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# An Improved Amplitude-frequency Formulation for Nonlinear Oscillators


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### Abstract

A brief introduction to amplitude-frequency formulae for nonlinear oscillators is given, an improved one is suggested.


**Keywords**: Nonlinear Oscillation, Duffing equation, period

We consider a generalized nonlinear oscillator in the form

$$u'' + f(u) = 0, \qquad u(0) = A, \qquad u'(0) = 0 \tag{1}$$

We use two trial functions

$$u_1(t) = A\cos\omega_1 t \tag{2}$$

and

$$u_2 = A\cos\omega_2 t\,, \tag{3}$$

The residuals are

$$R_1(t) = -\cos\omega_1 t + f(A\cos\omega_1 t) \tag{4}$$

and

$$R_2(\omega t) = -\omega_2^2\cos\omega_2 t + f(A\cos\omega_2 t) \tag{5}$$

The original frequency-amplitude formulation reads[1-4]

$$\omega^2 = \frac{\omega_1^2 R_2 - \omega_2^2 R_1}{R_2 - R_1} \tag{6}$$

In my previous publications[1-4], I just used the following formulation

$$\omega^2 = \frac{\omega_1^2 R_2(\omega_2 t = 0) - \omega_2^2 R_1(\omega_1 t = 0)}{R_2 - R_1} \tag{7}$$

Geng and Cai improved the formulation by choosing another location point[5]:

$$\omega^2 = \frac{\omega_1^2 R_2(\omega_2 t = \pi/3) - \omega_2^2 R_1(\omega_1 t = \pi/3)}{R_2 - R_1} \tag{8}$$

Generally we can locate at

$$\cos\omega_1 t = \cos\omega_2 t = k \tag{9}$$

To illustrate this shortcoming, we consider the Duffing equation

$$u'' + u + \varepsilon u^3 = 0 \tag{10}$$

Using trial functions

$$u_1(t) = A\cos t \tag{11}$$

and

$$u_2 = A\cos 2t \tag{12}$$

respectively for Eq.(1), we obtain the following residuals

$$R_1(t) = \varepsilon A^3\cos^3 t, \tag{13}$$

and

$$R_2(t) = -3A\cos 2t + \varepsilon A^3\cos^3 2t\,. \tag{14}$$

Locating at $\cos t_1 = \cos 2t_2 = k$, we obtain

$$\omega^2 = \frac{-3Ak + \varepsilon A^3k^3 - 4\varepsilon A^3k^3}{-3Ak + \varepsilon A^3k^3 - \varepsilon A^3k^3} = 1 + \varepsilon A^3k^2 \tag{15}$$

Its approximate solution reads

$$u(t) = A\cos\left[(1+k^2\varepsilon A^2)^{1/2}t\right] \quad (16)$$

In view of the approximate solution, Eq.(16), we re-write Eq.(10) in the form

$$u'' + (1+k^2\varepsilon A^2)u = k^2\varepsilon A^2 u - \varepsilon u^3 \quad (17)$$

If, by chance, Eq.(16) is the exact solution, then the right hand side of Eq.(17) is vanishing completely. Since our approach is only an approximation to the exact solution, we set

$$\int_0^{T/4} (k^2\varepsilon A^2 u - \varepsilon u^3)\cos\omega t dt = 0\,, \quad (18)$$

where $T = 2\pi/\omega$. Substituting (16) in (18), we obtain

$$k^2 = 3/4 \quad (19)$$

Finally the frequency is obtained

$$\omega = \sqrt{1+\frac{3}{4}\varepsilon A^2}\,. \quad (20)$$

To improve its accuracy, we can use the following trial-functions:

$$u_1(t) = \sum_{i=1}^{m} A_i \cos\omega_i t \quad (21)$$

$$u_2(t) = \sum_{i=1}^{m} A_i \cos\Omega_i t\,, \quad (22)$$

or

$$u_1(t) = \frac{\sum_{i=1}^{m} A_i \cos\omega_i t}{\sum_{j=1}^{n} B_j \cos\omega_j t} \quad (23)$$

$$u_2(t) = \frac{\sum_{i=1}^{m} A_i \cos\Omega_i t}{\sum_{j=1}^{n} B_j \cos\Omega_j t}\,, \quad (24)$$

Most useful trial-functions are

$$u_1(t) = A\cos t \quad (25)$$

$$u_2 = a\cos\omega t + (A-a)\cos 3\omega t\,, \quad (26)$$

and

$$u_1(t) = A\cos t \quad (27)$$

$$u_2 = \frac{A(1+c)\cos\omega t}{1+c\cos 2\omega t}\,, \quad (28)$$

where a and c are unknown constants.
We can always set $\cos t = k$ in $u_1$, and $\cos\omega t = k$ in $u_2$.

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