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Solitary solutions, periodic solutions and compacton-like solutions using the Exp-function method

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Abstract

In this paper, the Exp-function method is used for seeking solitary solutions, periodic solutions, and compacton-like solutions of nonlinear differential equations. The combined KdV–MKdV equation and the Liouville equation are chosen to illustrate the effectiveness and convenience of the proposed method.

Keywords: Solitary solutions; Periodic solutions and compacton-like solutions; Exp-function method

1. Introduction

Recently many new approaches have been proposed to solve exact traveling wave solutions of nonlinear differential equations. For example, the variational iteration method [1–4], the homotopy perturbation method [5–10], the variational method [11,12], the exp-function method [13,14] the tanh-method [15–18], the extended tanh-method [19–23], the sinh-method [24,25], the homogeneous balance method [26–29], the F-expansion method [30–32], the modified extended Fan's sub-equation method [33,34], and so on. A complete review is given in Refs. [35,36]. Most methods mentioned above belong to a class of methods called the sub-equation method. Especially the modified extended Fan's sub-equation method presents more general solutions using auxiliary ordinary equations, for example

$$\phi' = A + \phi^2,$$
$$\phi'^2 = B\phi + C\phi^3 + D\phi^4,$$
$$\phi'^2 = A + B\phi^2 + D\phi^4,$$
$$\phi' = r + p\phi + q\phi^2$$

and others.

In this paper we will apply the Exp-function method [13,14] to obtain solitary and periodic solutions from traveling wave nonlinear differential equations. We will compare our solutions with those gained by the modified extended Fan's sub-equation method.

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2. Exp-function method [13,14]

The basic idea of the exp-function method is very simple; it is based on the assumption that traveling wave solutions can be expressed in the following form:

$$u(\xi) = \frac{\sum_{n=-c}^{d} a_n \exp(n\xi)}{\sum_{m=-p}^{q} b_n \exp(m\xi)}$$

where c, d, p and q are positive integers which are unknown to be further determined, and a_n and b_m are unknown constants.

Exp-function can be not only used directly, but also applied as a sub-equation method. To illustrate the basic idea of the proposed method, we consider first the combined KdV–MKdV equation.

$$u_t + \mu u_x u + \delta u^2 u_x + u_{xxx} = 0. \tag{1}$$

The equation simultaneously exists in practical problems such as quantum field theory and fluid physics, and it describes the wave propagation of bound particles, sound waves and thermal pulses.

According to the Exp-function method, we introduce a complex variation ξ defined as $\xi = kx + \omega t + \varphi$. Eq. (1) thus becomes an ordinary different equation, which reads

$$\omega u' + k\mu u u' + k\delta u^2 u' + k^3 u''' = 0 \tag{2}$$

where prime denotes the derivative with respect to ξ.

We suppose that the solution of Eq. (2) can be expressed in the following form:

$$u(\xi) = \frac{a_c \exp[c\xi] + \cdots + a_{-d} \exp[-d\xi]}{b_p \exp[p\xi] + \cdots + b_{-q} \exp[-q\xi]}. \tag{3}$$

To determine the values of c and p, we balance the linear term of highest order of Eq. (2) with the highest order nonlinear term. By simple calculation, we have

$$u''' = \frac{c_1 \exp[(7p + c)\xi] + \cdots}{c_2 \exp[8p\xi] + \cdots} \tag{4}$$

and

$$u^2 u' = \frac{c_3 \exp[(p + 3c)\xi] + \cdots}{c_2 \exp[4p\xi] + \cdots} = \frac{c_3 \exp[(5p + 3c)\xi] + \cdots}{c_2 \exp[8p\xi] + \cdots}. \tag{5}$$

Balancing highest order of Exp-function in Eqs. (4) and (5), we have $7p + c = 5p + 3c$, which leads to the result $p = c$.

Similarly, we balance the lowest order in Eq. (2) to determine values of d and q:

$$u''' = \frac{c_1 \exp[-(7q + d)\xi] + \cdots}{c_2 \exp[-8q\xi] + \cdots} \tag{6}$$

and

$$u^2 u' = \frac{c_3 \exp[-(q + 3d)\xi] + \cdots}{c_2 \exp[-4q\xi] + \cdots} = \frac{c_3 \exp[-(5q + 3d)\xi] + \cdots}{c_2 \exp[-8q\xi] + \cdots}. \tag{7}$$

Therefore we can obtain the following relation

$$-(7q + d) = (-5q + 3d) \tag{8}$$

which leads to the result $q = d$.

For simplicity, we set $p = c = 1$ and $q = d = 1$, then Eq. (3) reduces to

$$u(x,t) = \frac{a_1 \exp[\xi] + a_0 + a_{-1} \exp[-\xi]}{b_1 \exp[\xi] + b_0 + b_{-1} \exp[-\xi]}. \tag{9}$$

In case $b_1 \neq 0$, Eq. (9) can be simplified as

$$u(x,t) = \frac{a_1 \exp[\xi] + a_0 + a_{-1} \exp[-\xi]}{\exp[\xi] + b_0 + b_{-1} \exp[-\xi]}. \tag{10}$$

Substituting Eq. (10) into Eq. (2), we have

$$\frac{1}{A}[C_3 \exp(3\xi) + C_2 \exp(2\xi) + C_1 \exp(\xi) + C_0 + C_{-1} \exp(-\xi) + C_{-2} \exp(-2\xi) + C_{-3} \exp(-3\xi)] = 0 \tag{11}$$

where

$$\begin{aligned}
&A = (\exp(\xi) + b_0 + b_{-1} \exp(-\xi))^4 \\
&C_3 = -\omega a_0 + \mu k a_1^2 b_0 + \delta k a_1^3 b_0 + k^3 a_1 b_0 - \delta k a_1^2 a_0 - \mu k a_0 a_1 - k^3 a_0 + \omega a_1 b_0 \\
&C_2 = -2\delta k a_1 a_0^2 - 2\omega a_{-1} + \mu k a_1^2 b_0^2 + 2\mu k a_1^3 b_{-1} + 2\mu k a_1^2 a_0 b_0 - 8k^3 a_{-1} + 2\omega a_1 b_{-1} \\
&\quad - 4k^3 a_1 b_0^2 + 2\mu k a_1^2 b_{-1} - \mu k a_0^2 - 2\omega a_0 b_0 - 2\mu k a_{-1} a_1 - 2\delta k a_1^2 a_{-1} + 2\omega a_1 b_0^2 \\
&\quad + 4k^3 a_0 b_0 + 8k^3 a_1 b_{-1} \\
&C_1 = -5\omega a_{-1} b_0 + \delta k a_1^2 a_{-1} b_0 + 5\delta k a_1^2 a_0 b_{-1} + \omega a_1 b_0^3 - \mu k a_0^2 b_0 + 6\omega a_1 b_0 b_{-1} - 5k^3 a_{-1} b_0 \\
&\quad + 2\mu k a_1 b_{-1} a_0 - \delta k a_0^3 - k^3 a_0 b_0^2 + 3\mu k a_1^2 b_0 b_{-1} + \mu k a_1 b_0^2 a_0 + 23k^3 a_0 b_{-1} - 3\mu k a_{-1} a_0 \\
&\quad - 2\mu k a_{-1} a_1 b_0 - \omega a_0 b_{-1} + \delta k a_0^2 a_1 b_0 - \omega a_0 b_0^2 - 6\delta k a_1 a_0 a_{-1} - 18k^3 a_1 b_0 b_{-1} + k^3 a_1 b_0^3 \\
&C_0 = -4\omega a_{-1} b_{-1} + 32k^3 a_{-1} b_{-1} - 32k^3 a_1 b_{-1}^2 - 4\mu k a_{-1} a_0 b_0 - 4\delta k a_0^2 a_{-1} - 2\mu k a_{-1}^2 \\
&\quad - 4\delta k a_1 a_{-1}^2 - 4\omega a_{-1} b_0^2 - 4k^3 a_{-1} b_0^2 + 4\omega a_1 b_0^2 b_{-1} + 2\mu k a_1^2 b_{-1}^2 + 4\delta k a_1 a_0^2 b_{-1} \\
&\quad + 4\omega a_1 b_{-1}^2 + 4k^3 a_1 b_0^2 b_{-1} + 4\delta k a_1^2 a_{-1} b_{-1} \\
&C_{-1} = -6\omega a_{-1} b_0 b_{-1} - 5\delta k a_0 a_{-1}^2 - \mu k a_{-1} b_0^2 a_0 + 2\mu k a_{-1} b_{-1} a_1 b_0 + \delta k a_0^3 b_{-1} - 2\mu k a_{-1} a_0 b_{-1} \\
&\quad + 18k^3 a_{-1} b_0 b_{-1} - 3\mu k a_{-1}^2 b_0 - k^3 a_{-1} b_0^3 + \mu k a_0^2 b_{-1} b_0 + 5k^3 a_1 b_0 b_{-1}^2 - \delta k a_1 a_{-1}^2 b_0 \\
&\quad - 23k^3 a_0 b_{-1}^2 + \omega a_0 b_{-1}^2 + k^3 a_0 b_{-1} b_0^2 - \omega a_{-1} b_0^3 + 5\omega a_1 b_0 b_{-1}^2 - \delta k a_0^2 a_{-1} b_0 \\
&\quad + \omega a_0 b_{-1} b_0^2 + 3\mu k a_1 b_{-1}^2 a_0 + 6\delta k a_1 a_{-1} a_0 b_{-1} \\
&C_{-2} = 2\omega a_0 b_{-1}^2 b_0 + 2\mu k a_1 b_{-1}^2 a_{-1} + 4k^3 a_{-1} b_0^2 b_{-1} - 2\mu k a_{-1}^2 b_{-1} - 2\omega a_{-1} b_{-1}^2 - 2\omega a_{-1} b_0^2 b_{-1} \\
&\quad + 2\omega a_1 b_{-1}^3 - 2\delta k a_{-1}^3 + \mu k a_0^2 b_{-1}^2 - \mu k a_{-1}^2 b_0^2 - 2\delta k a_0 a_{-1}^2 b_0 + 2\delta k a_{-1}^2 a_1 b_{-1} \\
&\quad + 2\delta k a_0^2 a_{-1} b_{-1} - 4k^3 a_0 b_{-1}^2 b_0 - 8k^3 a_{-1} b_{-1}^2 + 8k^3 a_1 b_{-1}^3 \\
&C_{-3} = k^3 a_0 b_{-1}^3 + \omega a_0 b_{-1}^3 + \mu k a_0 b_{-1}^2 a_{-1} - \mu k a_{-1}^2 b_0 b_{-1} - \omega a_{-1} b_0 b_{-1}^2 - \delta k a_{-1}^3 b_0 - k^3 a_{-1} b_0 b_{-1}^2 \\
&\quad + \delta k a_{-1}^2 a_0 b_{-1}.
\end{aligned}$$

Setting the coefficients of $\exp(n\xi)$ to be zero, we have

$$\begin{cases} C_3 = 0, \qquad C_2 = 0, \qquad C_1 = 0 \\ C_0 = 0 \\ C_{-3} = 0, \qquad C_{-2} = 0, \qquad C_{-1} = 0. \end{cases} \tag{12}$$

