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Exp-function method for nonlinear wave equations

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Abstract

In this paper, a new method, called Exp-function method, is proposed to seek solitary solutions, periodic solutions and compacton-like solutions of nonlinear differential equations. The modified KdV equation and Dodd–Bullough–Mikhailov equation are chosen to illustrate the effectiveness and convenience of the suggested method.

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1. Introduction

Recently many new approaches to nonlinear wave equations have been proposed, for example, tanh-function method [1–6], F-expansion method [7–9], Jacobian elliptic function method [10–12], variational iteration method [13,14], Adomian method [15–18], variational approach [19–21], and homotopy perturbation method [22–24]. All methods mentioned above have limitation in their applications. In this paper we suggest a novel method called Exp-function method (or Exp-method for short) to search for solitary solutions, compact-like solutions and periodic solutions of various nonlinear wave equations.

2. Basic idea of Exp-function method

In order to illustrate the basic idea of the suggested method, we consider first the following nonlinear dispersive equation of the form [13,25–27]:

$$u_t + u^2 u_x + u_{xxx} = 0. \tag{1}$$

This equation is called modified KdV equation, which arises in the process of understanding the role of nonlinear dispersion and in the formation of structures like liquid drops, and it exhibits compactons: solitons with compact support.

Introducing a complex variation η defined as

$$\eta = kx + \omega t. \tag{2}$$

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We have

$$\omega u' + ku^2u' + k^3u''' = 0, \tag{3}$$

where prime denotes the differential with respect to η.

The Exp-function method is very simple and straightforward, it is based on the assumption that traveling wave solutions can be expressed in the following form:

$$u(\eta) = \frac{\sum_{n=-c}^{d} a_n \exp(n\eta)}{\sum_{m=-p}^{q} b_m \exp(m\eta)}, \tag{4}$$

where c, d, p, and q are positive integers which are unknown to be further determined, a_n and b_m are unknown constants.

We suppose that the solution of Eq. (3) can be expressed as

$$u(\eta) = \frac{a_c \exp(c\eta) + \cdots + a_{-d} \exp(-d\eta)}{a_p \exp(p\eta) + \cdots + a_{-q} \exp(-q\eta)}. \tag{5}$$

To determine values of c and p, we balance the linear term of highest order in Eq. (3) with the highest order nonlinear term. By simple calculation, we have

$$u''' = \frac{c_1 \exp[(7p+c)\eta] + \cdots}{c_2 \exp[8p\eta] + \cdots} \tag{6}$$

and

$$u^2u' = \frac{c_3 \exp[(p+3c)\eta] + \cdots}{c_4 \exp[4p\eta] + \cdots} = \frac{c_3 \exp[(5p+3c)\eta] + \cdots}{c_4 \exp[8p\eta] + \cdots}, \tag{7}$$

where c_i are determined coefficients only for simplicity.

Balancing highest order of Exp-function in Eqs. (6) and (7), we have

$$7p + c = 5p + 3c, \tag{8}$$

which leads to the result

$$p = c. \tag{9}$$

Similarly to determine values of d and q, we balance the linear term of lowest order in Eq. (3)

$$u''' = \frac{\cdots + d_1 \exp[-(7q+d)\eta]}{\cdots + d_2 \exp[-8q\eta]} \tag{10}$$

and

$$u^2u' = \frac{\cdots + d_3 \exp[-(q+3d)\eta]}{\cdots + d_4 \exp[-4q\eta]} = \frac{\cdots + d_3 \exp[-(5q+3d)\eta]}{\cdots + d_4 \exp[-8q\eta]}, \tag{11}$$

where d_i are determined coefficients only for simplicity.

