas:} & U = I/\bar{P} \end{array} \right.$$

To derive the expenditure function:

$$I_D = U \cdot \bar{P}$$

$$I - \sum^{N} P_j b_j = U \cdot \bar{P}$$

$$E(P,U) = U \cdot \bar{P} + \sum^{N} P_j b_j \ ,$$

a function of utility, prices, and parameters a_j, b_j.

$$\underline{\text{Expenditure Functions}}\left\{\begin{array}{lll}\underline{\text{Displaced CES}}: & E = U\cdot\bar{P} + \sum_{j=1}^{N} P_j b_j & \\ & & \bar{P} = \left[\sum a_j P_j^{1-\sigma}\right]^{\frac{1}{1-\sigma}} \\ \underline{\text{CES}}: & E = U\cdot\bar{P} & \\ \underline{\text{Stone-Geary}}: & E = U\cdot\bar{P} + \sum^{N} P_j b_j & \\ & & \bar{P} = \prod_{j=1}^{N}\left(\frac{P_j}{a_j}\right)^{a_j} \\ \underline{\text{Cobb-Douglas}}: & E = U\cdot\bar{P} & \end{array}\right.$$

(where, for Displaced CES and CES: $\bar{P} = \left[\sum a_j P_j^{1-\sigma}\right]^{\frac{1}{1-\sigma}}$; where, for Stone-Geary and Cobb-Douglas: $\bar{P} = \prod_{j=1}^{N}\left(\frac{P_j}{a_j}\right)^{a_j}$)

WELFARE EVALUATIONS

First set up the benchmark equilibrium. We know prices = 1, (or 1+t), and we have expenditure data, so these give us quantities X_j. We use some extraneous parameters and these data to <u>impose</u> the other parameters by backward solution. Second, calculate utility in the status quo. Third, we can simulate a policy change and calculate new prices and new utility levels for each consumer. Finally, we can calculate:

<u>Compensating Variation and Compensating Gain</u>:

$$CV = E(P^1,V^0) - E(P^0,V^0)$$

$$CG = E(P^1,V^1) - E(P^1,V^0)$$

$$= [U^1\cdot\bar{P}^1 + \sum P_j^1 b_j] - [U^0\cdot P^1 + \sum P_j^1 b_j]$$

$$= [U^1 - U^0]\cdot\bar{P}^1$$

Equivalent Variation and Equivalent Gain:

$$EV = E(P^1,V^1) - E(P^0,V^1)$$

$$EG = E(P^0,V^1) - E(P^0,V^0)$$

$$= [U^1 \cdot \bar{P}^0 + \sum P_j^0 b_j] - [U^0 \cdot \bar{P}^0 + \sum P_j^0 b_j]$$