Solving the system, Eq. (12), simultaneously, we obtain two sets of solutions

$$\begin{cases} a_0 = \dfrac{b_0(2\delta a_1^2 + \mu a_1 + 6k^2)}{2\delta a_1 + \mu}, \qquad \omega = -k\delta a_1^2 - \mu k a_1 - k^3 \\ b_{-1} = \dfrac{b_0^2(6\delta k^2 + 4\delta^2 a_1^2 + 4\delta a_1 \mu + \mu^2)}{4(4\delta^2 a_1^2 + 4\delta a_1 \mu + \mu^2)} \\ a_{-1} = \dfrac{a_1 b_0^2(6\delta k^2 + 4\delta^2 a_1^2 + 4\delta a_1 \mu + \mu^2)}{4(4\delta^2 a_1^2 + 4\delta a_1 \mu + \mu^2)} \end{cases} \tag{13a}$$

and

$$\begin{cases} a_1 = \left(-\frac{\mu}{2\delta} + k\frac{\sqrt{-6\delta}}{\delta}\right), \qquad a_{-1} = -b_{-1}\left(\frac{\mu}{2\delta} + k\frac{\sqrt{-6\delta}}{\delta}\right) \\ \omega = \frac{1}{4\delta}k(\mu^2 + 8\delta k^2), \qquad b_0 = 0, \qquad a_0 = 0. \end{cases} \tag{13b}$$

By Eq. (13a), we obtain the following solution:

$$\begin{aligned} u &= \frac{a_1 \exp(\xi) + \frac{b_0(2\delta a_1^2+\mu a_1+6k^2)}{2\delta a_1+\mu} + \frac{a_1 b_0^2(6\delta k^2+4\delta^2 a_1^2+4\delta a_1\mu+\mu^2)}{4(4\delta^2 a_1^2+4\delta a_1\mu+\mu^2)}\exp(-\xi)}{\exp(\xi) + b_0 + \frac{b_0^2(6\delta k^2+4\delta^2 a_1^2+4\delta a_1\mu+\mu^2)}{4(4\delta^2 a_1^2+4\delta a_1\mu+\mu^2)}\exp(-\xi)} \\ &= a_1 + \frac{\frac{6b_0k^2}{2\delta a_1+\mu}}{\exp(\xi) + b_0 + \frac{b_0^2[6\delta k^2+(2\delta a_1+\mu)^2]}{4(2\delta a_1+\mu)^2}\exp(-\xi)} \end{aligned} \tag{14a}$$

where $\xi = kx + (-k\delta a_1^2 - \mu k a_1 - k^3)t + \varphi$, a_1, b_0, φ are parameters, $b_0 \neq 0$, and k is a free real number.

In case $b_0 = 0$, we could obtain the following solution from Eq. (13a)

$$u = \frac{\left(-\frac{\mu}{2\delta} + k\frac{\sqrt{-6\delta}}{\delta}\right)\exp\left[kx + \frac{1}{4\delta}k(\mu^2 + 8\delta k^2)t + \varphi\right] - b_{-1}\left(\frac{\mu}{2\delta} + k\frac{\sqrt{-6\delta}}{\delta}\right)\exp\left[-kx - \frac{1}{4\delta}k(\mu^2 + 8\delta k^2)t - \varphi\right]}{\exp\left[kx + \frac{1}{4\delta}k(\mu^2 + 8\delta k^2)t + \varphi\right] + b_{-1}\exp\left[-kx - \frac{1}{4\delta}k(\mu^2 + 8\delta k^2)t - \varphi\right]}. \tag{14b}$$

When k is an imaginary number, using the transformation

$$\begin{cases} k = \mathrm{i}K \\ \exp(\mathrm{i}K\xi) = \cos(K\xi) + \mathrm{i}\sin(K\xi) \\ \exp(-\mathrm{i}K\xi) = \cos(K\xi) - \mathrm{i}\sin(K\xi). \end{cases}$$

Eq. (14a) can be translated into a periodic or compact-like solution:

$$u = a_1 + \frac{\frac{-6b_0K^2}{2\delta a_1+\mu}}{\left(1 + \frac{b_0^2[-6\delta K^2+(2\delta a_1+\mu)^2]}{4(2\delta a_1+\mu)^2}\right)\cos(\xi) + b_0 + \mathrm{i}\left(1 - \frac{b_0^2[-6\delta K^2+(2\delta a_1+\mu)^2]}{4(2\delta a_1+\mu)^2}\right)\sin(\xi)} \tag{15a}$$

where $\xi = Kx - K(\delta a_1^2 + \mu a_1 - K^2)t + \varphi$, b_0, φ are parameters, $b_0 \neq 0$, and K is a free real number.

The periodic solution, Eq. (15a), might have some potential applications. For practical use, we eliminate the imaginary part in Eq. (15a), this requires

$$\frac{b_0^2[-6\delta K^2 + (2\delta a_1 + \mu)^2]}{4(2\delta a_1 + \mu)^2} = 1, \quad \text{or} \quad K^2 = \frac{(b_0^2 - 4)(2\delta a_1 + \mu)^2}{6b_0^2\delta}.$$

Eq. (15a) is reduced to periodic solution:

$$u = a_1 - \frac{(b_0^2 - 4)(2\delta a_1 + \mu)}{b_0\delta(2\cos(Kx - K(\delta a_1^2 + \mu a_1 - K^2)t + \varphi) + b_0)} \tag{15b}$$

in which $K = \pm\sqrt{\frac{(b_0^2-4)(2\delta a_1+\mu)^2}{6b_0^2\delta}}$.

We can also set $\frac{b_0^2[-6\delta K^2+(2\delta a_1+\mu)^2]}{4(2\delta a_1+\mu)^2} = \pm\mathrm{i}$ to eliminate imaginary part in Eq. (15a).

Similarly, Eq. (14b) can be translated into periodic or compact-like solutions as follows:

$$u = \frac{\left[-\frac{\mu}{2\delta}(1 + b_{-1}) + \mathrm{i}K\frac{\sqrt{-6\delta}}{\delta}(1 - b_{-1})\right]\cos(\xi) - \mathrm{i}\left[-\frac{\mu}{2\delta}(1 - b_{-1}) + \mathrm{i}K\frac{\sqrt{-6\delta}}{\delta}(1 + b_{-1})\right]\sin(\xi)}{(1 + b_{-1})\cos(\xi) + \mathrm{i}(1 - b_{-1})\sin(\xi)} \tag{16a}$$

where $\xi = Kx + \frac{1}{4\delta}K(\mu^2 - 8\delta K^2)t + \varphi$, b_{-1}, φ are parameters, $b_{-1} \neq 0$, and K is a free real number.

Elimination of the imaginary part in Eq. (16a) requires

$$b_{-1} = \pm 1, \quad \text{or} \quad b_{-1} = \pm \mathrm{i}, \quad \text{or} \quad b_{-1} = \pm \frac{\frac{u}{2\delta} - \mathrm{i}K\frac{\sqrt{-6\delta}}{\delta}}{\frac{u}{2\delta} + \mathrm{i}K\frac{\sqrt{-6\delta}}{\delta}} \quad \text{and so on.}$$

Setting $b_{-1} = 1$ for simplicity, we obtain the following periodic solution:

$$u = \frac{-\frac{\mu}{2\delta}\cos\left(Kx + \frac{1}{4\delta}K(\mu^2 - 8\delta K^2)t + \varphi\right) + K\frac{\sqrt{-6\delta}}{\delta}\sin\left(Kx + \frac{1}{4\delta}K(\mu^2 - 8\delta K^2)t + \varphi\right)}{\cos\left(Kx + \frac{1}{4\delta}K(\mu^2 - 8\delta K^2)t + \varphi\right)}. \tag{16b}$$

Exp-function method can not only be employed directly but also be used in a sub-equation way. Supposing that

$$u = \sum_{i=0}^{n} a_i \phi^i(k\xi),$$

where ϕ satisfies the general Riccati equation

$$\frac{\mathrm{d}\phi}{\mathrm{d}\xi} = r + p\phi + q\phi^2. \tag{17}$$

In case $p = 0$, it reduces to homogeneous balance method [26–29]; if $p = 0, r = k, q = -k$ it becomes the tanh method [15,16] (for $\tanh(k\xi)' = k - k\tanh^2(k\xi)$).

We assume that the solution of Eq. (17) can be expressed in the form

$$\phi = \frac{a_1 \exp[k\xi + \varphi] + a_0 + a_{-1}\exp[-k\xi - \varphi]}{\exp[k\xi + \varphi] + b_0 + b_{-1}\exp[-k\xi - \varphi]}.$$

By the Exp-function method, we can easily obtain

$$\phi = \frac{\frac{-p-\sqrt{p^2-4qr}}{2q}\exp\left(\frac{\sqrt{p^2-4qr}}{2}\xi + \varphi\right) + b_{-1}\frac{-p+\sqrt{p^2-4qr}}{2q}\exp\left(-\frac{\sqrt{p^2-4qr}}{2}\xi - \varphi\right)}{\exp\left(\frac{\sqrt{p^2-4qr}}{2}\xi + \varphi\right) + b_{-1}\exp\left(-\frac{\sqrt{p^2-4qr}}{2}\xi - \varphi\right)} \tag{18}$$

where φ is free parameter.

In case $p^2 - 4qr < 0$, we can use the following transformation

$$\begin{cases} \sqrt{p^2 - 4qr} = \mathrm{i}\sqrt{4qr - p^2} \\ \exp(\mathrm{i}k\xi) = \cos(k\xi) + \mathrm{i}\sin(k\xi) \\ \exp(-\mathrm{i}k\xi) = \cos(k\xi) - \mathrm{i}\sin(k\xi). \end{cases}$$

Eq. (18) is reduced to periodic solutions:

$$\begin{aligned} \phi &= \frac{1}{2q} \times \frac{[-p(1+b_{-1}) - 2\mathrm{i}k(1-b_{-1})]\cos(k\xi+\varphi) + \mathrm{i}[-p(1-b_{-1}) - 2\mathrm{i}k(1+b_{-1})]\sin(k\xi+\varphi)}{(1+b_{-1})\cos(k\xi+\varphi) + \mathrm{i}(1-b_{-1})\sin(k\xi+\varphi)} \\ &= \frac{1}{2q}\left[-p + \frac{-2\mathrm{i}k(1-b_{-1})\cos(k\xi+\varphi) + 2k(1+b_{-1})\sin(k\xi+\varphi)}{(1+b_{-1})\cos(k\xi+\varphi) + \mathrm{i}(1-b_{-1})\sin(k\xi+\varphi)}\right] \end{aligned} \tag{19}$$

where $k = \frac{\sqrt{4qr-p^2}}{2}$.

We will prove that Eqs. (18) and (19) can produce all periodic and solitary solutions obtained by Emmanuel Yomba [33].