Balancing lowest order of Exp-function in Eqs. (10) and (11), we have

$$-(7q + d) = -(5q + 3d), \tag{12}$$

which leads to the result

$$q = d. \tag{13}$$

For simplicity, we set $p = c = 1$ and $q = d = 1$, so Eq. (5) reduces to

$$u(\eta) = \frac{a_1 \exp(\eta) + a_0 + a_{-1} \exp(-\eta)}{\exp(\eta) + b_0 + a_{-1} \exp(-\eta)}. \tag{14}$$

Substituting Eq. (14) into Eq. (3), and by the help of Matlab, we have

$$\frac{1}{A}[C_3 \exp(3\eta) + C_2 \exp(2\eta) + C_1 \exp(\eta) + C_0 + C_{-1} \exp(-\eta) + C_{-2} \exp(-2\eta) + C_{-3} \exp(4\eta)] = 0, \tag{15}$$

where

$$\begin{aligned}
&A = (\exp(\eta) + b_{-1}\exp(-\eta) + b_0)^4,\\
&C_3 = \omega a_1 b_0 + k a_1^3 b_0 - k^3 a_0 - w a_0 - k a_1^2 a_0 + k^3 a_1 b_0,\\
&C_2 = 8k^3 a_1 b_{-1} + 2k a_1^3 b_{-1} - 4k^3 a_1 b_0^2 - 2w a_{-1} - 2k a_1 a_0^2 + 2w a_1 b_{-1} + 4k^3 a_0 b_0 - 2k a_1^2 a_{-1}\\
&\qquad + 2k a_1^2 a_0 b_0 + 2\omega a_1 b_0^2 - 2\omega a_0 b_0 - 8k^3 a_{-1},\\
&C_1 = \omega a_1 b_0^3 + 6\omega a_1 b_0 b_{-1} - \omega a_0 b_0^2 - k^3 a_0 b_0^2 - 18k^3 a_1 b_0 b_{-1} - 6k a_1 a_0 a_{-1} + k a_1 a_0^2 b_0 - k a_0^3\\
&\qquad + 23k^3 a_0 b_{-1} - \omega a_0 b_{-1} - 5\omega a_{-1} b_0 + k^3 a_1 b_0^3 - 5k^3 a_{-1} b_0 + k a_1^2 a_{-1} b_0 + 5k a_1^2 a_0 b_{-1},\\
&C_0 = 4\omega a_1 b_{-1}^2 - 4k a_1 a_{-1}^2 + 32k^3 a_{-1} b_{-1} + 4k a_1 a_0^2 b_{-1} - 32k^3 a_1 b_{-1}^2 + 4k^3 a_1 b_0^2 b_{-1} - 4\omega a_{-1} b_{-1}\\
&\qquad - 4k^3 a_{-1} b_0^2 - 4k a_0^2 a_{-1} - 4\omega a_{-1} b_0^2 + 4k a_1^2 a_{-1} b_{-1} + 4\omega a_1 b_0^2 b_{-1},\\
&C_{-1} = 18k^3 a_{-1} b_0 b_{-1} - 6\omega a_{-1} b_0 b_{-1} - k^3 a_{-1} b_0^3 + k^3 a_0 b_{-1} b_0^2 + \omega a_0 b_{-1}^2 - 5k a_0 a_{-1}^2 + 5\omega a_1 b_0 b_{-1}^2\\
&\qquad + \omega a_0 b_{-1} b_0^2 - \omega a_{-1} b_0^3 - k a_1 a_{-1}^2 b_0 - 23k^3 a_0 b_{-1}^2 - k a_0^2 a_{-1} b_0 + 5k^3 a_1 b_0 b_{-1}^2 + k a_0^3 b_{-1}\\
&\qquad + 6k a_1 a_0 a_{-1} b_{-1},\\
&C_{-2} = 2\omega a_0 b_{-1}^2 b_0 - 2\omega a_{-1} b_{-1}^2 - 2k a_{-1}^3 + 2k a_1 a_{-1}^2 b_{-1} + 2\omega a_1 b_{-1}^3 - 4k^3 a_0 b_{-1}^2 b_0 - 2\omega a_{-1} b_0^2 b_{-1}\\
&\qquad + 4k^3 a_{-1} b_0^2 b_{-1} - 8k^3 a_{-1} b_{-1}^2 + 2k a_0^2 a_{-1} b_{-1} - 2k a_0 a_{-1}^2 b_0 + 8k^3 a_1 b_{-1}^3,\\
&C_{-3} = k a_0 a_{-1}^2 b_{-1} + \omega a_0 b_{-1}^3 - k a_{-1}^3 b_0 + k^3 a_0 b_{-1}^3 - \omega a_{-1} b_0 b_{-1}^2 - k^3 a_{-1} b_0 b_{-1}^2.
\end{aligned}$$

