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Max-Min Approach to Nonlinear Oscillators

Ji-Huan He

Modern Textile Institute, Donghua University, Shanghai 200051, China
Email: jhhe@dhu.edu.cn

Abstract

This paper suggests a novel method called max-min method. Maximal and minimal solution thresholds of a nonlinear problem can be easily found, and an approximate solution of the nonlinear equation can be easily deduced using He Chengtian's interpolation, which has millennia history. Application of the method to nonlinear oscillators is systematically illustrated, illustrating examples show the present technology is very convenient and effective.

Keywords: He Chengtian inequality, Ancient Chinese Mathematics, Nonlinear Oscillation, Duffing equation, period

1. Introduction

In most engineering problems, it is easy to find maximum/minimum thresholds of a solution of a nonlinear equation.

We first consider a simple example in the form:

$$y' + y^2 = 0, \quad y(0) = 1. \tag{1}$$

The exact solution is $y_{ex}(x) = 1/(1+x)$. We give two crude trial functions:

$$y_1(x) = y(0) = 1 \tag{2}$$

and

$$y_2(x) = y(0) + y'(0)x = 1 - x. \tag{3}$$

It is obvious that

$$1 - x = y_2(x) < y(x) < y_1(x) = 1. \tag{4}$$

From this maximum-minimum relationship, Eq.(4), we can use an old inequality, He Chengtian inequality, which has millennia history, to find an approximate solution.

2. He Chengtian Inequality

In an ancient history book, it writes(the Chinese version of this text can be found in Refs.[1,2])

He Chengtian uses 26/49 as the strong, and 9/17 as the weak. Among the strong and the weak, Chengtian finds the fractional day(399/752) of the Moon by using the strong factor 15 *and the weak factor* 1.

The statement is rather cryptic, in modern mathematical term, the statement can be explained as follows.

According to the observation data, He Chengtian finds that

$$29\frac{26}{49} \text{ days} > 1 \quad \text{Moon} > 29\frac{9}{17} \text{ days}.$$

Using the weighting factors (15 and 1), He Chengtian(369?~447AD) obtains

$$\text{The fractional day} = \frac{26 \times 15 + 9 \times 1}{49 \times 15 + 17 \times 1} = \frac{399}{752},$$

so

$$1 \quad \text{Moon} = 29\frac{399}{752} \text{ days}.$$

He Chengtian actually uses the following inequality:

If

$$\frac{a}{b} < x < \frac{d}{c}, \tag{6}$$

where *a*,*b*,*c* and *d* are real numbers, then

$$\frac{a}{b} < \frac{ma+nd}{mb+nc} < \frac{d}{c}, \tag{7}$$

and x is approximated by

$$x = \frac{ma+nd}{mb+nc}, \tag{8}$$

where m and n are weighting factors.

The proof of the inequality can be found in details in Ref.[3], fascinating applications of the technology can be found in Refs.[4,5,6,7].

He Chengtian (369?~447AD) is a famous ancient Chinese mathematics and astronomer, he is an extremely important figure in development of mathematics, yet our Western colleagues know little about his mathematical achievements. Great classics, when revisited in the light of new developments, may reveal hidden pearls, as it the case with He Chengtian's interpolation.

We re-write (4) in the form

$$\frac{1-x}{1} < y(x) < \frac{1}{1} \tag{9}$$

According to He Chengtian's interpolation, we set

$$y(x;m,n) = \frac{m(1-x)+n}{m+n}, \tag{10}$$

or

$$y(x;k) = \frac{1-x+k}{1+k}, \tag{11}$$

where m and n are weighting factors, $k=n/m$.
The value of k can be approximately determined by various approximate methods[8,9,10]. Among others, hereby we use the residual method. Substituting (11) into (1) results in the following residual:

$$R(x;k) = \frac{-1}{1+k} + \left[\frac{1-x+k}{1+k}\right]^2. \tag{12}$$

Locating at $x_0 = 0.5$

$$R(0.5;k) = \frac{-1}{1+k} + \left[\frac{0.5+k}{1+k}\right]^2 = 0 \tag{13}$$

we obtain $k = 0.866$. The approximate result at x=0.5 is $y(0.5;0.866) = 0.732$. The 9.85% accuracy is remarkable good in view of the crudeness of the two trial functions. We can, of course, obtain a much better result by suitable choice of trial functions.