Let $\xi = x + ct$. Eq. (1) will be changed into

$$cu' + \mu u'u + \delta u^2 u' + u''' = 0. \tag{20}$$

By balancing u''' and u^2u' in Eq. (20), we obtain $n = 1$. We further suppose that Eq. (20) has the following form of solution:

$$u = a_0 + a_1\phi(\xi). \tag{21}$$

In view of Eqs. (17) and (21), we can easily obtain the following relations

$$u' = a_1(r + p\phi + q\phi^2) \tag{22}$$

$$u'' = a_1[p(r + p\phi + q\phi^2) + 2q\phi p(r + p\phi + q\phi^2)] = 2a_1q^2\phi^3 + 3a_1pq\phi^2 + a_1(p^2 + 2qr)\phi + a_1pr \tag{23}$$

and

$$u''' = 6a_1q^2\phi^2(r + p\phi + q\phi^2) + 6a_1pq\phi(r + p\phi + q\phi^2) + a_1(p^2 + 2qr)(r + p\phi + q\phi^2). \tag{24}$$

Substituting the above results into Eq. (20) gives

$$C_4\phi^4 + C_3\phi^3 + C_2\phi^2 + C_1\phi + C_0 = 0 \tag{25}$$

where $C_4 = 6a_1q^3 + na_1^3q$

$$C_3 = 12a_1q^2p + 2\delta a_0a_1^2q + \delta a_1^3p + \mu a_1^2q$$
$$C_2 = \delta a_0^2a_1q + 2\delta a_0a_1^2p + \delta a_1^3r + \mu a_1^2p + \mu a_1qa_0 + ca_1q + a_1(p^2 + 2qr)q + 6a_1q^2r + 6a_1p^2q$$
$$C_1 = 6a_1pqr + \mu a_1^2r + \mu a_1pa_0 + ca_1p + a_1(p^2 + 2qr)p + \delta a_0^2a_1p + 2\delta a_0a_1^2r$$
$$C_0 = ca_1r + a_1(p^2 + 2qr)r + \mu a_1ra_0 + \delta a_1a_0^2r.$$

Setting the coefficients of ϕ to be zero, we have

$$\begin{cases} C_4 = 0, \quad C_3 = 0, \quad C_2 = 0, \\ C_1 = 0, \quad C_0 = 0. \end{cases} \tag{26}$$

Solving the system (26) we have

$$\begin{cases} a_0 = \dfrac{-\mu + \sqrt{-6\delta}p}{2\delta} \\ a_1 = -\dfrac{\sqrt{-6\delta}q}{6} \\ c = \dfrac{1}{4\delta}[\mu^2 + 4\delta(p^2 - 4qr)]. \end{cases} \tag{27}$$

By Eq. (21) we have

$$\begin{aligned} u &= a_0 + a_1\phi \\ &= \frac{-\mu + \sqrt{-6\delta}p}{2\delta} - \frac{\sqrt{-6\delta}q}{6} \times \frac{\frac{-p-\sqrt{p^2-4qr}}{2q}\exp\left(\frac{\sqrt{p^2-4qr}}{2}\xi\right) + b_1\frac{-p+\sqrt{p^2-4qr}}{2q}\exp\left(-\frac{\sqrt{p^2-4qr}}{2}\xi\right)}{\exp\left(\frac{\sqrt{p^2-4qr}}{2}\xi\right) + b_1\exp\left(-\frac{\sqrt{p^2-4qr}}{2}\xi\right)} \\ &= \frac{\left(-\frac{\mu}{2\delta} + \frac{\sqrt{p^2-4qr}}{2}\frac{\sqrt{-6\delta}}{\delta}\right)\exp\left[\frac{\sqrt{p^2-4qr}}{2}\xi\right] - b_{-1}\left(\frac{\mu}{2\delta} + \frac{\sqrt{p^2-4qr}}{2}\frac{\sqrt{-6\delta}}{\delta}\right)\exp\left[-\frac{\sqrt{p^2-4qr}}{2}\xi\right]}{\exp\left[\frac{\sqrt{p^2-4qr}}{2}\xi\right] + b_{-1}\exp\left[-\frac{\sqrt{p^2-4qr}}{2}\xi\right]} \end{aligned} \tag{28}$$

where $\xi = x + \frac{1}{4\delta}[\mu^2 + 4\delta(p^2 - 4qr)]t + \phi$.

If we let $k = \frac{\sqrt{p^2-4qr}}{2}$, Eq. (27) becomes

$$u = \frac{\left(-\frac{\mu}{2\delta} + k\frac{\sqrt{-6\delta}}{\delta}\right)\exp\left[kx + \frac{1}{4\delta}k(\mu^2 + 8\delta k^2)t + \varphi\right] - b_{-1}\left(\frac{\mu}{2\delta} + k\frac{\sqrt{-6\delta}}{\delta}\right)\exp\left[-kx - \frac{1}{4\delta}k(\mu^2 + 8\delta k^2)t - \varphi\right]}{\exp\left[kx + \frac{1}{4\delta}k(\mu^2 + 8\delta k^2)t + \varphi\right] + b_{-1}\exp\left[-kx - \frac{1}{4\delta}k(\mu^2 + 8\delta k^2)t - \varphi\right]} \tag{29}$$

It is interesting to note that Eq. (29) is the same as Eq. (16b), but we cannot find a solution like Eq. (15b) in this way.

3. Liouville equation

Let us consider the Liouville equation

$$u_{xt} = -e^u. \tag{30}$$

Using the transformation $v = \exp(u)$ and $\xi = kx + \omega t + \varphi$, Eq. (30) becomes

$$kwvv'' - kw(v')^2 + v^3 = 0. \tag{31}$$

Let $M = kw$, and Eq. (31) turns out to be

$$Mvv'' - M(v')^2 + v^3 = 0. \tag{32}$$

Supposing that the solution of Eq. (32) can be expressed as

$$u(\xi) = \frac{a_c \exp[c\xi] + \cdots + a_{-d}\exp[-d\xi]}{b_p \exp[p\xi] + \cdots + b_{-q}\exp[-q\xi]}.$$

By balancing vv'' and v^3 in Eq. (32), we obtain $c = p$ and $d = q$.

We can freely choose the values of c and d, but we will illustrate that the final solution does not strongly depends upon the choice of values of c and d.

Case 1. Let $c = p = 1$ and $d = q = 1$, and the solution of Eq. (32) can be expressed as

$$v = \frac{a_1\exp(\xi) + a_0 + a_{-1}\exp(\xi)}{\exp(\xi) + b_0 + b_{-1}\exp(\xi)}. \tag{33}$$

Substituting (33) into (32), we obtain a polynomial of $\exp(n\xi)$, equating the coefficients of $\exp(n\xi)$ to be zero, and solving this system, we obtain

$$u(x,t) = \ln\left[\frac{a_0}{\exp(kx + \omega t + \varphi) + \frac{a_0}{k\omega} + \frac{a_0^2}{4k^2\omega^2}\exp(-kx - \omega t - \varphi)}\right] \tag{34}$$

where a_0, k, ω are non-zero free parameters.

We compare Eq. (34) with the solution in [15] by Wazwaz, which reads

$$u(x,t) = \ln\left[K\,\mathrm{sech}^2\left(\sqrt{-\frac{K}{2c}}(x - ct)\right)\right]. \tag{35}$$

We rewrite it in the following form:

$$u(x,t) = \ln\left[\frac{K}{\exp\left[2\sqrt{-\frac{K}{2c}}(x - ct)\right] + 2 + \exp\left[2\sqrt{-\frac{K}{2c}}(x - ct)\right]}\right]. \tag{36}$$

It is obvious that if we let $k = \sqrt{-\frac{K}{2c}}$, $\omega = -\sqrt{-\frac{K}{2c}} \times c$, and $a_0 = 2k\omega = K$, Eq. (34) reduces to Eq. (36).

By using the transformation

$$\begin{cases} \omega = \mathrm{i}W \qquad k = \mathrm{i}K \\ \exp(\mathrm{i}Kx + \mathrm{i}Wt) = \cos(Kx + Wt) + \mathrm{i}\sin(Kx + Wt) \\ \exp(-\mathrm{i}Kx - \mathrm{i}Wt) = \cos(Kx + Wt) - \mathrm{i}\sin(Kx + Wt). \end{cases} \tag{37}$$

Eq. (36) can be also translated into a periodic solution

$$u(x,t) = \ln\left[\frac{a_0}{\left(1 + \frac{a_0^2}{4K^2W^2}\right)\cos(Kx + Wt + \varphi) - \frac{a_0}{KW} + \mathrm{i}\left(1 - \frac{a_0^2}{4K^2W^2}\right)\sin(Kx + Wt + \varphi)}\right] \tag{38}$$

where a_0, K, W, φ are free parameters.

For example, if we let $a_0 = 2KW$

$$u(x,t) = \ln\left[\frac{KW}{\cos(Kx + Wt + \varphi) - 1}\right]. \tag{39}$$

Let $c = KW$, and Eq. (39) will be

$$u(x,t) = \ln\left[\frac{c}{\cos\left(Kx + \frac{c}{K}t + \varphi\right) - 1}\right]. \tag{40}$$

Case 2. Let $c = p = 2$ and $d = q = 2$, and we can assume the function to be

$$v = \frac{a_2\exp(2\xi) + a_1\exp(\xi) + a_0 + a_{-1}\exp(-\xi) + a_{-2}\exp(-2\xi)}{\exp(2\xi) + b_1\exp(\xi) + b_0 + b_{-1}\exp(-\xi) + b_{-2}\exp(-2\xi)}. \tag{41}$$

Substituting (41) into (32), we obtain polynomials of $\exp(n\xi)$, equating the coefficients of $\exp(n\xi)$ to be zero, and solving this system, we have three set of solutions:

$$\begin{cases} a_2 = 0; \quad a_1 = 0; \quad a_{-1} = 0; \quad a_{-2} = 0 \\ b_1 = 0; \quad b_0 = \dfrac{a_0}{4k\omega}; \quad b_{-1} = 0; \quad b_{-2} = \dfrac{a_0^2}{64k^2\omega^2} \end{cases} \tag{42a}$$

$$\begin{cases} a_2 = 0; \quad a_0 = -\dfrac{a_1^2}{k\omega}; \quad a_{-1} = \dfrac{a_1^3}{4k^2\omega^2}; \quad a_{-2} = 0 \\ b_1 = 0; \quad b_0 = -\dfrac{a_1^2}{2k^2\omega^2}; \quad b_{-1} = 0; \quad b_{-2} = \dfrac{a_1^4}{16k^4\omega^4} \end{cases} \tag{42b}$$

$$\begin{cases} a_2 = 0; \quad a_1 = \dfrac{4b_{-2}(k\omega)^2}{a_{-1}}; \quad a_0 = \dfrac{a_{-1}(k\omega b_{-1} - a_{-1})}{k\omega b_{-2}}; \quad a_{-2} = 0 \\ b_1 = \dfrac{16b_{-2}^3(k\omega)^4 + k\omega b_{-1}a_{-1}^3 - a_1^4}{4a_{-1}(k\omega)^3 b_{-2}^2}; \\ b_0 = \dfrac{4a_{-1}^3 b_{-1}k\omega + 16b_{-2}^3(k\omega)^4 - 3a_{-1}^4}{4a_{-1}^2(k\omega)^2 b_{-2}}. \end{cases} \tag{42c}$$

By (42a), we obtain

$$u(x,t) = \ln\left[\frac{a_0}{\exp(2kx + 2\omega t + 2\varphi) + \frac{a_0}{4k\omega} + \frac{a_0^2}{64k^2\omega^2}\exp(-2kx - 2\omega t - 2\varphi)}\right] \tag{43a}$$

because a_0, k, ω are free parameters, and Eq. (43a) is the same as Eq. (34).