Equating the coefficients of $\exp(n\eta)$ to be zero, we have

$$\begin{cases} C_3 = 0, \quad C_2 = 0, \quad C_1 = 0,\\ C_0 = 0,\\ C_{-2} = 0, \quad C_{-3} = 0, \quad C_{-4} = 0. \end{cases} \tag{16}$$

Solving the system, Eq. (16), simultaneously, we obtain

$$\begin{cases} a_0 = a_1 b_0 + \frac{3k^2 b_0}{a_1}, \quad a_{-1} = \frac{b_0^2(3k^2+2a_1^2)}{8a_1},\\ b_{-1} = \frac{b_0^2(3k^2+2a_1^2)}{8a_1^2}, \quad \omega = -k a_1^2 - k^3, \end{cases} \tag{17}$$

where a_1 and b_0 are free parameters.

We, therefore, obtain the following solution:

$$\begin{aligned} u(x,t) &= \frac{a_1\exp[kx - (k a_1^2 + k^3)t] + a_1 b_0 + \frac{3k^2 b_0}{a_1} + \frac{b_0^2(3k^2+2a_1^2)}{8a_1}\exp(-kx + (k a_1^2 + k^3)t)}{\exp[kx - (k a_1^2 + k^3)t] + b_0 + \frac{b_0^2(3k^2+2a_1^2)}{8a_1^2}\exp(-kx + (k a_1^2 + k^3)t)}\\ &= a_1 + \frac{\frac{3k^2 b_0}{a_1}}{\exp[kx - (k a_1^2 + k^3)t] + b_0 + \frac{b_0^2(3k^2+2a_1^2)}{8a_1^2}\exp(-kx + (k a_1^2 + k^3)t)}. \end{aligned} \tag{18}$$

Generally a_1, b_0, and k are real numbers, and the obtained solution, Eq. (18), is a generalized solitonary solution.

In case k is an imaginary number, the obtained solitonary solution can be converted into periodic solution or compact-like solution. We write

$$k = \mathrm{i}K. \tag{19}$$

Use the transformation

$$\exp[kx - (k a_1^2 + k^3)t] = \exp[\mathrm{i}Kx - \mathrm{i}(K a_1^2 - K^3)t] = \cos[Kx - (K a_1^2 - K^3)t] + \mathrm{i}\sin[Kx - (K a_1^2 - K^3)t]$$

and

$$\exp[-kx + (k a_1^2 + k^3)t] = \exp[-\mathrm{i}Kx + \mathrm{i}(K a_1^2 - K^3)t] = \cos[Kx - (K a_1^2 - K^3)t] - \mathrm{i}\sin[Kx - (K a_1^2 - K^3)t].$$

Eq. (18) becomes

$$u(x,t) = a_1 + \frac{-\frac{3K^2 b_0}{a_1}}{(1+p)\cos[Kx - (K a_1^2 - K^3)t] + b_0 + \mathrm{i}(1-p)\sin[Kx - (K a_1^2 - K^3)t]}, \tag{20}$$

where $p = \frac{b_0^2(-3k^2+2a_1^2)}{8a_1^2}$.

If we search for a periodic solution or compact-like solution, the imaginary part in the denominator of Eq. (20) must be zero, that requires that

$$1 - p = 1 - \frac{b_0^2(-3K^2 + 2a_1^2)}{8a_1^2} = 0. \tag{21}$$

Solving b_0 from Eq. (21) we obtain

$$b_0 = \pm\sqrt{\frac{8}{(-3K^2 + 2a_1^2)}}. \tag{22}$$

Substituting Eq. (22) into Eq. (20) results in a compact-like solution, which reads

$$u(x,t) = a_1 + \frac{\mp 3K^2\sqrt{\frac{8}{(-3K^2+2a_1^2)}}}{2\cos[Kx - (Ka_1^2 - K^3)t] \pm \sqrt{\frac{8}{(-3K^2+2a_1^2)}}a_1} = a_1 + \frac{\mp 3K^2\sqrt{\frac{2}{(-3K^2+2a_1^2)}}}{\cos[Kx - (Ka_1^2 - K^3)t] \pm \sqrt{\frac{2}{(-3K^2+2a_1^2)}}a_1}, \tag{23}$$

where a_1 and K are free parameters, and it requires that $2a_1^2 > 3K^2$.