It is interesting to note that m and n can also be functions of x. If we set m=1, and n=x, then (6) happens to be the exact solution!

$$y(x;1,x) = \frac{1-x+x}{1+x} = \frac{1}{1+x}. \tag{14}$$

3. Application to Nonlinear Oscillations

Consider Duffing equation which reads

$$u'' + u + \varepsilon u^3 = 0, \quad u(0) = A, \quad u'(0) = 0, \tag{15}$$

where ε needs not be small in the present study, i.e. $0 \le \varepsilon < \infty$.

We re-write Eq.(15) in the form

$$u'' + (1+\varepsilon u^2)u = 0. \tag{16}$$

We choose a trial-function in the form

$$u = A\cos\omega t, \tag{17}$$

where ω is the frequency to be determined.

Observe that the square of frequency, ω^2, is never less than that in the solution

$$\varphi_1(t) = A\cos t \tag{18}$$

of the following oscillation

$$u'' = -(1+\varepsilon u_{\min}^2)u = -u. \tag{19}$$

In addition, ω^2 never exceeds the square of frequency of the solution

$$\varphi_2(t) = A\cos\sqrt{1+\varepsilon A^2}\,t \tag{20}$$

of the following oscillation

$$u'' = -(1+\varepsilon u_{\max}^2)u = -(1+\varepsilon A^2)u. \tag{21}$$

Hence, it follows that

$$\frac{1}{1} < \omega^2 < \frac{1+\varepsilon A^2}{1}. \tag{22}$$

According to He Chengtian's interpolation, we have

$$\omega^2 = \frac{m+n(1+\varepsilon A^2)}{m+n} = 1 + k\varepsilon A^2\,, \qquad (23)$$

where *m* and *n* are weighting factors, $k = n/(m+n)$.

So the frequency can be approximated as

$$\omega = \sqrt{1+k\varepsilon A^2}\ . \qquad (24)$$

Its approximate solution reads

$$u(t) = A\cos\left[(1+k\varepsilon A^2)^{1/2}t\right] \qquad (25)$$

In view of the approximate solution, Eq.(25), we re-write Eq.(15) in the form

$$u'' + (1+k\varepsilon A^2)u = k\varepsilon A^2 u - \varepsilon u^3 \qquad (26)$$

If, by chance, Eq.(25) is the exact solution, then the right hand side of Eq.(26) is vanishing completely. Since our approach is only an approximation to the exact solution, we set

$$\int_0^{T/4} (k\varepsilon A^2 u - \varepsilon u^3)\cos\omega t dt = 0\,, \qquad (27)$$

where $T = 2\pi/\omega$. Substituting (25) in (27), we obtain

$$k = 3/4 \qquad (28)$$

Finally the frequency is obtained

$$\omega = \sqrt{1+\frac{3}{4}\varepsilon A^2}\ . \qquad (29)$$

To illustrate the remarkable accuracy of the obtained result, we compare the approximate period

$$T = \frac{2\pi}{\sqrt{1+3\varepsilon A^2/4}} \qquad (30)$$

with the exact one [17]

$$T_{ex} = \frac{4}{\sqrt{1+\varepsilon A^2}}\int_0^{\pi/2}\frac{\mathrm{d}x}{\sqrt{1-k\sin^2 x}}\,, \text{ with}$$

$$k = \frac{\varepsilon A^2}{2(1+\varepsilon A^2)}\,. \qquad (31)$$

What is rather surprising about the remarkable range of validity of (18) is that the actual asymptotic period as $\varepsilon \to \infty$ is also of high accuracy.

$$\lim_{\varepsilon A^2 \to \infty}\frac{T_{ex}}{T} = \frac{\sqrt{3}}{\pi}\int_0^{\pi/2}\frac{\mathrm{d}x}{\sqrt{1-0.5\sin^2 x}} = 0.929 \qquad (32)$$

Therefore, for any value of ε, it can be easily proved that the maximal relative error is less than 5.66%.