By (42b), we obtain

$$u(x,t) = \ln\left\{\frac{a_1\exp(kx+\omega t+\varphi) - \frac{a_1^2}{k\omega} + \frac{a_1^3}{4k^2\omega^2}\exp(-kx-\omega t-\varphi)}{\exp(2kx+2\omega t+2\varphi) - \frac{a_1^2}{2k^2\omega^2} + \frac{a_1^4}{16k^4\omega^4}\exp(-2kx-2\omega t-2\varphi)}\right\}. \tag{43b}$$

Simplifying Eq. (43b) yields

$$\begin{aligned} u &= \ln\left\{\frac{a_1\left[\exp(kx+\omega t+\varphi) + \frac{a_1^2}{4k^2\omega^2}\exp(-kx-\omega t) - \frac{a_1}{k\omega} - \varphi\right]}{\left[\exp(kx+\omega t+\varphi) + \frac{a_1^2}{4k^2\omega^2}\exp(-kx-\omega t-\varphi) - \frac{a_1}{k\omega}\right]\left[\exp(kx+\omega t+\varphi) + \frac{a_1^2}{4k^2\omega^2}\exp(-kx-\omega t-\varphi) + \frac{a_1}{k\omega}\right]}\right\} \\ &= \ln\left[\frac{a_1}{\exp(kx+\omega t+\varphi) + \frac{a_1^2}{4k^2\omega^2}\exp(-kx-\omega t-\varphi) + \frac{a_1}{k\omega}}\right]. \end{aligned} \tag{43c}$$

It is also the same as the solution, Eq. (34), in Case 1.

By (42c), we have

$$u(x,t) = \ln\left(\frac{A}{B}\right). \tag{44a}$$

where

$$\begin{aligned} A &= \frac{4b_{-2}(k\omega)^2}{a_{-1}}\exp(kx+\omega t+\varphi) + \frac{a_{-1}(k\omega b_{-1} - a_{-1})}{k\omega b_{-2}} + a_{-1}\exp(-kx-\omega t-\varphi) \\ B &= \exp(2kx+2\omega t+2\varphi) + b_{-2}\exp(-2kx-2\omega t-2\varphi) + \frac{4a_{-1}^3 b_{-1}k\omega + 16b_{-2}^3(k\omega)^4 - 3a_{-1}^4}{4a_{-1}^2(k\omega)^2 b_{-2}} \\ &\quad + \frac{16b_{-2}^3(k\omega)^4 + k\omega b_{-1}a_{-1}^3 - a_1^4}{4a_{-1}(k\omega)^3 b_{-2}^2}\exp(kx+\omega t+\varphi) + b_{-1}\exp(-kx-\omega t-\varphi). \end{aligned}$$

We simplify Eq. (44a) as follows

$$\begin{aligned} u(x,t) &= \ln\left\{\frac{A}{A\times[a_{-1}^2\exp(kx+\omega t) + 4a_{-1}k\omega b_{-2} + 4b_{-2}^2k^2w^2\exp(-kx-\omega t)]\times\frac{1}{4k^2w^2a_{-1}b_{-2}}}\right\} \\ &= \ln\left[\frac{4k^2w^2a_{-1}b_{-2}}{a_{-1}^2\exp(kx+\omega t) + 4a_{-1}k\omega b_{-2} + 4b_{-2}^2k^2w^2\exp(-kx-\omega t)}\right] \\ &= \ln\left[\frac{\frac{4k^2w^2b_{-2}}{a_1}}{\exp(kx+\omega t) + \frac{4k\omega b_{-2}}{a_1} + \frac{4b_{-2}^2k^2w^2}{a_1^2}\exp(-kx-\omega t)}\right] \end{aligned} \tag{44b}$$

which is the same as Eq. (34) in Case 1.

Case 3. Supposing that $v = \sum_{i=0}^{n} a_i\phi^i(k\xi)$, where ϕ satisfies the general Riccati equation $\mathrm{d}\phi/\mathrm{d}\xi = r + p\phi + q\phi^2$, we have

$$\phi = \frac{\frac{-p-\sqrt{p^2-4qr}}{2q}\exp\left(\frac{\sqrt{p^2-4qr}}{2}\xi+\varphi\right) + b_{-1}\frac{-p+\sqrt{p^2-4qr}}{2q}\exp\left(-\frac{\sqrt{p^2-4qr}}{2}\xi-\varphi\right)}{\exp\left(\frac{\sqrt{p^2-4qr}}{2}\xi+\varphi\right) + b_{-1}\exp\left(-\frac{\sqrt{p^2-4qr}}{2}\xi-\varphi\right)}.$$

Let $\xi = x + ct$, and Eq. (32) will be changed into

$$cvv'' - c(v')^2 + v^3 = 0. \tag{45}$$

By balancing v''' and vv'' in Eq. (45), we obtain $n = 2$, so we suppose that Eq. (45) can be expressed in the form

$$v = a_0 + a_1\phi(\xi) + a_2\phi(\xi)^2. \tag{46}$$

Substituting Eq. (46) into Eq. (45), we obtain a polynomial of ϕ^n. Setting the coefficients of ϕ^n to be zero, and solving this system, we obtain

$$a_0 = -2qrc, \qquad a_1 = 0, \qquad a_2 = -2cq^2, \qquad p = 0.$$

$$\begin{aligned}\text{So } v &= a_0 + a_1\phi(\xi) + a_2\phi(\xi)^2 \\ &= -2qrc - 2cq^2\left[\frac{\frac{-\sqrt{-4qr}}{2q}\exp\left(\frac{\sqrt{-4qr}}{2}\xi+\varphi\right) + b_{-1}\frac{\sqrt{-4qr}}{2q}\exp\left(-\frac{\sqrt{-4qr}}{2}\xi-\varphi\right)}{\exp\left(\frac{\sqrt{-4qr}}{2}\xi+\varphi\right) + b_{-1}\exp\left(-\frac{\sqrt{-4qr}}{2}\xi-\varphi\right)}\right]^2 \\ &= \frac{-8cqrb_{-1}}{\exp(\sqrt{-4qr}(x+ct)+2\varphi) + 2b_{-1} + b_{-1}^2\exp(-\sqrt{-4qr}(x+ct)-2\varphi)}\end{aligned} \tag{47}$$

and

$$u(x,t) = \ln\left[\frac{-8cqrb_{-1}}{\exp(\sqrt{-4qr}(x+ct)+2\varphi) + 2b_{-1} + b_{-1}^2\exp(-\sqrt{-4qr}(x+ct)-2\varphi)}\right]. \tag{48}$$

If we let $k = \sqrt{-4qr}$, $\omega = \sqrt{-4qr}c$, $\tilde{\varphi} = 2\varphi$ $a_0 = 2k\omega b_{-1}$, Eq. (48) can be translated into

$$u = \ln\left[\frac{a_0}{\exp(kx+\omega t+\tilde{\varphi}) + \frac{a_0}{k\omega} + \frac{a_0^2}{4k^2\omega^2}\exp(-kx-\omega t-\tilde{\varphi})}\right]. \tag{49}$$

It is obvious that Eq. (49) is the same as Eq. (34).

4. Comparison between the Exp-method and the extended modified Fan sub-equation method

Yomba [33] has found 36 solutions for the generalized Riccati equation $\phi' = r + p\phi + q\phi^2$. Now we can compare Eqs. (18) and (19) with the solutions obtained by Yomba [33]:

$$\begin{aligned}\phi_1^{\mathrm{I}} &= \frac{-1}{2q}\left[p + \sqrt{p^2-4qr}\tanh\left(\frac{\sqrt{p^2-4qr}}{2}\xi\right)\right] \\ &= \frac{-1}{2q}\left[p + \sqrt{p^2-4qr}\,\frac{\exp\left(\frac{\sqrt{p^2-4qr}}{2}\xi\right) - \exp\left(-\frac{\sqrt{p^2-4qr}}{2}\xi\right)}{\exp\left(\frac{\sqrt{p^2-4qr}}{2}\xi\right) + \exp\left(-\frac{\sqrt{p^2-4qr}}{2}\xi\right)}\right] \\ &= \frac{\frac{p+\sqrt{p^2-4qr}}{-2q}\exp\left(\frac{\sqrt{p^2-4qr}}{2}\xi\right) + \frac{p-\sqrt{p^2-4qr}}{-2q}\exp\left(-\frac{\sqrt{p^2-4qr}}{2}\xi\right)}{\exp\left(\frac{\sqrt{p^2-4qr}}{2}\xi\right) + \exp\left(-\frac{\sqrt{p^2-4qr}}{2}\xi\right)}.\end{aligned}$$

It is obviously that ϕ_1^{I} is only a special case when $b_{-1} = 1$ and $\varphi = 0$ in Eq. (18),

$$\begin{aligned}\phi_2^{\mathrm{I}} &= \frac{-1}{2q}\left[p + \sqrt{p^2-4qr}\coth\left(\frac{\sqrt{p^2-4qr}}{2}\xi\right)\right] \\ &= \frac{\frac{p+\sqrt{p^2-4qr}}{-2q}\exp\left(\frac{\sqrt{p^2-4qr}}{2}\xi\right) - \frac{(p-\sqrt{p^2-4qr})}{-2q}\exp\left(-\frac{\sqrt{p^2-4qr}}{2}\xi\right)}{\exp\left(\frac{\sqrt{p^2-4qr}}{2}\xi\right) - \exp\left(-\frac{\sqrt{p^2-4qr}}{2}\xi\right)}\end{aligned}$$

and ϕ_2^{I} is only a special case, when $b_{-1} = -1$ and $\varphi = 0$ in Eq. (18)

$$\begin{aligned}
\phi_3^{\mathrm{I}} &= \frac{-1}{2q}\left[p+\sqrt{p^2-4qr}\left[\tanh\left(\sqrt{p^2-4qr}\xi\right) \pm \mathrm{i}\,\mathrm{sech}\left(\sqrt{p^2-4qr}\xi\right)\right]\right] \\
&= \frac{-1}{2q}\left[p+\sqrt{p^2-4qr}\left[\tanh\left(\sqrt{p^2-4qr}\xi\right) \pm \mathrm{i}\,\mathrm{sech}\left(\sqrt{p^2-4qr}\xi\right)\right]\right] \\
&= \frac{-1}{2q}\left[p+\sqrt{p^2-4qr}\left[\frac{\exp\left(\sqrt{p^2-4qr}\xi\right)-\exp\left(-\sqrt{p^2-4qr}\xi\right)\pm 2\mathrm{i}}{\exp\left(\sqrt{p^2-4qr}\xi\right)+\exp\left(-\sqrt{p^2-4qr}\xi\right)}\right]\right] \\
&= \frac{-1}{2q}\left[p+\sqrt{p^2-4qr}\right. \\
&\quad \left.\times\left\{\frac{\left[\exp\left(\frac{\sqrt{p^2-4qr}\xi}{2}\right)\pm \mathrm{i}\exp\left(\frac{-\sqrt{p^2-4qr}\xi}{2}\right)\right]^2}{\left[\exp\left(\frac{\sqrt{p^2-4qr}\xi}{2}\right)+\mathrm{i}\exp\left(\frac{-\sqrt{p^2-4qr}\xi}{2}\right)\right]\times\left[\exp\left(\frac{\sqrt{p^2-4qr}\xi}{2}\right)-\mathrm{i}\exp\left(\frac{-\sqrt{p^2-4qr}\xi}{2}\right)\right]}\right\}\right] \\
&= \frac{-1}{2q}\left[p+\sqrt{p^2-4qr}\right. \\
&\quad \left.\times\left\{\frac{\exp\left(\frac{\sqrt{p^2-4qr}\xi}{2}\right)\pm \mathrm{i}\exp\left(\frac{-\sqrt{p^2-4qr}\xi}{2}\right)}{\exp\left(\frac{\sqrt{p^2-4qr}\xi}{2}\right)\mp \mathrm{i}\exp\left(\frac{-\sqrt{p^2-4qr}\xi}{2}\right)}\right\}\right] \\
&= \frac{\frac{p+\sqrt{p^2-4qr}}{-2q}\exp\left(\frac{\sqrt{p^2-4qr}\xi}{2}\right)+\frac{\mp p\mathrm{i}\pm\mathrm{i}\sqrt{p^2-4qr}}{-2q}\exp\left(\frac{-\sqrt{p^2-4qr}\xi}{2}\right)}{\exp\left(\frac{\sqrt{p^2-4qr}\xi}{2}\right)\mp \mathrm{i}\exp\left(\frac{-\sqrt{p^2-4qr}\xi}{2}\right)}.
\end{aligned}$$