To compare our result, Eq. (23), with that in open literature, we write down Zhu et al.'s solutions [28,29], which read

$$u(x,t) = \frac{4\sqrt{2}k\sin^2(kx - 4k^3t)}{3 - 2\sin^2(kx - 4k^3t)}. \tag{24a}$$

and

$$u(x,t) = \frac{4\sqrt{2}k\cos^2(kx - 4k^3t)}{3 - 2\cos^2(kx - 4k^3t)}. \tag{24b}$$

We re-write Eqs. (24a) and (24b), respectively, in the forms

$$u(x,t) = -2\sqrt{2}k + \frac{6\sqrt{2}k}{2 + \cos(2kx - 8k^3t)} \tag{25a}$$

and

$$u(x,t) = -2\sqrt{2}k + \frac{6\sqrt{2}k}{2 - \cos(2kx - 8k^3t)}. \tag{25b}$$

If we choose $a_1 = -2\sqrt{2}k$, our solution, Eq. (23), turns out to be Zhu et al.'s solutions as expressed in Eqs. (25a) and (25b).

So the suggested Exp-function method can obtain easily the generalized solitonary solution and compact-like solution for nonlinear wave equations. To illustrate its effectives and convenience, we consider in the next section the Dodd–Bullough–Mikhailov equation.

3. An example

Now we consider the Dodd–Bullough–Mikhailov equation [4]

$$u_{xt} + \mathrm{e}^{u} + \mathrm{e}^{-2u} = 0. \tag{26}$$

This equation plays a significant role in many scientific applications such as solid state physics, nonlinear optics and quantum field theory. By the transformation $u = \ln v$, Eq. (26) becomes

$$vv_{xt} - v_x v_t + v^3 + 1 = 0. \tag{27}$$

Introducing a complex variation η defined as $\eta = kx + \omega t$, we have

$$f(v) \equiv k\omega vv'' - k\omega(v')^2 + v^3 + 1 = 0, \tag{28}$$

where prime denotes the differential with respect to η.

We suppose that the solution of Eq. (28), can be expressed as

$$u(x,t) = \frac{a_c\exp[c(kx + \omega t)] + \cdots + a_{-d}\exp[-d(kx + \omega t)]}{a_p\exp[p(kx + \omega t)] + \cdots + a_{-q}\exp[-q(kx + \omega t)]}. \tag{29}$$

By the same manipulation as illustrated in the previous section, we can determine values of c and p by balancing vv'' and v^3 in Eq. (28)

$$vv'' = \frac{c_1 \exp[(3p+2c)\eta] + \cdots}{c_2 \exp[5p\eta] + \cdots} \tag{30}$$

and

$$v^3 = \frac{c_3 \exp[3c\eta] + \cdots}{c_4 \exp[3p\eta] + \cdots} = \frac{c_1 \exp[(2p+3c)\eta] + \cdots}{c_4 \exp[5p\eta] + \cdots}. \tag{31}$$

Balancing the highest order of Exp-function in Eqs. (30) and (31), we have

$$3p + 2c = 2p + 3c, \tag{32}$$

which leads to the result

$$p = c. \tag{33}$$

By a similar derivation as illustrated in the previous section, we obtain

$$d = q. \tag{34}$$

3.1. Case 1: $p = c = 1$, $d = q = 1$

We can freely choose the values of c and d, but we will illustrate that the final solution does not strongly depends upon the choice of values of c and d. For simplicity we choose $p = c = 1$ and $d = q = 1$, the trial-function, Eq. (29), becomes

$$v(\eta) = \frac{a_1 \exp(\eta) + a_0 + a_{-1} \exp(-\eta)}{\exp(\eta) + b_0 + a_{-1} \exp(-\eta)}. \tag{35}$$