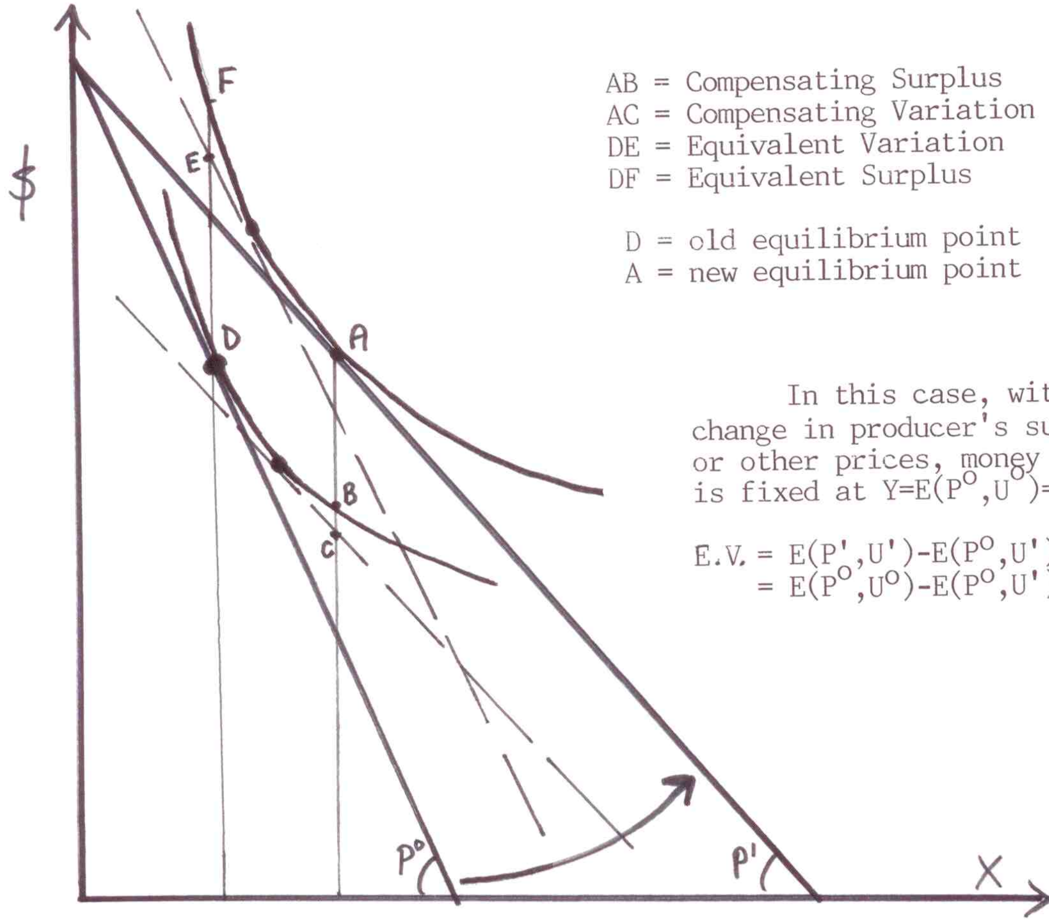
$$= [U^1 - U^0] \cdot \bar{P}^0$$

The equivalent variation is the amount that could be taken away from the consumer at P^0 to make her as well off as she would be at P^1. The compensating variation is the amount we'd have to pay the consumer at prices P^1 to make her just as happy as she was at P^0.

The compensating and equivalent variations, as usually defined, include only the change in consumer surplus, the area under a Hicksian demand curve (compensated to V^0 or to V^1) while varying the price from P^0 to P^1. See diagram. If there is no change in producer surplus nor any other change to money incomes, then $I = E(P^0,V^0) = E(P^1,V^1)$, so CG = -CV and EG = -EV. See M.A. King (1983), "Welfare Analysis of Tax Reforms using Household Data," Journal of Public Economics 21, July, page 192. In a general equilibrium model, however, there may be simultaneous changes to consumer surplus and income (such as a change of "virtual" income in a progressive income tax reform). Only the compensating or equivalent gain (CG or EG) include all effects on actual utility (from V^0 to V^1), measured either at new prices (CG) or at old prices (EG).

The CV or CG shows, in new prices, the dollar value of the utility change. Because each set of new prices is different, these measures are not comparable for alternative policy changes. See J.A. Kay (JPubE, 1980). The