When $b_{-1} = \pm\mathrm{i}$ and $\varphi = 0$, Eq. (18) reduces to ϕ_3^{I}

$$\begin{aligned}
\phi_4^{\mathrm{I}} &= \frac{-1}{2q}\left[p+\sqrt{p^2-4qr}\left[\coth\left(\sqrt{p^2-4qr}\xi\right) \pm \mathrm{csch}\left(\sqrt{p^2-4qr}\xi\right)\right]\right] \\
&= \frac{\frac{p+\sqrt{p^2-4qr}}{-2q}\exp\left(\frac{\sqrt{p^2-4qr}\xi}{2}\right)+\frac{\mp p\pm\sqrt{p^2-4qr}}{-2q}\exp\left(\frac{-\sqrt{p^2-4qr}\xi}{2}\right)}{\exp\left(\frac{\sqrt{p^2-4qr}\xi}{2}\right)\mp \exp\left(\frac{-\sqrt{p^2-4qr}\xi}{2}\right)}.
\end{aligned}$$

When $b_{-1} = \pm 1$ and $\varphi = 0$, Eq. (18) reduces to ϕ_4^{I}

$$\begin{aligned}
\phi_5^{\mathrm{I}} &= \frac{-1}{4q}\left[2p+\sqrt{p^2-4qr}\left[\tanh\left(\frac{\sqrt{p^2-4qr}\xi}{4}\right)+\coth\left(\frac{\sqrt{p^2-4qr}\xi}{4}\right)\right]\right] \\
&= \frac{-1}{4q}\left[2p+\sqrt{p^2-4qr}\left[\frac{2\exp\left(\frac{\sqrt{p^2-4qr}\xi}{2}\right)+2\exp\left(\frac{-\sqrt{p^2-4qr}\xi}{2}\right)}{\exp\left(\frac{\sqrt{p^2-4qr}\xi}{2}\right)-\exp\left(\frac{-\sqrt{p^2-4qr}\xi}{2}\right)}\right]\right]
\end{aligned}$$

$$= \frac{\frac{p+\sqrt{p^2-4qr}}{-2q}\exp\left(\frac{\sqrt{p^2-4qr}\xi}{2}\right) - \frac{p-\sqrt{p^2-4qr}}{-2q}\exp\left(\frac{-\sqrt{p^2-4qr}\xi}{2}\right)}{\exp\left(\frac{\sqrt{p^2-4qr}\xi}{2}\right) - \exp\left(\frac{-\sqrt{p^2-4qr}\xi}{2}\right)}$$

ϕ_5^{I} is only a special case in Eq. (18), when $b_{-1} = -1$

$$\phi_6^{\mathrm{I}} = \frac{1}{2q}\left[-p + \frac{\sqrt{\left(A^2+B^2\right)\left(p^2-4qr\right)} - A\sqrt{\left(p^2-4qr\right)}\coth\left(\sqrt{p^2-4qr}\xi\right)}{A\sinh\left(\sqrt{p^2-4qr}\xi\right)+B}\right]$$

where A and B are two non-zero real constants, and satisfies $B^2 > A^2$.

Let $L = \frac{B}{A}$, so $L > 1$ or $L < -1$ in case $A < 0$, $\frac{\sqrt{A^2+B^2}}{\mathrm{A}} = -\sqrt{1+L^2}$, and in case $A > 0$, $\frac{\sqrt{A^2+B^2}}{A} = \sqrt{1+L^2}$, thus we have

$$\phi_6^{\mathrm{I}} = \frac{1}{2q}\left[-p + \sqrt{\left(p^2-4qr\right)}\frac{\pm 2\sqrt{1+L^2} - \exp\left(\sqrt{p^2-4qr}\xi\right) - \exp\left(\sqrt{p^2-4qr}\xi\right)}{\exp\left(\sqrt{p^2-4qr}\xi\right) - \exp\left(-\sqrt{p^2-4qr}\xi\right) + 2L}\right].$$

In case $A > 0$, we have

$$\phi_6^{\mathrm{I}} = \frac{1}{2q}\left[-p + \sqrt{\left(p^2-4qr\right)}\frac{2\sqrt{1+L^2} - \exp\left(\sqrt{p^2-4qr}\xi\right) - \exp\left(\sqrt{p^2-4qr}\xi\right)}{\exp\left(\sqrt{p^2-4qr}\xi\right) - \exp\left(-\sqrt{p^2-4qr}\xi\right) + 2L}\right]$$

$$= \frac{1}{2q}\left[-p - \sqrt{\left(p^2-4qr\right)}\frac{\exp\left(2\sqrt{p^2-4qr}\xi\right) - 2\sqrt{1+L^2}\exp\left(\sqrt{p^2-4qr}\xi\right) + 1}{\exp\left(2\sqrt{p^2-4qr}\xi\right) + 2L\exp\left(\sqrt{p^2-4qr}\xi\right) - 1}\right]$$

$$= \frac{1}{2q}\left[-p - \sqrt{\left(p^2-4qr\right)}\frac{\left(\exp\left(\sqrt{p^2-4qr}\xi\right) - \sqrt{1+L^2}\right)^2 + 1 - 1 + L^2}{\left(\exp\left(\sqrt{p^2-4qr}\xi\right) + L\right)^2 - L^2 - 1}\right]$$

$$= \frac{1}{2q}\left[-p - \sqrt{\left(p^2-4qr\right)}\right.$$

$$\left.\times\ \frac{\left(\exp\left(\sqrt{p^2-4qr}\xi\right) - \sqrt{1+L^2} - L\right) \times \left(\exp\left(\sqrt{p^2-4qr}\xi\right) - \sqrt{1+L^2} + L\right)}{\left(\exp\left(\sqrt{p^2-4qr}\xi\right) + L - \sqrt{1+L^2}\right) \times \left(\exp\left(\sqrt{p^2-4qr}\xi\right) + L + \sqrt{1+L^2}\right)}\right]$$

$$= \frac{1}{2q}\left[-p - \sqrt{\left(p^2-4qr\right)}\frac{\left(\exp\left(\sqrt{p^2-4qr}\xi\right) - \sqrt{1+L^2} - L\right)}{\left(\exp\left(\sqrt{p^2-4qr}\xi\right) + L + \sqrt{1+L^2}\right)}\right]$$

$$= \frac{1}{2q}\left[-p - \sqrt{\left(p^2-4qr\right)}\frac{\left(\exp\left(\sqrt{p^2-4qr}\xi\right) - M\right)}{\left(\exp\left(\sqrt{p^2-4qr}\xi\right) + M\right)}\right]$$

where $M = L + \sqrt{1+L^2}$, so $0 < M < \sqrt{2} - 1$, or $M > \sqrt{2} + 1$

$$= \frac{1}{2q}\left[-p - \sqrt{p^2-4qr}\frac{\left(\exp\left(\frac{\sqrt{p^2-4qr}\xi}{2}\right) - M\exp\left(-\frac{\sqrt{p^2-4qr}\xi}{2}\right)\right)}{\left(\exp\left(\frac{\sqrt{p^2-4qr}\xi}{2}\right) + M\exp\left(-\frac{\sqrt{p^2-4qr}\xi}{2}\right)\right)}\right]$$

$$= \frac{\left(\frac{p+\sqrt{p^2-4qr}}{-2q}\exp\left(\frac{\sqrt{p^2-4qr}\xi}{2}\right) - M\frac{p-\sqrt{p^2-4qr}}{-2q}\exp\left(-\frac{\sqrt{p^2-4qr}\xi}{2}\right)\right)}{\left(\exp\left(\frac{\sqrt{p^2-4qr}\xi}{2}\right) + M\exp\left(-\frac{\sqrt{p^2-4qr}\xi}{2}\right)\right)}.$$

It is only a special case in Eq. (18), when $0 < b_{-1} < \sqrt{2}-1$, or $b_{-1} > \sqrt{2}+1b_{-1} > 0$, and $\varphi = 0$. In case $A < 0$, we obtain

$$\phi_6^{\mathrm{I}} = \frac{1}{2q}\left[-p + \sqrt{(p^2-4qr)}\frac{-2\sqrt{1+L^2} - \exp\left(\sqrt{p^2-4qr}\xi\right) - \exp\left(-\sqrt{p^2-4qr}\xi\right)}{\exp\left(\sqrt{p^2-4qr}\xi\right) - \exp\left(-\sqrt{p^2-4qr}\xi\right) + 2L}\right]$$

$$= \frac{1}{2q}\left[-p - \sqrt{(p^2-4qr)}\frac{\exp\left(2\sqrt{p^2-4qr}\xi\right) + 2\sqrt{1+L^2}\exp\left(\sqrt{p^2-4qr}\xi\right) + 1}{\exp\left(2\sqrt{p^2-4qr}\xi\right) + 2L\exp\left(\sqrt{p^2-4qr}\xi\right) - 1}\right]$$

$$= \frac{1}{2q}\left[-p - \sqrt{(p^2-4qr)}\frac{\left(\exp\left(\sqrt{p^2-4qr}\xi\right) + \sqrt{1+L^2} - L\right)}{\left(\exp\left(\sqrt{p^2-4qr}\xi\right) + L - \sqrt{1+L^2}\right)}\right]$$

$$= \frac{1}{2q}\left[-p - \sqrt{(p^2-4qr)}\frac{\left(\exp\left(\sqrt{p^2-4qr}\xi\right) - M\right)}{\left(\exp\left(\sqrt{p^2-4qr}\xi\right) + M\right)}\right]$$

where $M = L - \sqrt{1+L^2}$, so $1-\sqrt{2} < M < 0$, or $M < -\sqrt{2}-1$

$$= \frac{\left(\frac{p+\sqrt{p^2-4qr}}{-2q}\exp\left(\frac{\sqrt{p^2-4qr}\xi}{2}\right) - M\frac{p-\sqrt{p^2-4qr}}{-2q}\exp\left(-\frac{\sqrt{p^2-4qr}\xi}{2}\right)\right)}{\left(\exp\left(\frac{\sqrt{p^2-4qr}\xi}{2}\right) + M\exp\left(-\frac{\sqrt{p^2-4qr}\xi}{2}\right)\right)}.$$

It is only a special case in Eq. (18), when $1-\sqrt{2} < b_{-1} < 0$, or $b_{-1} < -\sqrt{2}-1$ and $\varphi = 0$

$$\phi_7^{\mathrm{I}} = \frac{1}{2q}\left[-p - \frac{\sqrt{(B^2-A^2)(p^2-4qr)} - A\sqrt{(p^2-4qr)}\sinh\left(\sqrt{p^2-4qr}\xi\right)}{A\cosh\left(\sqrt{p^2-4qr}\xi\right) + B}\right]$$

where A and B are two non-zero real constants, and satisfies $B^2 > A^2$

$$\phi_7^{\mathrm{I}} = \frac{1}{2q}\left[-p - \sqrt{(p^2-4qr)}\frac{\pm 2\sqrt{L^2-1} + \exp\left(\sqrt{p^2-4qr}\xi\right) - \exp\left(\sqrt{p^2-4qr}\xi\right)}{\exp\left(\sqrt{p^2-4qr}\xi\right) + \exp\left(-\sqrt{p^2-4qr}\xi\right) + 2L}\right]$$

where $L = \frac{B}{A}$, in case $A > 0$, $\frac{\sqrt{B^2-A^2}}{A} = \sqrt{L^2-1}$;

in case $A < 0$, $\frac{\sqrt{B^2-A^2}}{A} = -\sqrt{L^2-1}$.