Substituting (35) into (28), by help of Matlab, we have

$$\begin{aligned} &\frac{1}{A}\{C_4 \exp(4\eta) + C_3 \exp(3\eta) + C_2 \exp(2\eta) + C_1 \exp(\eta) + C_0 + C_{-1} \exp(-\eta) + C_{-2} \exp(-2\eta) + C_{-3} \exp(-3\eta) \\ &\quad + C_{-4} \exp(-4\eta)\} = 0, \end{aligned} \tag{36}$$

where

$$\begin{aligned}
A &= (\exp(\eta) + b_{-1} \exp(-\eta) + b_0)^4, \\
C_4 &= a^3 + 1, \\
C_3 &= 3a_1^2 a_0 + k\omega a_0 a_1 - k\omega a_1^2 b_0 + a_1^3 b_0 + 4b_0, \\
C_2 &= 3a_1 a_0^2 + 4b_{-1} + 4k\omega a_1 a_{-1} + 6b_0^2 - 4k\omega a_1^2 b_{-1} + a_1^3 b_{-1} + 3a_1^2 a_0 b_0 + 3a_1^2 a_{-1}, \\
C_1 &= a_0^3 - 6k\omega a_1 b_{-1} a_0 + 6k\omega a_1 b_0 a_{-1} + 12 b_0 b_{-1} - k\omega a_0^2 b_0 - k\omega a_1^2 b_0 b_{-1} + 6a_1 a_0 a_{-1} \\
&\quad + k\omega a_0 a_1 b_0^2 + 3a_1 a_0^2 b_0 + 4b_0^3 + 3a_1^2 a_0 b_{-1} + 3a_1^2 a_{-1} b_0 + k\omega a_0 a_{-1}, \\
C_0 &= 3a_1^2 a_{-1} b_{-1} + 6b_{-1}^2 + 12 b_0^2 b_{-1} - 4k\omega a_0^2 b_{-1} + 3a_1 a_{-1}^2 + 3a_0^2 a_{-1} + 6a_1 a_0 a_{-1} b_0 + b_0^4 \\
&\quad + 4k\omega a_1 a_{-1} b_0^2 + a_0^3 b_0 + 3a_1 a_0^2 b_{-1}, \\
C_{-1} &= a_0^3 b_{-1} + 3a_0 a_{-1}^2 + 3a_0^2 a_{-1} b_0 + k\omega a_0 a_{-1} b_0^2 + 6k\omega a_{-1} b_0 a_1 b_1 + k\omega a_0 b_{-1}^2 a_1 + 4b_0^3 b_{-1} \\
&\quad - k\omega a_0^2 b_{-1} b_0 + 6a_1 a_0 a_{-1} b_{-1} - 6k\omega a_0 a_{-1} b_{-1} - k\omega a_{-1}^2 b_0 + 3a_1 a_{-1}^2 b_0 + 12 b_0 b_{-1}^2, \\
C_{-2} &= 6b_0^2 b_{-1}^2 - 4k\omega a_{-1}^2 b_{-1} + a_{-1}^3 + 3a_0^2 a_{-1} b_{-1} + 3a_1 a_{-1}^2 b_{-1} + 4b_{-1}^3 + 4k\omega a_{-1} a_1 b_{-1}^2 + 3a_0 a_{-1}^2 b_0, \\
C_{-3} &= 3a_0 a_{-1}^2 b_{-1} - k\omega a_{-1}^2 b_0 b_{-1} + a_{-1}^3 b_0 + 4b_0 b_{-1}^3 + k\omega a_{-1} a_0 b_{-1}^3, \\
C_{-4} &= b_{-1}^4 + a_{-1}^3 b_1.
\end{aligned}$$

Equating the coefficients of $\exp(n\eta)$ in Eq. (36) to be zero, we have

$$\begin{cases} C_4 = 0, \quad C_3 = 0, \quad C_2 = 0, \quad C_1 = 0, \\ C_0 = 0, \\ C_{-1} = 0, \quad C_{-2} = 0, \quad C_{-3} = 0, C_{-4} = 0. \end{cases} \tag{37}$$

Solving the system, Eq(37), simultaneously yields

$$a_1 = -1; \quad a_0 = 2b_0; \quad a_{-1} = \frac{-b_0^2}{4}; \quad b_{-1} = \frac{b_0^2}{4}; \quad \omega = \frac{3}{k}. \tag{38}$$

