2008

Appendix to Failure of the No-Arbitrage Principle

Kristof Madarasz, *London School of Economics and Political Science*

Appendix for "A Failure of the No-Arbitrage Principle"

Botond Kőszegi, Kristóf Madarász, and Máté Matolcsi

September 2007

This appendix shows that there is an arbitrage opportunity in the week in question, and also identifies all balanced arbitrage strategies. We first develop some notation and terminology. We call any sports event a "match" and reserve the term "event" for the statistical concept. Hence, for instance, a match being a tie is an event. Crucial to the arbitrage opportunity we analyze is that there may be multiple ways to subdivide outcomes of a match for wagering purposes. We call one such division a partition. For a Chicago-Dallas soccer match, for instance, a partition is whether Chicago wins, Dallas wins, or the match is a tie. The following lemma helps us identify an arbitrage opportunity.

Lemma 1 *An arbitrage involving a given set of partitions exists if and only if a balanced arbitrage involving the same set of partitions exists.*

Proof. The "if" part is trivial. To see the "only if" part, suppose there is an arbitrage opportunity. Take the contingency that yields the lowest winnings. Decrease wagers on other contingencies until all contingencies yield the same winnings. This clearly yields a balanced arbitrage. ■

Hence, without loss of generality, we look for balanced arbitrage strategies that yield 1 HUF in all contingencies. To do so, we calculate how much a "balanced betting strategy" costs a bettor—that is, how much she needs to pay to make sure that she wins exactly 1 HUF in all contingencies when betting on a set of partitions. Let O_k^i be the odds of the i-th outcome of the k-th partition, C_k the cost of a balanced betting strategy on the k-th partition P_k, and C^K the cost of a balanced strategy involving partitions $\{P_k\}_{k=1}^K$. Finally, we denote by $E^K(i_1, i_2, ..., i_K)$ the event that corresponds to the intersection of the i_1-th outcome of the first partition, the i_2-th outcome of the second partition, and so on up to the i_K-th outcome of the K-th partition. A crucial notion for the existence of an arbitrage opportunity is whether there is a logical connection between the partitions that can enter wagers:

Definition 1 *A set of partitions is logically independent if no combination of events in a subset of the partitions rules out an event in another partition.*

Formally, a set of K partitions is logically independent if there is no $i_1, ..., i_K$ such that $E^K(i_1, ..., i_K) = \emptyset$. If a set of K partitions is not logically independent then we say that it is logically connected. Our next lemma shows how to calculate the cost of a balanced strategy for K independent partitions.

Lemma 2 *If partitions $\{P_k\}_{k=1}^K$ are independent then the cost of a balanced strategy involving the K partitions is equal to the product of the cost of the balanced strategies on each partition alone.*

Proof. Using that $C_k = \sum_{i_k} 1/O_k^{i_k}$, we have

$$C^K = \sum_{i_1,...,i_K} \frac{1}{O_1^{i_1} \cdot ... \cdot O_K^{i_K}} = \left(\sum_{i_1} \frac{1}{O_1^{i_1}} \right) \cdot ... \cdot \left(\sum_{i_K} \frac{1}{O_1^{i_K}} \right) = \prod_{k=1}^{K} C_k.$$

■

This lemma implies that if partitions are independent and there is no arbitrage possibility on any single partition, then there is no arbitrage possibility on multiple partitions either. As a straightforward extension of this lemma, consider K partitions of which the first L are independent from the last $K - L$ in the above sense that there is no combination of outcomes in the first L partitions that rules out a combination of outcomes in the last $K - L$ partitions. Given such K partitions, the cost of a balanced strategy is equal to the product of the two costs:

Corollary 1 *For K partitions such that the first L are independent from the last $K - L$, we have $C^K = C^L C^{K-L}$.*

If there is logical dependence between the partitions then the above lemma does not hold anymore, since there are combinations of outcomes that are logically impossible. This implies that one does not need to bet money on such events, so that:

Corollary 2 $C^K = \prod_{k=1}^{K} C_k - C^D$, *where*

$$C^D = \sum_{E^K(i_1,\ldots,i_K)=\emptyset} \frac{1}{O_1^{i_1} \cdot \ldots \cdot O_K^{i_K}}.$$

Having established these facts, we can now turn to the identification of balanced arbitrage strategies.