Hicks' Four Consumer Surpluses

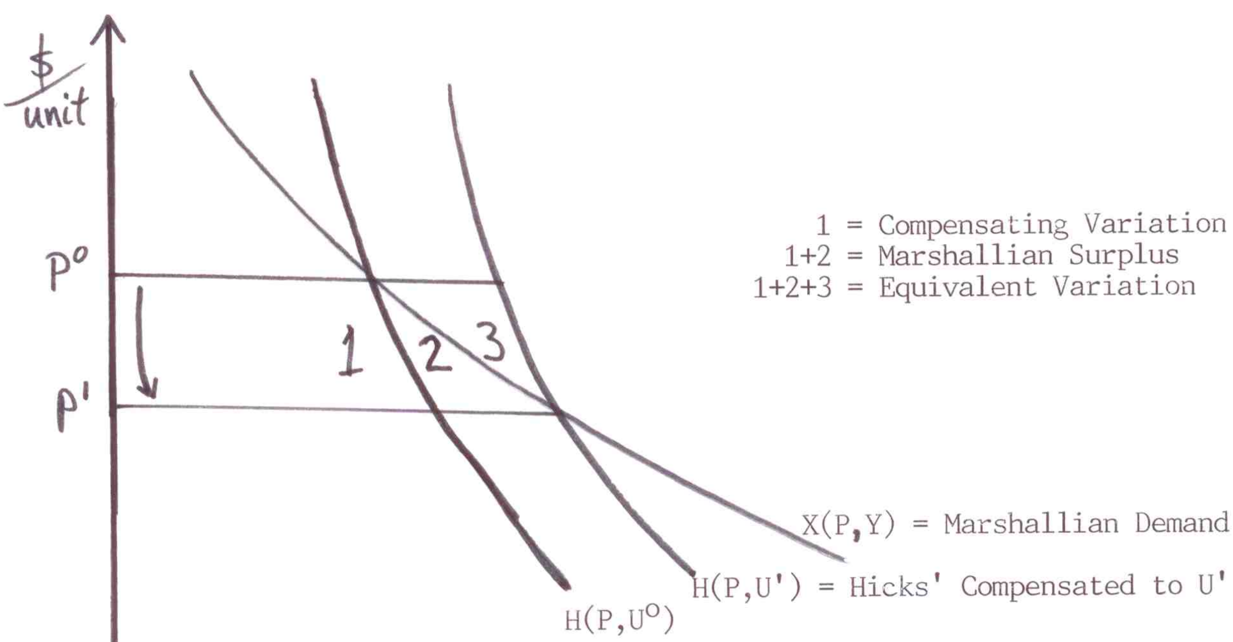


AB = Compensating Surplus
AC = Compensating Variation
DE = Equivalent Variation
DF = Equivalent Surplus

D = old equilibrium point
A = new equilibrium point

In this case, with no change in producer's surplus or other prices, money income is fixed at $Y=E(P^O,U^O)=E(P',U')$

$$E.V. = E(P',U')-E(P^O,U')$$
$$= E(P^O,U^O)-E(P^O,U')$$



1 = Compensating Variation
1+2 = Marshallian Surplus
1+2+3 = Equivalent Variation

equivalent gain expresses the entire change in utility in dollars of the original equilibrium, so alternative policy changes are comparable.

NESTING

Forget the minimum purchases b_j for the moment, so we have CES forms:

Utility: $$U = \left[\sum_{j=1}^{N} a_j^{\frac{1}{\sigma}} X_j^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}$$

Demand: $$X_j = \frac{a_j I}{P_j^{\sigma}\left[\sum_{j=1}^{N} a_j P_j^{1-\sigma}\right]}$$

Price Index: $$\overline{P} = \left[\sum_{j=1}^{N} a_j P_j^{1-\sigma}\right]^{\frac{1}{1-\sigma}}$$

Indirect Utility: $$U = I/\overline{P}$$

Expenditure Function: $$I = U \cdot \overline{P}$$

Backward Solution: $$a_j = P_j^{\sigma} X_j \Big/ \left[\sum_{j=1}^{N} P_j^{\sigma} X_j\right]$$

First, note that the consumer "buys" a composite good, utils, which each cost a composite price $\overline{P}$. The expenditure function is just a quantity of U times its price. With this in mind, we can also interpret one of the X_j as a "composite" good, made up of a lot of other goods Z_i for $i = 1...M$. For example, suppose the first good X_1 is transportation, made up of a combination of auto, air, sea, and rail transport. This is useful if we want more detail in one (or more) sector(s), with a different (higher) elasticity of substitution among components. Then P_1 is price of the composite good, and X_1P_1 is expenditure on the composite good. The <u>next</u> problem is to allocate the X_1P_1 expenditures among the Z_i. Use a subutility function, and:

$$\text{maximize:} \quad X_1 = \left[\sum_{i=1}^{M} \alpha_i^{\frac{1}{\epsilon}} Z_i^{\frac{\epsilon-1}{\epsilon}}\right]^{\frac{\epsilon}{\epsilon-1}} \quad \text{subject to} \quad X_1P_1 = \sum_{i=1}^{M} P_i^z Z_i$$

For the inner nest, functional forms are exactly analogous to those above:

Subutility: $$X_1 = \left[\sum_{i=1}^{M} \alpha_i^{\frac{1}{\epsilon}} Z_i^{\frac{\epsilon-1}{\epsilon}}\right]^{\frac{\epsilon}{\epsilon-1}}$$

Demand: $$Z_i = \frac{\alpha_i (X_1P_1)}{(P_i^z)^{\epsilon}\left[\sum_{i=1}^{M} \alpha_i (P_i^z)^{1-\epsilon}\right]}$$

Price Index: $$P_1 = \left[\sum_{i=1}^{M} \alpha_i (P_i^z)^{1-\epsilon}\right]^{\frac{1}{1-\epsilon}}$$

Indirect Subutility: $$X_1 = (\text{Expend})/P_1$$

Expenditure Function: $$\text{Expend} = X_1P_1$$

Backward Solution: $$\alpha_i = (P_i^z)^{\epsilon} Z_i \Big/ \left[\sum_{i=1}^{M} (P_i^z)^{\epsilon} Z_i\right]$$

Strategy:

1. Use data on expenditures of inner nested goods ($Z_i P_i^Z$), set prices P_i^Z equal to one, net of tax, or $(1 + t_i^Z)$ gross of tax. Then derive α_j from knowledge of P_i^Z, Z_i and ϵ.