We, therefore, obtain

$$v(x,t) = \frac{-\exp\left(kx + \frac{3}{k}t\right) + 2b_0 - \frac{b_0^2}{4}\exp\left(-kx - \frac{3}{k}t\right)}{\exp\left(kx + \frac{3}{k}t\right) + b_0 + \frac{b_0^2}{4}\exp\left(-kx - \frac{3}{k}t\right)}, \tag{39}$$

where k and b_0 are non-zero free parameters

The solution of the Dodd–Bullough–Mikhailov equation can be expressed as follows:

$$u(x,t) = \ln v = \ln\left[-1 + \frac{3b_0}{\left(\exp\left(\frac{k}{2}x + \frac{3}{2k}t\right) + \frac{b_0}{2}\exp\left(-\frac{k}{2}x - \frac{3}{2k}t\right)\right)^2}\right]. \tag{40}$$

Hereby it requires that $b_0 > 0$ and $\ln\left(\frac{\sqrt{3}-1}{2}\sqrt{b_0}\right) < \left(\frac{k}{2}x + \frac{3}{2k}t\right) < \ln\left(\frac{\sqrt{3}+1}{2}\sqrt{b_0}\right)$.

To compare our result with the known ones in literature, we write down Wazwaz's solution [4], which reads

$$v(x,t) = \frac{1}{2}\left(1 - 3\tanh^2\left[\frac{1}{2}\sqrt{-\frac{3}{c}}(x - ct)\right]\right) \tag{41}$$

We re-write it in the form

$$\begin{aligned} v(x,t) &= \frac{1}{2}\left(1 - 3 \times \left[\frac{\exp\left(\frac{1}{2}\sqrt{-\frac{3}{c}} \times x + \frac{1}{2}\sqrt{-3c} \times t\right) - \exp\left(-\frac{1}{2}\sqrt{-\frac{3}{c}} \times x - \frac{1}{2}\sqrt{-3c} \times t\right)}{\exp\left(\frac{1}{2}\sqrt{-\frac{3}{c}} \times x - \frac{1}{2}\sqrt{-3c} \times t\right) + \exp\left(-\frac{1}{2}\sqrt{-\frac{3}{c}} \times x - \frac{1}{2}\sqrt{-3c} \times t\right)}\right]^2\right) \\ &= \frac{-\exp\left(\sqrt{-\frac{3}{c}}x + \sqrt{-3c}t\right) + 4 - \exp\left(-\sqrt{-\frac{3}{c}}x - \sqrt{-3c}t\right)}{\exp\left(\sqrt{-\frac{3}{c}}x + \sqrt{-3c}t\right) + 2 + \exp\left(-\sqrt{-\frac{3}{c}}x - \sqrt{-3c}t\right)}. \end{aligned} \tag{42}$$

It is obvious that in case $b_0 = 2$, our solution reduces to Wazwaz's.

As illustrated above that the obtained solitonary solution can be converted into periodic solution or compact-like solution if k is chosen as an imaginary number. If $k = \mathrm{i}K$, then Eq. (39) becomes

$$v(x,t) = \frac{-\left(1 + \frac{b_0^2}{4}\right)\cos\left[Kx - \frac{3}{K}t\right] - \mathrm{i}\left(1 - \frac{b_0^2}{4}\right)\sin\left[Kx - \frac{3}{K}t\right] + 2b_0}{\left(1 + \frac{b_0^2}{4}\right)\cos\left[Kx - \frac{3}{K}t\right] + \mathrm{i}\left(1 - \frac{b_0^2}{4}\right)\sin\left[Kx - \frac{3}{K}t\right] + b_0}. \tag{43}$$

Elimination of the imaginary part requires that

$$b_0 = 2. \tag{44}$$

Eq. (43) reduces to

$$v(x,t) = \frac{-\cos\left[Kx - \frac{3}{K}t\right] + 2}{\cos\left[Kx - \frac{3}{K}t\right] + 1} = -1 + \frac{3}{\cos\left[Kx - \frac{3}{K}t\right] + 1}. \tag{45}$$

We, therefore, obtain a periodic solution, which reads

$$u(x,t) = \ln v = \ln\left(-1 + \frac{3}{\cos\left[Kx - \frac{3}{K}t\right] + 1}\right), \tag{46}$$

it requires that $Kx - \frac{3}{K}t \neq (2n+1)\pi$.