Claim 1 *There are balanced arbitrage strategies.*

The prove this claim, consider the balanced arbitrage opportunity we mentioned in Section 3. The partitions and their respective odds appear in the top part of Table 1 at the end of the appendix. Note that partitions 1 and 6 are logically connected since the outcome of partition 6 perfectly determines the outcome of partition 1. Furthermore, partitions 2 and 6 and 3 and 6 are also connected: if Argentina (Croatia) scores no more than one goal, then Batistuta (Suker) does not score more than one goal, and if Argentina (Croatia) is scoreless, then Batistuta (Suker) does not score. Finally, Partitions 4 and 5 are also connected. If the United States does not lose against Yugoslavia, then the outcome of partition 5 must be that Yugoslavia does not win. Hence, combining these two games there are only four possible events, and the cost of a balanced strategy involving these two partitions is $C^{Y,US} = 1/(1.15 \cdot 1.35) + 1/(1.15 \cdot 3.3) + 1/(4.2 \cdot 5) + 1/(7.3 \cdot 5) = 0.982\,64$.

To make our calculations more transparent, we introduce the symbol S to stand for the cost of covering partition 2 conditional on the event that Croatia scores less than two goals. Since this implies that Suker scores less than 2 goals, $S = 1/1.4 + 1/2.65 = 1.091\,6$. Similarly, we introduce the symbol B to stand for the cost of covering partition 3 conditional on the fact that Argentina scores less than two goals. This cost is $B = 1/1.7 + 1/2.15 = 1.053\,4$. The following calculations show the parts of the costs of a balanced strategy involving partitions 1 to 3 and 6, conditional on the different outcomes of partition 6 that need to be covered.[1]

[1]For instance, the cost of covering event 1 of partition 6, denoted $C^*_{6.1}$, is $1/(6.65 \cdot 2.9 \cdot 1.4 \cdot 1.7)$ because one can put money on the combined event that (i) Argentina ties Croatia (odds of 2.9); (ii) Suker does not score (odds of 1.4); (iii) Batistuta does not score (odds of 1.7); and (iv) the score of the Argentina-Croatia game is 0-0 (odds of 6.65).

$$
\begin{aligned}
C^*_{6.1} &= 1/(6.65 \cdot 2.9 \cdot 1.4 \cdot 1.7) \\
C^*_{6.2} &= 1/(4 \cdot 1.7 \cdot 1.4) \\
C^*_{6.3} &= 1/(5.7 \cdot 3.2 \cdot 1.7) \\
C^*_{6.4} &= 1/(4 \cdot 2.9) \cdot B \\
C^*_{6.5} &= 1/(5 \cdot 1.7 \cdot 1.4) \\
C^*_{6.6} &= 1/(10 \cdot 3.2 \cdot 1.7) \\
C^*_{6.7} &= 1/(5.35 \cdot 1.7) \cdot S \\
C^*_{6.8} &= 1/(10 \cdot 3.2) \cdot B \\
C^*_{6.9} &= 1/(10 \cdot 2.9) \cdot 1.25 \\
C^*_{6.10} &= 1/(11.45 \cdot 1.7 \cdot 1.4) \\
C^*_{6.11} &= 1/(20 \cdot 3.2 \cdot 1.7) \\
C^*_{6.12} &= 1/(8 \cdot 1.7) \cdot S \\
C^*_{6.13} &= 1/(16 \cdot 3.2) \cdot B \\
C^*_{6.14} &= 1/13.35 \cdot 1.25^2
\end{aligned}
$$

Hence, the total cost for the strategy based on the combination of these partitions is $C^* = \sum_{i=1}^{14} C^*_{6.i} = 0.813\,31$. Since Szjrt required partition 6 to be part of at least quintuple bets, we combine these partitions with partitions 4 and 5.[2] By virtue of Lemma 3, the cost of a balanced strategy that bets on partitions 1 to 6 and delivers 1 HUF for sure is given by $C^6 = C^* \cdot C^{Y,US} = 0.799\,19$. This means that there is a 25.1 percent risk-less return on this strategy, proving our claim that there is an arbitrage strategy.

To show the other balanced arbitrage strategies we mentioned in the text, consider the last two partitions in Table 1. The relationship between these two partitions is very similar to that between the two partitions based on the Yugoslavia-United States match. Since Germany cannot win without Iran losing, the cost of a strategy which yields 1 HUF and bets on partitions 7 and 8 is $C^{Ger,Iran} = 1/(1.1 \cdot 1.3) + 1/(1.1 \cdot 3.5) + 1/(4.7 \cdot 5.3) + 1/(8 \cdot 5.3) = 1.022\,8$. This implies that a strategy which yields 1 HUF for sure and bets on partitions $1, 2, 3, 6, 7, 8$ costs

[2] SzjRt had restrictions in place on the minimum number of partitions that had to be involved in the wagers. Certain partitions could be bet as single wagers, others only in combination with a minimum of either two more or four more partitions. The partitions that were necessary for arbitrage were such that they required at least quintuple combined bets.