In case $A > 0$, we have

$$\phi_7^{\mathrm{I}} = \frac{1}{2q}\left[-p - \sqrt{(p^2-4qr)}\frac{2\sqrt{L^2-1} + \exp\left(\sqrt{p^2-4qr}\xi\right) - \exp\left(\sqrt{p^2-4qr}\xi\right)}{\exp\left(\sqrt{p^2-4qr}\xi\right) + \exp\left(-\sqrt{p^2-4qr}\xi\right) + 2L}\right]$$

$$= \frac{1}{2q}\left[-p - \sqrt{(p^2-4qr)}\frac{\left(\exp\left(\sqrt{p^2-4qr}\xi\right) - M\right)}{\left(\exp\left(\sqrt{p^2-4qr}\xi\right) + M\right)}\right]$$

$$\text{where } M = L - \sqrt{L^2-1}, \text{ so } 0 < M < 1, \text{ or } M < -1$$

$$= \frac{\left(\frac{p+\sqrt{p^2-4qr}}{-2q}\exp\left(\frac{\sqrt{p^2-4qr}\xi}{2}\right) - M\frac{p-\sqrt{p^2-4qr}}{-2q}\exp\left(-\frac{\sqrt{p^2-4qr}\xi}{2}\right)\right)}{\left(\exp\left(\frac{\sqrt{p^2-4qr}\xi}{2}\right) + M\exp\left(-\frac{\sqrt{p^2-4qr}\xi}{2}\right)\right)}.$$

It is only a special case when $0 < b_{-1} < 1$, or $b_{-1} < -\sqrt{2}-1$ and $\varphi = 0$.
In case $A < 0$, we obtain

$$\phi_7^{\mathrm{I}} = \frac{1}{2q}\left[-p - \sqrt{(p^2-4qr)}\frac{-2\sqrt{L^2-1} + \exp\left(\sqrt{p^2-4qr}\xi\right) - \exp\left(\sqrt{p^2-4qr}\xi\right)}{\exp\left(\sqrt{p^2-4qr}\xi\right) + \exp\left(-\sqrt{p^2-4qr}\xi\right) + 2L}\right]$$

$$= \frac{\left(\frac{p+\sqrt{p^2-4qr}}{-2q}\exp\left(\frac{\sqrt{p^2-4qr}\xi}{2}\right) - M\frac{p-\sqrt{p^2-4qr}}{-2q}\exp\left(-\frac{\sqrt{p^2-4qr}\xi}{2}\right)\right)}{\left(\exp\left(\frac{\sqrt{p^2-4qr}\xi}{2}\right) + M\exp\left(-\frac{\sqrt{p^2-4qr}\xi}{2}\right)\right)}$$

$$\text{where } M = L + \sqrt{L^2-1}, \text{ so } M > 1, \text{ or } -1 < M < 0.$$

It is only a special case when $-1 < b_{-1} < 0$, or $b_{-1} > 1$ and $\varphi = 0$

$$\phi_8^{\mathrm{I}} = \frac{2r\cosh\left(\frac{\sqrt{p^2-4qr}\xi}{2}\right)}{\sqrt{p^2-4qr}\sinh\left(\frac{\sqrt{p^2-4qr}\xi}{2}\right) - p\cosh\left(\frac{\sqrt{p^2-4qr}\xi}{2}\right)}$$

$$= \frac{2r\exp\left(\frac{\sqrt{p^2-4qr}\xi}{2}\right) + 2r\exp\left(-\frac{\sqrt{p^2-4qr}\xi}{2}\right)}{\left(\sqrt{p^2-4qr}-p\right)\exp\left(\frac{\sqrt{p^2-4qr}\xi}{2}\right) - \left(\sqrt{p^2-4qr}+p\right)\cosh\left(\frac{\sqrt{p^2-4qr}\xi}{2}\right)}.$$

If we let $b_{-1} = \frac{-p-\sqrt{p^2-4qr}}{-p+\sqrt{p^2-4qr}}$, Eq. (18) reduces to ϕ_8^{I}

$$\phi_9^{\mathrm{I}} = \frac{-2r\sinh\left(\frac{\sqrt{p^2-4qr}\xi}{2}\right)}{p\sinh\left(\frac{\sqrt{p^2-4qr}\xi}{2}\right) - \sqrt{p^2-4qr}\cosh\left(\frac{\sqrt{p^2-4qr}\xi}{2}\right)}$$

$$= \frac{2r\exp\left(\frac{\sqrt{p^2-4qr}\xi}{2}\right) - 2r\exp\left(-\frac{\sqrt{p^2-4qr}\xi}{2}\right)}{\left(\sqrt{p^2-4qr}-p\right)\exp\left(\frac{\sqrt{p^2-4qr}\xi}{2}\right) + \left(\sqrt{p^2-4qr}+p\right)\exp\left(-\frac{\sqrt{p^2-4qr}\xi}{2}\right)}.$$

If we let $b_{-1} = \frac{p+\sqrt{p^2-4qr}}{-p+\sqrt{p^2-4qr}}$ and $\varphi = 0$, Eq. (18) becomes ϕ_9^{I}

$$\phi_{10}^{\mathrm{I}} = \frac{2r\cosh\left(\sqrt{p^2-4qr}\xi\right)}{\sqrt{p^2-4qr}\sinh\left(\sqrt{p^2-4qr}\xi\right) - p\cosh\left(\sqrt{p^2-4qr}\right) \pm \mathrm{i}\sqrt{p^2-4qr}}$$

$$= \frac{2r\exp\left(\sqrt{p^2-4qr}\xi\right) + 2r\exp\left(-\sqrt{p^2-4qr}\xi\right)}{\left(\sqrt{p^2-4qr}-p\right)\exp\left(\sqrt{p^2-4qr}\xi\right) - \left(\sqrt{p^2-4qr}+p\right)\exp\left(-\sqrt{p^2-4qr}\xi\right) \pm 2\mathrm{i}\sqrt{p^2-4qr}}$$

$$= \frac{2r\exp\left(2\sqrt{p^2-4qr}\xi\right)+2r}{\left(\sqrt{p^2-4qr}-p\right)\exp\left(2\sqrt{p^2-4qr}\xi\right)\pm 2\mathrm{i}\sqrt{p^2-4qr}\exp\left(\sqrt{p^2-4qr}\xi\right)-\left(\sqrt{p^2-4qr}+p\right)}$$

$$= \frac{\frac{2r}{\left(\sqrt{p^2-4qr}-p\right)}\left[\exp\left(2\sqrt{p^2-4qr}\xi\right)+1\right]}{\exp\left(2\sqrt{p^2-4qr}\xi\right)\pm 2\mathrm{i}\frac{\sqrt{p^2-4qr}}{\left(\sqrt{p^2-4qr}-p\right)}\exp\left(\sqrt{p^2-4qr}\xi\right)-\frac{\left(\sqrt{p^2-4qr}+p\right)}{\left(\sqrt{p^2-4qr}-p\right)}}$$

$$= \frac{\frac{2r}{\left(\sqrt{p^2-4qr}-p\right)}\left[\exp\left(2\sqrt{p^2-4qr}\xi\right)+1\right]}{\exp\left(2\sqrt{p^2-4qr}\xi\right)\pm 2\mathrm{i}\frac{\sqrt{p^2-4qr}}{\left(\sqrt{p^2-4qr}-p\right)}\exp\left(\sqrt{p^2-4qr}\xi\right)-\frac{\left(\sqrt{p^2-4qr}+p\right)}{\left(\sqrt{p^2-4qr}-p\right)}}$$

$$= \frac{\frac{2r}{\left(\sqrt{p^2-4qr}-p\right)}\left[\exp\left(\sqrt{p^2-4qr}\xi\right)+\mathrm{i}\right]\left[\exp\left(\sqrt{p^2-4qr}\xi\right)-\mathrm{i}\right]}{\left[\exp\left(\sqrt{p^2-4qr}\xi\right)\pm\mathrm{i}\frac{\sqrt{p^2-4qr}}{\left(\sqrt{p^2-4qr}-p\right)}\right]^2+\frac{p^2}{\left(\sqrt{p^2-4qr}-p\right)^2}}$$

$$= \frac{\frac{2r}{\left(\sqrt{p^2-4qr}-p\right)}\left[\exp\left(\sqrt{p^2-4qr}\xi\right)+\mathrm{i}\right]\left[\exp\left(\sqrt{p^2-4qr}\xi\right)-\mathrm{i}\right]}{\left[\exp\left(\sqrt{p^2-4qr}\xi\right)+\mathrm{i}\frac{\pm\sqrt{p^2-4qr}+p}{\left(\sqrt{p^2-4qr}-p\right)}\right]\left[\exp\left(\sqrt{p^2-4qr}\xi\right)+\mathrm{i}\frac{\pm\sqrt{p^2-4qr}-p}{\left(\sqrt{p^2-4qr}-p\right)}\right]}$$

$$= \frac{\frac{2r}{\left(\sqrt{p^2-4qr}-p\right)}\left[\exp\left(\sqrt{p^2-4qr}\xi\right)\mp\mathrm{i}\right]}{\left[\exp\left(\sqrt{p^2-4qr}\xi\right)\pm\mathrm{i}\frac{\sqrt{p^2-4qr}+p}{\left(\sqrt{p^2-4qr}-p\right)}\right]}$$

$$= \frac{\frac{2r}{\left(\sqrt{p^2-4qr}-p\right)}\left[\exp\left(\frac{\sqrt{p^2-4qr}\xi}{2}\right)\mp\mathrm{i}\exp\left(-\frac{\sqrt{p^2-4qr}\xi}{2}\right)\right]}{\left[\exp\left(\frac{\sqrt{p^2-4qr}\xi}{2}\right)\pm\mathrm{i}\frac{\sqrt{p^2-4qr}+p}{\left(\sqrt{p^2-4qr}-p\right)}\exp\left(-\frac{\sqrt{p^2-4qr}\xi}{2}\right)\right]}.$$