3.2. Case 2: $p = c = 2$, $d = q = 2$

As mentioned above the values of c and d can be freely chosen, now we set $p = c = 2$ and $d = q = 2$, then the trial-function, Eq. (29), becomes

$$v(\eta) = \frac{a_2\exp(2\eta) + a_1\exp(\eta) + a_0 + a_{-1}\exp(-\eta) + a_{-2}\exp(-2\eta)}{\exp(2\eta) + b_1\exp(\eta) + b_0 + b_{-1}\exp(-\eta) + b_{-2}\exp(-\eta)}. \tag{47}$$

There are some free parameters in Eq. (47), we set $b_1 = b_{-1} = 0$ for simplicity, the trial-function (47) is simplified as follows:

$$v(\eta) = \frac{a_2\exp(2\eta) + a_1\exp(\eta) + a_0 + a_{-1}\exp(-\eta) + a_{-2}\exp(-2\eta)}{\exp(2\eta) + b_0 + b_{-2}\exp(-\eta)}. \tag{48}$$

By the same manipulation as illustrated above, we obtain

$$\begin{cases} a_2 = -1; \quad a_0 = -\frac{5}{18}a_1^2; \quad a_{-1} = \frac{a_1^3}{36}; \quad a_{-2} = -\frac{a_1^4}{1296}, \\ b_{-2} = \frac{a_1^4}{1296}; \quad \omega = \frac{3}{k}, \end{cases} \tag{49a}$$

or

$$\begin{cases} a_2 = -1; \quad a_1 = 0; \quad a_0 = 2b_0; \quad a_{-1} = 0; \quad a_{-2} = -\frac{b_0^2}{4}, \\ b_{-2} = \frac{b_0^2}{4}; \quad \omega = \frac{3}{4k}. \end{cases} \tag{49b}$$

Eq. (49b) leads to the following solution:

$$v(x,t) = \frac{-\exp\left(2kx+\frac{6}{k}t\right) + a_1\exp\left(kx+\frac{3}{k}t\right) - \frac{5}{18}a_1^2 + \frac{a_1^3}{36}\exp\left(-kx-\frac{3}{k}t\right) - \frac{a_1^4}{1296}\left(-2kx-\frac{6}{k}t\right)}{\exp\left(2kx+\frac{6}{k}t\right) - \frac{a_1^2}{18} + \frac{a_1^4}{1296}\left(-2kx-\frac{6}{k}t\right)}. \tag{50}$$

Setting $a_1 = 6a$, Eq. (50) is simplified as follows:

$$\begin{aligned} v(x,t) &= \frac{-\exp\left(2kx+\frac{6}{k}t\right) + 6a\exp\left(kx+\frac{3}{k}t\right) - 10a^2 + 6a^3\exp\left(-kx-\frac{3}{k}t\right) - a^4\left(-2kx-\frac{6}{k}t\right)}{\exp\left(2kx+\frac{6}{k}t\right) - 2a^2 + a^4\left(-2kx-\frac{6}{k}t\right)} \\ &= \frac{-\left[\exp\left(kx+\frac{3}{k}t\right) + a^2\exp\left(-kx-\frac{3}{k}t\right) - 2a\right]\left[\exp\left(kx+\frac{3}{k}t\right) + a^2\exp\left(-kx-\frac{3}{k}t\right) - 4a\right]}{\left[\exp\left(kx+\frac{3}{k}t\right) + a^2\exp\left(-kx-\frac{3}{k}t\right)\right]^2 - 4a^2} \\ &= \frac{-\left[\exp\left(kx+\frac{3}{k}t\right) + a^2\exp\left(-kx-\frac{3}{k}t\right) - 4a\right]}{\exp\left(kx+\frac{3}{k}t\right) + a^2\exp\left(-kx-\frac{3}{k}t\right) + 2a}. \end{aligned} \tag{51a}$$

Substituting Eq. (49b) into Eq. (48) results in another solution, which reads

$$v(x,t) = \frac{-\exp\left(kx+\frac{3}{k}t\right) + 2b_0 - \frac{b_0^2}{4}\exp\left(-kx-\frac{3}{k}t\right)}{\exp\left(kx+\frac{3}{k}t\right) + b_0 + \frac{b_0^2}{4}\exp\left(-kx-\frac{3}{k}t\right)}. \tag{51b}$$

It is interesting that we find the same solutions as those in the Case 1.