2. Use these α_i to calculate subutility X_1 and price P_1.

3. Then go to outer nest and solve backward for all a_i using all X_i, P_i, and σ.

Nesting of Displaced CES forms:

Use the strategy above to determine quantities X_j (in the CES form) for a first stage, but interpret one (or any number) of X_j as a composite good made up of lots of other discretionary goods $(Z_i - \beta_i)$ for $i = 1\ldots M$, where β_i are minimum required purchases of Z_i, and P_j is a composite price made up of a lot of P_i^Z, $i = 1\ldots M$. Note that the required part of Z_i (the β_i) do not enter subutility function X_j, nor utility U. The (discretionary) expenditure on the j^{th} composite good is:

$$X_j P_j = \sum_{i=1}^{M} P_i^Z (Z_i - \beta_i)$$

It may be that only one of the X_j is a composite, if that is the sector (e.g. transportation) of particular interest (where Z_i are railroads, airlines, trucking, autos, etc.). Or, other X_j also may be composites of their own Z_i, $i = 1\ldots$any number.

Simplify notation by considering only one inner nest, for X_j. In the first stage, max $U = \left[\sum_{j=1}^{N} a_j^{\frac{1}{\sigma}} X_j^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}$ subject to $\sum^{N} X_j P_j = I_D \equiv I - \sum_{i=1}^{M} P_i^z \beta_i$. Then I_D is income available for buying X_j (where X_j is net of the β_i parts of Z_i). Restate all above functions for displaced CES forms, with I_D in place of I.

Demand: $$X_j = \frac{a_j (I_D)}{P_j^{\sigma}\left[\sum_{j=1}^{N} a_j P_j^{1-\sigma}\right]}$$

Composite price: (price of utility) $$\bar{P} = P_U = \left[\sum_{j=1}^{N} a_j P_j^{1-\sigma}\right]^{\frac{1}{1-\sigma}}$$

Indirect Utility: $$U = I_D/\bar{P}$$

Expenditure Function: $$U \cdot \bar{P} + \sum_{i=1}^{M} P_i^z \beta_i$$

The consumer first uses I_D and prices P_j to determine $(X_j P_j)$, total expenditures on the $(Z_i - \beta_i)$, i = 1...M. In the second stage, the consumer can decide how to break it down among the Z_i, by maximizing a subutility function:

$$\max X_j = \left[\sum_{i=1}^{M} \alpha_i^{\frac{1}{\epsilon}} (Z_i - \beta_i)^{\frac{\epsilon-1}{\epsilon}}\right]^{\frac{\epsilon}{\epsilon-1}} \text{ subject to } \sum_{i=1}^{M} P_i^z (Z_i - \beta_i) = (X_j P_j)$$

Here, because the budget constraint is a little different from the usual one for a displaced CES form, it is useful to define $Z'_i \equiv Z_i - \beta_i$. Then

$$\max \quad L = \left[\sum_{i=1}^{M} \alpha_i^{\frac{1}{\epsilon}} (Z'_i)^{\frac{\epsilon-1}{\epsilon}}\right]^{\frac{\epsilon}{\epsilon-1}} + \lambda\left[(X_j P_j) - \sum_{i=1}^{M} P_i^z Z'_i\right]$$

which looks exactly like the CES form and has demand functions:

$$Z'_i = \frac{\alpha_i (X_j P_j)}{(P_i^z)^{\epsilon}\left[\sum_{i=1}^{M} \alpha_i (P_i^z)^{1-\epsilon}\right]}$$

and therefore demand:

$$Z_i = \beta_i + \frac{\alpha_i (X_j P_j)}{(P_i^z)^{\epsilon}\left[\sum_{i=1}^{M} \alpha_i (P_i^z)^{1-\epsilon}\right]} \quad \text{where} \quad (X_j P_j) = \sum_{i=1}^{M} P_i^z (Z_i - \beta_i)$$

Composite Price: $\quad P_j = \left[\sum_{i=1}^{M} \alpha_i (P_i^z)^{1-\epsilon}\right]^{\frac{1}{1-\epsilon}}$

gets used as P_j in demand for X_j above

Indirect Subutility: $\quad X_j = (X_j P_j)/P_j = (\text{Income for subutility } X_j)/P_j$

Expenditure Function: Given subutility X_j, required expenditures are $X_j P_j$.