If we let $b_{-1}=\pm\frac{-p-\sqrt{p^2-4qr}}{-p+\sqrt{p^2-4qr}}\mathrm{i}$ and $\varphi=0$, Eq. (18) turns out to be ϕ_{10}^{I}

$$\phi_{11}^{\mathrm{I}} = \frac{2r\sinh\left(\sqrt{p^2-4qr}\xi\right)}{-p\sinh\left(\sqrt{p^2-4qr}\xi\right)+\sqrt{p^2-4qr}\cosh\left(\sqrt{p^2-4qr}\right)\pm\sqrt{p^2-4qr}}$$

$$= \frac{\frac{2r}{\left(\sqrt{p^2-4qr}-p\right)}\left[\exp\left(\frac{\sqrt{p^2-4qr}\xi}{2}\right)\mp\exp\left(-\frac{\sqrt{p^2-4qr}\xi}{2}\right)\right]}{\left[\exp\left(\frac{\sqrt{p^2-4qr}\xi}{2}\right)\pm\frac{\sqrt{p^2-4qr}+p}{\left(\sqrt{p^2-4qr}-p\right)}\exp\left(-\frac{\sqrt{p^2-4qr}\xi}{2}\right)\right]}.$$

If we let $b_{-1}=\pm\frac{-p-\sqrt{p^2-4qr}}{-p+\sqrt{p^2-4qr}}$ and $\varphi=0$, Eq. (18) is equivalent to ϕ_{11}^{I}

$$\phi_{12}^{\mathrm{I}} = \frac{4r\cosh\left(\frac{\sqrt{p^2-4qr}\xi}{4}\right)\sinh\left(\frac{\sqrt{p^2-4qr}\xi}{4}\right)}{-2p\cosh\left(\frac{\sqrt{p^2-4qr}\xi}{4}\right)\sinh\left(\frac{\sqrt{p^2-4qr}\xi}{4}\right)+2\sqrt{p^2-4qr}\cosh^2\left(\frac{\sqrt{p^2-4qr}\xi}{4}\right)-\sqrt{p^2-4qr}}$$
$$= \frac{2\exp\left(\frac{\sqrt{p^2-4qr}\xi}{2}\right)-2\exp\left(-\frac{\sqrt{p^2-4qr}\xi}{2}\right)}{\left(\sqrt{p^2-4qr}-p\right)\exp\left(\frac{\sqrt{p^2-4qr}\xi}{2}\right)+\left(\sqrt{p^2-4qr}+p\right)\exp\left(-\frac{\sqrt{p^2-4qr}\xi}{2}\right)}.$$

If we let $b_{-1} = -\frac{-p-\sqrt{p^2-4qr}}{-p+\sqrt{p^2-4qr}}$ and $\varphi = 0$, Eq. (18) becomes ϕ_{12}^{I}

$$\phi_{13}^{1} = \frac{1}{2q}\left[-p+\sqrt{4qr-p^2}\tan\left(\frac{\sqrt{4qr-p^2}}{2}\xi\right)\right].$$

If we let $b_{-1} = 1$ and $\varphi = 0$, Eq. (19) leads to ϕ_{13}^{1}

$$\phi_{14}^{1} = -\frac{1}{2q}\left[p+\sqrt{4qr-p^2}\cot\left(\frac{\sqrt{4qr-p^2}}{2}\xi\right)\right].$$

It is only a special case of Eq. (19) when $b_{-1} = -1$ and $\varphi = 0$

$$\phi_{15}^{\mathrm{I}} = \frac{1}{2q}\left\{-p+\sqrt{4qr-p^2}\left[\tan\left(\sqrt{4qr-p^2}\xi\right)\pm\sec\left(\sqrt{4qr-p^2}\xi\right)\right]\right\}$$
$$= \frac{1}{2q}\left[-p+\sqrt{4qr-p^2}\frac{\sin\left(\sqrt{4qr-p^2}\xi\right)\pm 1}{\cos\left(\sqrt{4qr-p^2}\xi\right)}\right]$$
$$= \frac{1}{2q}\left\{-p-\sqrt{4qr-p^2}\right.$$
$$\left.\times\frac{\left[\cos\left(\frac{\sqrt{4qr-p^2}\xi}{2}\right)\pm\sin\left(\frac{\sqrt{4qr-p^2}\xi}{2}\right)\right]^2}{\left[\cos\left(\frac{\sqrt{4qr-p^2}\xi}{2}\right)+\sin\left(\frac{\sqrt{4qr-p^2}\xi}{2}\right)\right]\left[\cos\left(\frac{\sqrt{4qr-p^2}\xi}{2}\right)-\sin\left(\frac{\sqrt{4qr-p^2}\xi}{2}\right)\right]}\right\}$$
$$= \frac{1}{2q}\left\{-p-\sqrt{4qr-p^2}\frac{\cos\left(\frac{\sqrt{4qr-p^2}\xi}{2}\right)\pm\sin\left(\frac{\sqrt{4qr-p^2}\xi}{2}\right)}{\left[\cos\left(\frac{\sqrt{4qr-p^2}\xi}{2}\right)\mp\sin\left(\frac{\sqrt{4qr-p^2}\xi}{2}\right)\right]}\right\}.$$

If we let $b_{-1} = \pm\mathrm{i}$ and $\varphi = 0$, Eq. (19) becomes ϕ_{15}^{I}.

$$\phi_{16}^{\mathrm{I}} = \frac{1}{2q}\left\{-p+\sqrt{4qr-p^2}\left[\cot\left(\sqrt{4qr-p^2}\xi\right)\pm\csc\left(\sqrt{4qr-p^2}\xi\right)\right]\right\}$$
$$= \frac{1}{2q}\left(-p+\sqrt{4qr-p^2}\frac{\cos\left(\sqrt{4qr-p^2}\xi\right)\pm 1}{\sin\left(\sqrt{4qr-p^2}\xi\right)}\right)$$

$$= \frac{1}{2q}\left(-p+\sqrt{4qr-p^2}\frac{\cos\left(\frac{\sqrt{4qr-p^2}\xi}{2}\right)}{\sin\left(\frac{\sqrt{4qr-p^2}\xi}{2}\right)}\right) \text{ or } = \frac{1}{2q}\left(-p-\sqrt{4qr-p^2}\frac{\sin\left(\frac{\sqrt{4qr-p^2}\xi}{2}\right)}{\cos\left(\frac{\sqrt{4qr-p^2}\xi}{2}\right)}\right).$$

It is obviously that ϕ_{16}^{I} is only special case of Eq. (19) when $b_{-1} = \pm 1$ and $\varphi = 0$.

$$\phi_{17}^{\mathrm{I}} = \frac{1}{4q}\left\{-2p+\sqrt{4qr-p^2}\left[\tan\left(\frac{\sqrt{4qr-p^2}}{4}\xi\right)-\cot\left(\frac{\sqrt{4qr-p^2}}{4}\xi\right)\right]\right\}$$

$$= \frac{1}{4q}\left\{-2p+\sqrt{4qr-p^2}\left[\frac{\sin\left(\frac{\sqrt{4qr-p^2}}{4}\xi\right)^2-\cos\left(\frac{\sqrt{4qr-p^2}}{4}\xi\right)^2}{\sin\left(\frac{\sqrt{4qr-p^2}}{4}\xi\right)\cos\left(\frac{\sqrt{4qr-p^2}}{4}\xi\right)}\right]\right\}$$

$$= \frac{1}{4q}\left\{-2p-2\sqrt{4qr-p^2}\left[\frac{\cos\left(\frac{\sqrt{4qr-p^2}}{2}\xi\right)}{\sin\left(\frac{\sqrt{4qr-p^2}}{2}\xi\right)}\right]\right\}.$$

If we let $b_{-1} = -1$ and $\varphi = 0$, Eq. (19) turns out to be ϕ_{17}^{I}

$$\phi_{18}^{\mathrm{I}} = \frac{1}{2q}\left\{-p+\frac{\pm\sqrt{\left(A^2-B^2\right)\left(4qr-p^2\right)}-A\sqrt{4qr-p^2}\cos\left(\sqrt{4qr-p^2}\xi\right)}{A\sin\left(\sqrt{4qr-p^2}\right)+B}\right\}$$

where A, B are two non-zero real constants, and satisfies $A^2 - B^2 > 0$, we re-write ϕ_{18}^{I} in the form

$$\phi_{18}^{\mathrm{I}} = \frac{1}{2q}\left\{-p+\sqrt{4qr-p^2}\frac{\pm\sqrt{1-\left(\frac{B}{A}\right)^2}-\cos\left(\sqrt{4qr-p^2}\xi\right)}{\sin\left(\sqrt{4qr-p^2}\right)+\frac{B}{A}}\right\}.$$

Let $\sin\alpha = \frac{B}{A}$, so that $\pm\frac{\sqrt{A^2-B^2}}{A} = \cos\alpha$.
We further simplify ϕ_{18}^{I}

$$\phi_{18}^{\mathrm{I}} = \frac{1}{2q}\left\{-p+\sqrt{4qr-p^2}\times\frac{\cos(\alpha)-\cos\left(\sqrt{4qr-p^2}\xi\right)}{\sin\left(\sqrt{4qr-p^2}\xi\right)+\sin(\alpha)}\right\}$$

$$= \frac{1}{2q}\left\{-p+\sqrt{4qr-p^2}\times\frac{\sin\left(\frac{\sqrt{4qr-p^2}\xi}{2}+\frac{\alpha}{2}\right)\sin\left(\frac{\sqrt{4qr-p^2}\xi}{2}-\frac{\alpha}{2}\right)}{\sin\left(\frac{\sqrt{4qr-p^2}\xi}{2}+\frac{\alpha}{2}\right)\cos\left(\frac{\sqrt{4qr-p^2}\xi}{2}-\frac{\alpha}{2}\right)}\right\}$$

$$= \frac{1}{2q}\left\{-p+\sqrt{4qr-p^2}\times\frac{\sin\left(\frac{\sqrt{4qr-p^2}\xi}{2}-\frac{\alpha}{2}\right)}{\cos\left(\frac{\sqrt{4qr-p^2}\xi}{2}-\frac{\alpha}{2}\right)}\right\}.$$

If we let $\varphi = -\frac{\alpha}{2}$, $b_{-1} = 1$, Eq. (19) becomes ϕ_{18}^{I}

$$\phi_{19}^{\mathrm{I}} = \frac{1}{2q}\left\{-p-\frac{\pm\sqrt{\left(A^2-B^2\right)\left(4qr-p^2\right)}-A\sqrt{4qr-p^2}\sin\left(\sqrt{4qr-p^2}\xi\right)}{A\cos\left(\sqrt{4qr-p^2}\right)+B}\right\}$$

where A, B are two non-zero real constants, and satisfies $A^2 - B^2 > 0$.