Example 2

Consider the equation (Exercise 4.4 in Ref.[17])

$$(1+u^2)u'' + u = 0\,,\ u(0) = A, u'(0) = 0\ .(33)$$

We re-write (33) in the form

$$u'' = -\frac{1}{1+u^2}u\ . \qquad (34)$$

If we choose the trial-function in the form $u = A\cos\omega t$, where ω is the frequency, then the maximal and minimal values of $1/(1+u^2)$ are, respectively, 1 and $1/(1+A^2)$. So we immediately obtain

$$\frac{1}{1} < \omega^2 < \frac{1}{1+A^2}\,. \qquad (35)$$

According to He Chengtian's interpolation, we set

$$\omega^2 = \frac{m+n}{m+n(1+A^2)} = \frac{1}{1+kA^2}\,, \qquad (36)$$

where *m* and *n* are weighting factors, *k*=*n*/(*m*+*n*).

So the frequency can be approximated as

$$\omega = \frac{1}{\sqrt{1+kA^2}}\,. \qquad (37)$$

Similarly we re-write Eq.(33) in the form

$$(1+kA^2)u'' + u = kA^2u'' - u^2u'' \qquad (38)$$

Setting

$$\int_0^{T/4}(kA^2u'' - u^2u'')\cos\omega t dt = 0 \qquad (39)$$

we obtain

$$k=3/4 \tag{40}$$

Its approximate frequency reads

$$\omega = \frac{1}{\sqrt{1+\frac{3}{4}A^2}}\,. \tag{41}$$

Its approximate period can be expressed as

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{1+\frac{3}{4}A^2}\,. \tag{42}$$

Its exact period reads[18]

$$T_{ex} = 4\int_0^A \frac{\mathrm{d}u}{\sqrt{\ln(1+A^2)-\ln(1+u^2)}}$$

$$= 4\int_0^A \frac{\mathrm{d}u}{\sqrt{\ln\left[(1+A^2)/(1+u^2)\right]}}\,. \tag{43}$$

In case $A \to \infty$, we have

$$\lim_{A\to\infty} T_{ex} = 4\int_0^A \frac{\mathrm{d}u}{\sqrt{2(\ln A - \ln u)}}\,. \tag{44}$$

By transformation $u=As$, the above equation reduces to

$$\lim_{A\to\infty} T_{ex} = 2\sqrt{2}A\int_0^1 \frac{\mathrm{d}s}{\sqrt{\ln(1/s)}}\,. \tag{45}$$

By transformation $s=\exp(-x^2)$, we have

$$\lim_{A\to\infty} T_{ex} = 4\sqrt{2}A\int_0^\infty \exp(-x^2)\,\mathrm{d}x = 2\sqrt{2\pi}A\,. \tag{46}$$

In case $A \to \infty$, we have

$$\lim_{A\to\infty} \frac{T}{T_{ex}} = \frac{\pi\sqrt{3}A}{2\sqrt{2\pi}A} = \sqrt{\frac{3\pi}{8}} = 1.085\,. \tag{47}$$

The accuracy of 8.5% when $A \to \infty$ is remarkable good.

3. Conclusion

To conclude, we find that the ancient Chinese mathematics represents one of the most important fields of research in science and technology. There exist innumerable hidden pears in the great classics, such as *Jiuzhang Suanshu*(Nine Chapters), these pearls, when contacted with modern technologies, can shine marvelously.

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