$C^{6'} = C^{*} \cdot C^{Ger,Iran} = 0.831\,85$. This strategy then yields a risk-free return of 20.2 percent. Furthermore, from Lemma 3 it follows that a strategy which yields 1 HUF for sure and bets on all partitions from 1 to 8 costs $C^{8} = C^{*} \cdot C^{Y,US} \cdot C^{Ger,Iran} = 0.817\,41$ and delivers a risk-free return of 22.3 percent.

Finally, to show that our calculations regarding the upper bound on the amount spent on arbitrage strategies are valid, we show the following:

Claim 2 *Any arbitrage strategy involves partition* 6 *and either partition* 5 *or partition* 8.

Note first that for any partition k on which SzjRt took bets in Week 26 (including partitions not listed in Table 1), $C_k > 1.24$. It follows then that there are no arbitrage strategies based on independent partitions. Therefore, an arbitrage strategy must contain logically connected events. Besides the ones already listed, there were no more logically connected ones in the game on Week 26.[3] This means that an arbitrage strategy must have contained at least two connected partitions out of partitions 1 to 8. Partitions 4 and 5 are logically connected and a balanced strategy on a combined bet costs less than 1 HUF. These events, however, must have entered five-fold bets or more. Since $C^{Y,US} \cdot 1.24^{3} = 1.\,873\,5$, combining these partitions with three independent ones precludes arbitrage. This means that for these two partitions to be in an arbitrage strategy, they have to be combined with other logically connected partitions. The only option not involving partition 6 is one where partitions 4 and 5 and 7 and 8 and a fifth independent partition are combined. The cost of such a strategy, however, is at least $C^{Ger,Iran} \cdot C^{Y,US} \cdot 1.24 = 1.\,246\,2$. A similar argument shows that there is no risk-free arbitrage strategy that involves partitions $7-8$ but not 6. This implies that partition 6 must have entered any arbitrage strategy.

To prove that any arbitrage strategy must have included either partition 5 or partition 8, consider the case of the cheapest risk-free betting strategy including partitions 1 to 3 and 6. The cost of this strategy is $C^{*} = 0.813\,31$. Given the constraint that partition 6 could only be part of at least quintuple bets, we need to add two more partitions to this strategy. Clearly, adding two independent partitions would eliminate the positive risk-free return since $0.813\,31 \cdot 1.24^{2} = 1.\,250\,5$.

[3] There were other pairs of partitions based on the same match, but just like partitions 1 and 3 above, there was no logical connections between any of these pairs.

The only remaining option is to add logically connected partitions. The only logically connected ones, however, are partitions $4 - 5$ and $7 - 8$. This proves our claim.

Partition #	Event #	Event	Odds
1	1	Argentina beats Croatia	1.7
1	2	Argentina ties Croatia	2.9
1	3	Argentina loses to Croatia	3.2
2	1	Suker scores 0 goals	1.4
2	2	Suker scores 1 goal	2.65
2	3	Suker scores $>$ 1 goals	6.15
3	1	Batistuta scores 0 goals	1.7
3	2	Batistuta scores 1 goal	2.15
3	3	Batistuta scores $>$ 1 goals	5
4	1	Yugoslavia beats United States	1.15
4	2	Yugoslavia ties United States	4.2
4	3	Yugoslavia loses to United States	7.3
5	1	Yugoslavia wins by $\geq$2 goals	1.35
5	2	Yugoslavia wins by 1 goal	3.3
5	3	Yugoslavia does not win	5
6	1	Argentina-Croatia: {0,0}	6.65
6	2	Argentina-Croatia: {1,0}	4
6	3	Argentina-Croatia: {0,1}	5.7
6	4	Argentina-Croatia: {1,1}	4
6	5	Argentina-Croatia: {2,0}	5
6	6	Argentina-Croatia: {0,2}	10
6	7	Argentina-Croatia: {2,1}	5.35
6	8	Argentina-Croatia: {1,2}	10
6	9	Argentina-Croatia: {2,2}	10
6	10	Argentina-Croatia: {3,0}	11.45
6	11	Argentina-Croatia: {0,3}	20
6	12	Argentina-Croatia: {3,1}	8
6	13	Argentina-Croatia: {1,3}	16
6	14	Argentina-Croatia: ELSE	13.35
7	1	Germany beats Iran	1.1
7	2	Germany ties Iran	4.7
7	3	Germany loses to Iran	8
8	1	Germany wins by $\geq$2 goals	1.3
8	2	Germany wins by 1 goal	3.5
8	3	Germany does not win	5.3

Table 1: The Odds for the Logically Connected Partitions in Week 26