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LETTERS AND COMMENTS

Comment on 'He's frequency formulation for nonlinear oscillators'

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Abstract

Geng and Cai suggested a modified He's frequency–amplitude formulation for nonlinear oscillators (Geng and Cai 2007 *Eur. J. Phys.* **28** 923–31). We point out the accuracy of both the original one and the modified version depends upon the chosen location point, and so far we have no general rule for the location choice. The method of weighted residuals is suggested to overcome the shortcoming.

1. Introduction

Geng and Cai [1] suggested a modified He's frequency–amplitude formulation for nonlinear oscillators. The only difference between the original formulation [2–5] and the modified version [1] is the choice of the location point, the former simply chooses $t_1 = \omega t_2 = 0$, while the latter chooses $t_1 = \omega t_2 = \pi/3$. We find the accuracy depends upon the location choice. In this short comment, we suggest the method of weighted residuals to overcome the shortcoming.

2. The frequency–amplitude formulation

We consider a generalized nonlinear oscillator in the form

$$u'' + f(u) = 0, \qquad u(0) = A, \qquad u'(0) = 0. \tag{1}$$

We use two trial functions $u_1(t) = A\cos t$ and $u_2 = A\cos\omega t$, which are, respectively, the solutions of the following linear oscillator equations:

$$u'' + \omega_1^2 u = 0, \qquad \omega_1^2 = 1 \tag{2}$$

and

$$u'' + \omega_2^2 u = 0 \qquad \omega_2^2 = \omega^2, \tag{3}$$

where ω is assumed to be the frequency of the nonlinear oscillator, equation (1).

The residuals are

$$R_1(t) = -\cos t + f(A\cos t) \tag{4}$$

and

$$R_2(\omega t) = -\omega^2 \cos\omega t + f(A\cos\omega t). \tag{5}$$

The original frequency–amplitude formulation reads [2]

$$\omega^2 = \frac{\omega_1^2 R_2 - \omega_2^2 R_1}{R_2 - R_1}. \tag{6}$$

In our previous publications [2–5], we just used the following formulation:

$$\omega^2 = \frac{\omega_1^2 R_2(\omega t = 0) - \omega_2^2 R_1(t = 0)}{R_2 - R_1}. \tag{7}$$

Geng and Cai improved the formulation by choosing another location point [1]

$$\omega^2 = \frac{\omega_1^2 R_2(\omega t = \pi/3) - \omega_2^2 R_1(t = \pi/3)}{R_2 - R_1}. \tag{8}$$

The accuracy depends upon the chosen location point, and we have no general rule for the choice. To illustrate this shortcoming, we consider the Duffing equation

$$u'' + u + \varepsilon u^3 = 0. \tag{9}$$

Recently many new approaches to nonlinear oscillators have been proposed, for example, the variational iteration method [6, 7], the homotopy perturbation method [8–17], the parameter-expanding method [2, 18] and the exp-function method [19–21]. A review on recently developed analytical methods is available in [2, 3].

Using trial functions $u_1(t) = A\cos t$ and $u_2 = A\cos\omega t$ respectively for equation (1), we obtain the following residuals:

$$R_1(t) = \varepsilon A^3 \cos^3 t \tag{10}$$

and

$$R_2(t) = A(1 - \omega^2)\cos\omega t + \varepsilon A^3 \cos^3 \omega t. \tag{11}$$

Locating at $\cos t_1 = \cos\omega t_2 = k$ (where $0 < k < 1$, $k = 1$ for equation (7), while $k = 1/2$ for equation (8)), we obtain

$$\omega^2 = \frac{A(1-\omega^2)k + \varepsilon A^3 k^3 - \omega^2 \varepsilon A^3 k^3}{A(1-\omega^2)k + \varepsilon A^3 k^3 - \varepsilon A^3 k^3} = 1 + \varepsilon A^3 k^2, \qquad 0 < k < 1. \tag{12}$$

So it is clear the accuracy depends on the location point.