3.3. Case 3: p = c = 2, d = q = 1

Now we consider the case $p = c = 2$ and $d = q = 1$. Under such case, the trial-function can be expressed as follows:

$$v(\eta) = \frac{a_2\exp(2\eta) + a_1\exp(\eta) + a_0 + a_{-1}\exp(-\eta)}{\exp(2\eta) + b_1\exp(\eta) + b_0 + b_{-1}\exp(-\eta)}. \tag{52}$$

By simple calculation by Matlab, we have

$$\begin{cases} a_2 = -1; \quad a_0 = -\frac{a_1^2}{4} + \frac{a_1 b_1}{6} + \frac{5}{12}b_1^2; \quad a_{-1} = \frac{a_1^3}{108} - \frac{b_1^2 a_1}{36} - \frac{b_1^3}{54}, \\ b_0 = -\frac{a_1^2}{12} + \frac{a_1 b_1}{6} + \frac{b_1^2}{4}; \quad b_{-1} = -\frac{a_1^3}{108} + \frac{b_1^2 a_1}{36} + \frac{b_1^3}{54}, \\ \omega = \frac{3}{k}. \end{cases} \tag{53}$$

So the solution can be expressed as

$$\begin{aligned} v(x,t) &= \frac{-\exp\left(2kx+\frac{6}{k}t\right)+a_1\exp\left(kx+\frac{3}{k}t\right)-\frac{a_1^2}{4}+\frac{a_1b_1}{6}+\frac{5}{12}b_1^2+\left(\frac{a_1^3}{108}-\frac{b_1^2a_1}{36}-\frac{b_1^3}{54}\right)\exp\left(-kx-\frac{3}{k}t\right)}{\exp\left(2kx+\frac{6}{k}t\right)+b_1\exp\left(kx+\frac{3}{k}t\right)-\frac{a_1^2}{12}+\frac{a_1b_1}{6}+\frac{1}{4}b_1^2-\left(\frac{a_1^3}{108}-\frac{b_1^2a_1}{36}-\frac{b_1^3}{54}\right)\exp\left(-kx-\frac{3}{k}t\right)} \\ &= -1+\frac{(a_1+b_1)\exp\left(kx+\frac{3}{k}t\right)-\frac{a_1^2}{3}+\frac{a_1b_1}{3}+\frac{2}{3}b_1^2}{\exp\left(2kx+\frac{6}{k}t\right)+b_1\exp\left(kx+\frac{3}{k}t\right)-\frac{a_1^2}{12}+\frac{a_1b_1}{6}+\frac{1}{4}b_1^2-\left(\frac{a_1^3}{108}-\frac{b_1^2a_1}{36}-\frac{b_1^3}{54}\right)\exp\left(-kx-\frac{3}{k}t\right)} \\ &= -1+\frac{36(a_1+b_1)\left(3\exp\left(kx+\frac{3}{k}t\right)-a_1+2b_1\right)}{-\exp\left(-kx-\frac{3}{k}t\right)\times\left(3\exp\left(kx+\frac{3}{k}t\right)-a_1+2b_1\right)\times\left(6\exp\left(kx+\frac{3}{k}t\right)+a_1+b_1\right)^2} \\ &= -1+\frac{36(a_1+b_1)\exp\left(kx+\frac{3}{k}t\right)}{\left(6\exp\left(kx+\frac{3}{k}t\right)+a_1+b_1\right)^2}. \end{aligned} \tag{54}$$

If we set $6c = a_1 + b_1$, Eq. (54) reduces to

$$v(x,t) = -1+\frac{6c\exp\left(kx+\frac{3}{k}t\right)}{\left(\exp\left(kx+\frac{3}{k}t\right)+c\right)^2} = -1+\frac{6c}{\exp\left(kx+\frac{3}{k}t\right)+2c+c^2\exp\left(-kx-\frac{3}{k}t\right)}, \tag{55}$$

which is same as those obtained in Cases 1 and 2.

4. Conclusion

We give a very simple and straightforward method called Exp-function method for nonlinear wave equations. The suggest method has some pronounced merits:

(1) The method leads to both the generalized solitonary solutions and periodic solutions;
(2) The solution procedure, by help of Matlab, is of utter simplicity, and can be easily extended to all kinds of nonlinear equations.

The Exp-function method might become a promising and powerful new method for nonlinear equations.

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