Let $\cos\alpha = \frac{B}{A}$, so that $\pm\frac{\sqrt{A^2-B^2}}{A} = \sin\alpha$, ϕ_{19}^{I} becomes

$$\phi_{19}^{\mathrm{I}} = \frac{1}{2q}\left\{-p + \sqrt{4qr - p^2} \times \frac{\sin\left(\frac{\sqrt{4qr-p^2}\xi}{2} - \frac{\alpha}{2}\right)}{\cos\left(\frac{\sqrt{4qr-p^2}\xi}{2} - \frac{\alpha}{2}\right)}\right\}.$$

If we let $\varphi = -\frac{\alpha}{2}$, $b_{-1} = 1$, Eq. (19) becomes ϕ_{19}^{I}.

$$\phi_{20}^{\mathrm{I}} = -\frac{2r\cos\left(\frac{\sqrt{4qr-p^2}}{2}\xi\right)}{\sqrt{4qr-p^2}\sin\left(\frac{\sqrt{4qr-p^2}}{2}\xi\right) + p\cos\left(\frac{\sqrt{4qr-p^2}}{2}\xi\right)}.$$

In view of Eq. (19), we have

$$\phi = \frac{1}{2q} \times \frac{\left[-p\,(1+b_{-1}) - 2\mathrm{i}k\,(1-b_{-1})\right]\cos(k\xi+\varphi) + \mathrm{i}\left[-p\,(1-b_{-1}) - 2\mathrm{i}k\,(1+b_{-1})\right]\sin(k\xi+\varphi)}{(1+b_{-1})\cos(k\xi+\varphi) + \mathrm{i}\,(1-b_{-1})\sin(k\xi+\varphi)}$$

where $k = \frac{\sqrt{4qr-p^2}}{2}$.

If we let $b_{-1} = \frac{p+\mathrm{i}\sqrt{4qr-p^2}}{p-\mathrm{i}\sqrt{4qr-p^2}}$ and $\varphi = 0$, Eq. (19) becomes

$$\begin{aligned}\phi &= \frac{1}{2q} \times \frac{-\frac{2\left(\sqrt{4qr-p^2}^2+p^2\right)}{p-\mathrm{i}\sqrt{4qr-p^2}}\cos\left(\frac{\sqrt{4qr-p^2}}{2}\xi\right)}{\frac{2p}{p-\mathrm{i}\sqrt{4qr-p^2}}\cos\left(\frac{\sqrt{4qr-p^2}}{2}\xi\right) + \frac{2\sqrt{4qr-p^2}}{p-\mathrm{i}\sqrt{4qr-p^2}}\sin\left(\frac{\sqrt{4qr-p^2}}{2}\xi\right)}\\ &= -\frac{2r\cos\left(\frac{\sqrt{4qr-p^2}}{2}\xi\right)}{\sqrt{4qr-p^2}\sin\left(\frac{\sqrt{4qr-p^2}}{2}\xi\right) + p\cos\left(\frac{\sqrt{4qr-p^2}}{2}\xi\right)}.\end{aligned}$$

It reduces to ϕ_{20}^{I}.

$$\phi_{21}^{\mathrm{I}} = \frac{2r\sin\left(\frac{\sqrt{4qr-p^2}}{2}\xi\right)}{-p\sin\left(\frac{\sqrt{4qr-p^2}}{2}\xi\right) + \sqrt{4qr-p^2}\cos\left(\frac{\sqrt{4qr-p^2}}{2}\xi\right)}.$$

It is only a special case when $b_{-1} = -\frac{p+\mathrm{i}\sqrt{4qr-p^2}}{p-\mathrm{i}\sqrt{4qr-p^2}}$ and $\varphi = 0$

$$\phi_{22}^{\mathrm{I}} = \frac{2r\cos\left(\sqrt{4qr-p^2}\xi\right)}{\sqrt{4qr-p^2}\sin\left(\sqrt{4qr-p^2}\xi\right) + p\cos\left(\sqrt{4qr-p^2}\xi\right) \pm \sqrt{4qr-p^2}}.$$

Let $\frac{\sqrt{4qr-p^2}}{4qr} = \cos\alpha$ so $\sin\alpha = \frac{p}{4qr}$.
Consider first the case

$$\phi_{22}^{\mathrm{I}} = \frac{2r\cos\left(\sqrt{4qr-p^2}\xi\right)}{\sqrt{4qr-p^2}\sin\left(\sqrt{4qr-p^2}\xi\right) + p\cos\left(\sqrt{4qr-p^2}\xi\right) + \sqrt{4qr-p^2}}$$

$$= \frac{1}{2q}\frac{\cos\left(\sqrt{4qr-p^2}\xi\right)}{\cos\alpha\sin\left(\sqrt{4qr-p^2}\xi\right)+\sin\alpha\cos\left(\sqrt{4qr-p^2}\xi\right)+\cos\alpha}$$

$$= \frac{1}{2q}\frac{\sin\left(\frac{\pi}{2}+\sqrt{4qr-p^2}\xi\right)}{\sin\left(\sqrt{4qr-p^2}\xi+\alpha\right)+\sin\left(\frac{\pi}{2}-\alpha\right)}$$

$$= \frac{1}{2q}\frac{2\sin\left(\frac{\pi}{4}+\frac{\sqrt{4qr-p^2}}{2}\xi\right)\cos\left(\frac{\pi}{4}+\frac{\sqrt{4qr-p^2}}{2}\xi\right)}{2\sin\left(\frac{\pi}{4}+\frac{\sqrt{4qr-p^2}}{2}\xi\right)\cos\left(\frac{\sqrt{4qr-p^2}}{2}\xi+\alpha-\frac{\pi}{4}\right)}$$

$$= \frac{1}{2q}\frac{\cos\left(\frac{\pi}{4}+\frac{\sqrt{4qr-p^2}}{2}\xi\right)}{\cos\left(\frac{\sqrt{4qr-p^2}}{2}\xi+\alpha-\frac{\pi}{4}\right)} = \frac{1}{2q}\frac{\cos\left(\frac{\pi}{4}+\frac{\sqrt{4qr-p^2}}{2}\xi\right)}{-\sin\left(\frac{\sqrt{4qr-p^2}}{2}\xi+\alpha+\frac{\pi}{4}\right)}$$

$$= -\frac{1}{2q}\frac{\cos\left(\frac{\pi}{4}+\frac{\sqrt{4qr-p^2}}{2}\xi\right)}{\sin\left(\frac{\sqrt{4qr-p^2}}{2}\xi+\frac{\pi}{4}\right)\cos\alpha+\cos\left(\frac{\sqrt{4qr-p^2}}{2}\xi+\frac{\pi}{4}\right)\sin\alpha}$$

$$= -\frac{2r\cos\left(\frac{\pi}{4}+\frac{\sqrt{4qr-p^2}}{2}\xi\right)}{\sqrt{4qr-p^2}\sin\left(\frac{\sqrt{4qr-p^2}}{2}\xi+\frac{\pi}{4}\right)+p\cos\left(\frac{\sqrt{4qr-p^2}}{2}\xi+\frac{\pi}{4}\right)}.$$

Consider the other case of ϕ_{22}^{I}:

$$\phi_{22}^{\mathrm{I}} = \frac{2r\cos\left(\sqrt{4qr-p^2}\xi\right)}{\sqrt{4qr-p^2}\sin\left(\sqrt{4qr-p^2}\xi\right)+p\cos\left(\sqrt{4qr-p^2}\xi\right)-\sqrt{4qr-p^2}}$$

$$= \frac{2r\sin\left(\frac{\sqrt{4qr-p^2}}{2}\xi+\frac{\pi}{4}\right)}{-p\sin\left(\frac{\sqrt{4qr-p^2}}{2}\xi+\frac{\pi}{4}\right)+\sqrt{4qr-p^2}\cos\left(\frac{\sqrt{4qr-p^2}}{2}\xi+\frac{\pi}{4}\right)}.$$

Setting $b_{-1} = \pm\frac{p+\mathrm{i}\sqrt{4qr-p^2}}{p-\mathrm{i}\sqrt{4qr-p^2}}$ and $\varphi = \frac{\pi}{4}$ in Eq. (19) results in ϕ_{22}^{I}.

$$\phi_{23}^{\mathrm{I}} = \frac{2r\sin\left(\sqrt{4qr-p^2}\xi\right)}{-p\sin\left(\sqrt{4qr-p^2}\xi\right)+\sqrt{4qr-p^2}\cos\left(\sqrt{4qr-p^2}\xi\right)\pm\sqrt{4qr-p^2}}.$$

Similarly, setting $b_{-1} = \pm\frac{p+\mathrm{i}\sqrt{4qr-p^2}}{p-\mathrm{i}\sqrt{4qr-p^2}}$ and $\varphi = 0$ in Eq. (19) yields ϕ_{23}^{I}.

$$\phi_{24}^{\mathrm{I}} = \frac{4r\cos\left(\frac{\sqrt{4qr-p^2}}{4}\xi\right)\sin\left(\frac{\sqrt{4qr-p^2}}{4}\xi\right)}{-2p\cos\left(\frac{\sqrt{4qr-p^2}}{4}\xi\right)\sin\left(\frac{\sqrt{4qr-p^2}}{4}\xi\right)+2\sqrt{4qr-p^2}\cos^2\left(\frac{\sqrt{4qr-p^2}}{4}\xi\right)-\sqrt{4qr-p^2}}$$

$$= \frac{2r\sin\left(\frac{\sqrt{4qr-p^2}}{2}\xi\right)}{-p\sin\left(\frac{\sqrt{4qr-p^2}}{2}\xi\right)+\sqrt{4qr-p^2}\left(\cos\left(\frac{\sqrt{4qr-p^2}}{2}\xi\right)+1\right)-\sqrt{4qr-p^2}}$$

$$= \frac{2r\sin\left(\frac{\sqrt{4qr-p^2}}{2}\xi\right)}{-p\sin\left(\frac{\sqrt{4qr-p^2}}{2}\xi\right)+\sqrt{4qr-p^2}\cos\left(\frac{\sqrt{4qr-p^2}}{2}\xi\right)}.$$

It is obvious that ϕ_{24}^{I} is same with ϕ_{21}^{I}, so ϕ_{24}^{I} is only special case in Eq. (19) when $b_{-1} = -\frac{p+\mathrm{i}\sqrt{4qr-p^2}}{p-\mathrm{i}\sqrt{4qr-p^2}}$ and $\varphi = 0$.

It is obvious that ϕ_1^{Π} to ϕ_{12}^{Π} in [33] are only special cases in ϕ_1^{I} to ϕ_{12}^{I} when $p^2 = -2qr$, so they are also special cases in Eq. (18).

We can see that Eqs. (18) and (19) have involved all solutions obtained by the extended Fan sub-equation method.

5. Discussion and conclusion

We give a very simple and straightforward method called the Exp-function method for nonlinear wave equations. We make some important remarks on the method as follows.

(1) The method leads to both generalized solitonary solutions and periodic solutions.
(2) The expression of the Exp-function is more general than the sinh-function and the tanh-function, so we can found more general solutions in the Exp-function method.
(3) The solution procedure, using Matlab or Mathematica, is of utter simplicity.
(4) The Exp-function method can be employed in both the straightforward way and the sub-equation way. But we suggest that it is better to use this method directly, not only for its convenience, but also because it is sometimes possiblt to lose some information and solutions if we apply it in the sub-equation way.
(5) From the comparison mentioned above, we can see that the Exp-function method is more convenient and effective than the extended Fan sub-equation method.

The method might become a promising and powerful new method for nonlinear equations.

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