3. The method of weighted residuals

We use the method of weighted residuals [3] to overcome the shortcoming. To this end, we introduce two new residual variables $\tilde{R}_1$ and $\tilde{R}_2$ defined as

$$\tilde{R}_1 = \frac{4}{T_1}\int_0^{T_1/4} R_1(t)\cos\left(\frac{2\pi}{T_1}t\right)\,\mathrm{d}t \tag{13}$$

and

$$\tilde{R}_2 = \frac{4}{T_2}\int_0^{T_2/4} R_2(t)\cos\left(\frac{2\pi}{T_2}t\right)\,\mathrm{d}t. \tag{14}$$

We can approximately determine ω^2 in the form

$$\omega^2 = \frac{\omega_1^2 \tilde{R}_2 - \omega_2^2 \tilde{R}_1}{\tilde{R}_2 - \tilde{R}_1}. \tag{15}$$

For the Duffing equation, by a simple calculation, we obtain

$$\tilde{R}_1 = \frac{4}{T_1}\int_0^{T_1/4} R_1(t)\cos\left(\frac{2\pi}{T_1}t\right)\,\mathrm{d}t = \frac{2}{\pi}\int_0^{\pi/2} \varepsilon A^3\cos^4 t\,\mathrm{d}t = \frac{3}{4\pi} \tag{16}$$

and

$$\begin{aligned}\tilde{R}_2 &= \frac{4}{T_2}\int_0^{T_2/4} R_2(t)\cos\left(\frac{2\pi}{T_2}t\right)\,\mathrm{d}t \\ &= \frac{2\omega}{\pi}\int_0^{T_2/4}\{A(1-\omega^2)\cos^2\omega t + \varepsilon A^3\cos^4\omega t\}\,\mathrm{d}t \\ &= \frac{2}{\pi}\int_0^{\pi/2}\{A(1-\omega^2)\cos^2 s + \varepsilon A^3\cos^4 s\}\,\mathrm{d}s \\ &= A(1-\omega^2)\frac{1}{\pi} + \frac{3}{4\pi}\varepsilon A^3.\end{aligned} \tag{17}$$

Applying equation (15), we have

$$\begin{aligned}\omega^2 &= \frac{\omega_1^2 \tilde{R}_2 - \omega_2^2 \tilde{R}_1}{\tilde{R}_2 - \tilde{R}_1} = \frac{A(1-\omega^2)\frac{1}{\pi} + \frac{3}{4\pi}\varepsilon A^3 - \frac{3}{4\pi}\varepsilon A^3\omega^2}{A(1-\omega^2)\frac{1}{\pi} + \frac{3}{4\pi}\varepsilon A^3 - \frac{3}{4\pi}\varepsilon A^3} \\ &= \frac{(1-\omega^2) + \frac{3}{4}\varepsilon A^3 - \frac{3}{4}\varepsilon A^3\omega^2}{(1-\omega^2)} = 1 + \tfrac{3}{4}\varepsilon A^2.\end{aligned} \tag{18}$$

Its approximate frequency reads

$$\omega = \sqrt{1 + \frac{3}{4}\varepsilon A^2}. \tag{19}$$

Observe that for small ε, i.e. $\varepsilon \ll 1$, it follows that

$$\omega = 1 + \tfrac{3}{8}\varepsilon A^2. \tag{20}$$

Consequently, in this limit, the present method gives exactly the same results as the standard perturbation method [2, 3]. To illustrate the accuracy of the obtained result, we compare the approximate period

$$T = \frac{2\pi}{\sqrt{1 + 3\varepsilon A^2/4}}, \tag{21}$$

with the exact one

$$T_{\mathrm{ex}} = \frac{4}{\sqrt{1+\varepsilon A^2}}\int_0^{\pi/2}\frac{\mathrm{d}x}{\sqrt{1 - k\sin^2 x}}, \tag{22}$$

where $k = 0.5\varepsilon A^2/(1+\varepsilon A^2)$.

What is rather surprising about the remarkable range of validity of (19) is that the approximate period, equation (21), as $\varepsilon \to \infty$ is also of high accuracy,

$$\lim_{\varepsilon A^2 \to \infty}\frac{T_{\mathrm{ex}}}{T} = \frac{2\sqrt{3/4}}{\pi}\int_0^{\pi/2}\frac{\mathrm{d}x}{\sqrt{1 - 0.5\sin^2 x}} = 0.9294.$$

Therefore, for any value of $\varepsilon > 0$, it can be easily proved that the maximal relative error of the period (21) is less than 7.6%, i.e. $|T - T_{\mathrm{ex}}|/T_{\mathrm{ex}} < 7.6\%$.

4. Conclusion

We overcome the shortcoming of the original frequency–amplitude formulation for nonlinear oscillators. The solution procedure is simple, and its accuracy is high, and the obtained solution is valid for the whole solution domain. This method can serve as a useful mathematical tool for dealing with various nonlinear oscillators, and the present note can for serve as a paradigm for many other applications in searching for frequency–amplitude relationships of various nonlinear oscillators.

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