Income to spend on all the X_j is $I - \sum_{j=1}^{N} \sum_{i=1}^{M} P_{ij}^z \beta_{ij}$ (if multiple nests).

Strategy:

We have data on expenditures ($Z_i P_i^z$) and assume P_i^z prices = 1 or ($1 + t_i^z$) for each good in the nest (e.g. each type of transportation). We have an elasticity of substitution ϵ for each nest (i.e. among transportation types), and suppose the β_i parameters are given.

Backward solution:

First, calculate the α_i for the inner nest(s):

$$\alpha_i = \frac{(P_i^z)^{\epsilon}(Z_i - \beta_i)}{\sum_{i=1}^{M} (P_i^z)^{\epsilon}(Z_i - \beta_i)}$$

Then, with inner parameters available, calculate

$$X_j = \left[\sum_{i=1}^{M} \alpha_i^{\frac{1}{\epsilon}} (Z_i - \beta_i)^{\frac{\epsilon-1}{\epsilon}}\right]^{\frac{\epsilon}{\epsilon-1}}$$

$$P_j = \left[\sum_{i=1}^{M} \alpha_i (P_i^z)^{1-\epsilon}\right]^{\frac{1}{1-\epsilon}}$$

and use those composite goods and prices (with σ given) to get:

$$a_j = \frac{P_j^{\sigma} X_j}{\sum_{j=1}^{N} P_j^{\sigma} X_j} \qquad j = 1 \ldots N$$

Some of those X_j and P_j are data (e.g. for food, housing), those that are not composites.

Simulation (forward solution): From prices off the simplex, calculate P_i^z. Then use parameters α_i and ϵ to calculate P_j. Then $I_D = I - \sum_{i=1}^{M} P_i^z \beta_i$ and we can get demands X_j. Then use expenditure $(X_j P_j)$ to get demands for individual goods Z_i.

EXAMPLE: INTERTEMPORAL STONE-GEARY UTILITY

Suppose that we have a sub-utility function for each period:

$$X_j = \prod_{i=1}^{M} (Z_{ij} - \beta_i)^{\alpha_i} \qquad j = 1\ldots N \text{ (Number of periods)}$$

with the assumption that every period has the same set of minimum required purchases β_i on M goods, and the same set of marginal expenditure shares α_i for $i = 1\ldots M$.

These enter a lifetime utility function:

$$U = \prod_{j=1}^{N} X_j^{a_j} \qquad \text{where} \qquad a_j = \frac{1}{(1+\rho)^{j-1}} \Big/ \sum_{i=1}^{N} \frac{1}{(1+\rho)^{i-1}}$$

This particular choice for a_j implies that desired expenditures would be falling over time, all else equal, because future expenditures are discounted at ρ, the rate of time preference.

The normalization of a_j ensures that $\sum^N a_j = 1$, so that utility is really Cobb-Douglas.

Income Y_j in each period is exogenous, and is discounted at the interest rate r. The budget constraint is:

$$I \equiv \sum_{j=1}^{N} \frac{Y_j}{(1 + r)^{j-1}} = \sum_{j=1}^{N} \frac{\sum_{i=1}^{M} P_i^z Z_{ij}}{(1 + r)^{j-1}}$$

where individuals can borrow or lend at a given r, have no bequest motive, and spend their last cent on the day they die.

Note that X_j are not quantities, but composites of discretionary spending $(Z_{ij} - \beta_i)$. The amount of income available to spend on these X_j is:

$$I_D \equiv \sum_{j=1}^{N} \left[\frac{Y_j - \sum_{i=1}^{M} P_i^z \beta_i}{(1 + r)^{j-1}} \right]$$ which, by the budget constraint

$$= \sum_{j=1}^{N} \left[\frac{\sum_{i=1}^{M} P_i^z (Z_{ij} - \beta_i)}{(1 + r)^{j-1}} \right]$$

Suppose P_j is the "price" of X_j, to be derived below. The <u>first</u> problem of the consumer is to maximize:

$$L = \prod_{j=1}^{N} X_j^{a_j} + \lambda\left[I_D - \sum_{j=1}^{N} \frac{X_j P_j}{(1 + r)^{j-1}}\right]$$

This amended Cobb-Douglas problem generates simple known demand functions, the same as normal, but P_j is replaced by $P_j/(1 + r)^{j-1}$:

Demands: $$X_j = \frac{a_j I_D}{P_j/(1 + r)^{j-1}}$$

Composite Price: $$\overline{P} = \prod_{j=1}^{N} \left(\frac{P_j}{(1 + r)^{j-1}}\right)^{a_j} / \prod_{j=1}^{N} (a_i)^{a_i}$$

Indirect Utility Function: $U = I_D/\overline{P}$

Expenditure Function: $$I_D = U \cdot \overline{P}, \quad \text{or} \quad I = U \cdot \overline{P} + \sum_{j=1}^{N} \sum_{i=1}^{M} \frac{P_i^z \beta_i}{(1 + r)^{j-1}}$$

Having decided how to divide I_D among $(X_j P_j)$ for j = 1...N (expenditures on $(Z_{ij} - \beta_i)$ in all the periods j), the consumer goes to the <u>second stage</u> to allocate these expenditures:

$$\max L_j = \prod_{i=1}^{M} (Z_{ij} - \beta_i)^{\alpha_i} + \lambda\left[\frac{(X_j P_j)}{(1 + r)^{j-1}} - \frac{\sum_{i=1}^{M} P_j^z (Z_{ij} - \beta_i)}{(1 + r)^{j-1}}\right]$$

(for each j = 1...N period)

Letting $Z'_{ij} \equiv (Z_{ij} - \beta_i)$, this becomes

$$L_j = \prod_{i=1}^{M} (Z'_{ij})^{\alpha_i} + \lambda \left[\frac{X_j P_j}{(1 + r)^{j-1}} - \frac{\sum_{i=1}^{M} P_i^z Z'_{ij}}{(1 + r)^{j-1}} \right], \qquad \sum_{i=1}^{M} \alpha_i = 1$$

for which we have demand:

$$Z'_{ij} = \frac{\alpha_i \left[\dfrac{X_j P_j}{(1 + r)^{j-1}} \right]}{P_i^z (1 + r)^{j-1}}$$

so

$$Z_{ij} = \beta_i + \frac{\alpha_i (X_j P_j)}{P_i^z}$$

Composite Price: $P_j = \prod_{i=1}^{M} \left(\dfrac{P_i^z}{\alpha_i} \right)^{\alpha_i}$ is used for P_j in first stage above.

Indirect Utility: $X_j = EXP/P_j$

$\Big\}$ where EXP is (discretionary) spending on X_j

Expenditure Function: $EXP = X_j \cdot P_j$

Strategy:

With P_i^z original prices, calculate P_j composite indices for X_j. Then, with Y_j and P_i^z, calculate $I_D = \sum_{j=1}^{N} \left[\frac{Y_j - \sum_{i=1}^{M} P_i^z \beta_i}{(1 + r)^{j-1}} \right]$, and then demands for X_j. Finally, with expenditures $X_j P_j$, calculate individual demands Z_{ij}.

Each of the above equations can be used in a computer program to calculate these variables (P_j, I_D, X_j and Z_{ij}). We do <u>not</u> <u>need</u> to substitute ($X_j P_j$) into Z_{ij} to solve for the Z_{ij} demand as a single function of income and prices. If we <u>did</u> substitute the demand for X_j (or expenditures $X_j P_j$) into the demand for Z_{ij}, we would get:

$$Z_{ij} = \beta_i + \frac{\alpha_i}{P_i^z} \left[\frac{a_j I_D}{1/(1 + r)^{j-1}} \right]$$

and if we further substitute the definition of a_j, we have

$$Z_{ij} = \beta_i + \frac{\alpha_i}{P_i^z \left(\sum_{j=1}^{N} \frac{1}{(1 + \rho)^{j-1}} \right)} \left[\frac{1 + r}{1 + \rho} \right]^{j-1} \left[\sum_{j=1}^{N} \frac{Y_j - \sum_{i=1}^{M} P_i^z \beta_i}{(1 + r)^{j-1}} \right]$$

Notice that this demand looks like an LES demand (from Stone-Geary utility), where β_i is the minimum required purchase, the last term is I_D, the present value of discretionary income, and α_i is the (amended) share of I_D for Z_i.

The addition of the r and ρ terms means that future demands will increase with the rate of interest r to the degree that it makes those goods effectively cheaper, but decrease with the rate of time preference ρ, to the degree that it makes future goods worth less at the present time.

Finally, note that if $\rho = 0$, this demand reduces exactly to those in Claire Holton-Hammond's thesis. (See Roger Sherman to borrow a copy). She did not use nesting, but effectively plugged X_j into U and solved directly for Z_{ij}. It may not be necessary here, but nesting can make other complicated problems much easier.