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# Rebuild of King Fang 40 BC musical scales by He's inequality

Ji-Huan He *, Jie Tang

*College of Science, Donghua University, P.O. Box 471, 1882 Yan'an Xilu Road, Shanghai 200051, People's Republic of China*

**Abstract**

He Chengtian's inequality is used to re-build the King Fang's musical scales 41/24 and 53/31 in 40 BC. The technology, which might be little known to the West, has been used for millennia by ancient Chinese, and can be powerfully applied to nonlinear problems.


*Keywords:* Inequality; Musical scale; Nonlinear interpolation; Nonlinear equations; Jiuzhang Suanshu (nine chapters); Chinese lunar calendar; Pi; He Chengtian; King Fang; Zu Chongzhi; Liu Hui

## 1. Introduction

Considering the following musical equation [1]:

$$(3/2)^x = 2^y. \tag{1}$$

* Corresponding author.
*E-mail address:* jhhe@dhu.edu.cn (J.-H. He).

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The so-called musical scale ($x$:$y$) is to search for integers or rational numbers for $x$ and $y$, that best satisfy (1). The exact solution is

$$M = \frac{x}{y} = \frac{\log 2}{\log(3/2)} \approx 1.709511291.$$

The most audible will be the lowest two, which are in a 3:2 ratio. This is the pure fifth. Before 40 BC, ancient Chinese used 12:7, which is generally called as ancient Chinese musical scale.

In 40 BC King Fang (77–27 BC), instead with 12:7, gave the following scales: 41:24 and 53:31. Note that $3^{53} \approx 2 \times 10^{25}$, it is astonishing how ancient Chinese got such an accurate solution. We will re-build the possible solution procedure by He Chengtian's technology [2].

## 2. He's inequality

A brief introduction to the He Chengtian's work on inequality was given in Ref. [2], and its modern applications can be found in Refs. [3,4]. He Chengtian (369–447 AD) was a famous ancient Chinese mathematician and astronomer. He Chengtian's theory was called *Tiaorifa* (调日法, lit. method for correcting denominator), which was successfully applied to seeking a high accurate *Metonic* cycle in Chinese lunar calendar. Before He Chengtian, it knew that

$$29\frac{26}{49} \quad \text{days} > 1 \text{ Moon} > 29\frac{9}{17} \quad \text{days}.$$

Using the weighting factors (15 and 1), He Chengtian obtained

$$\text{The fractional day} = \frac{26 \cdot 15 + 9 \cdot 1}{49 \cdot 15 + 17 \cdot 1} = \frac{399}{752},$$

so

$$1 \text{ Moon} = 29\frac{399}{752} \text{ days},$$

which is of very high accuracy (the error is about 0.1 s) even in the modern view.

By the same technology, He Chengtian estimated the year as 365 (150/608) replacing the 4 year Metonic cycle by a 19 year cycle with 7 intercalary months. Zu Chongzhi (430–501) [5] further improved the cycle by a 391 year cycle with 144 (intercalary months).

Now consider the following inequality:

If

$$\frac{a}{b} < x < \frac{d}{c}, \tag{2}$$

where $a, b, c$ and $d$ are positive integers, then

$$\frac{a}{b} < \frac{ma+nd}{mb+nc} < \frac{d}{c}, \tag{3}$$

where $m$ and $n$ are nonzero positive integers.

**Proof** [3]. In view of the inequality (2), we have $ac - bd < 0$, so we have

$$\frac{a}{b} - \frac{ma+nd}{mb+nc} = \frac{(ac-bd)n}{b(mb+nc)} < 0$$

and

$$\frac{ma+nd}{mb+nc} - \frac{d}{c} = \frac{(ac-bd)m}{c(mb+nc)} < 0.$$

The inequality (3) is, therefore, proved. He Chengtian actually approximated $x$ in (2) by

$$x = \frac{ma+nd}{mb+nc}, \tag{4}$$

where $m$ and $n$ are weighting factors, generally nonzero positive integers. It is obvious that $\lim_{m\to\infty} = a/b$ ($n$ is finite) and $\lim_{n\to\infty} = d/c$ ($m$ is finite).

In case $\frac{m}{n} = \frac{c}{b}$, Eq. (4) reduces to an average value

$$x = \frac{ma+nd}{mb+nc} = \frac{1}{2}\left(\frac{a}{b} + \frac{d}{c}\right). \tag{5}$$

So He Chengtian's inequality theory is actually a nonlinear interpolation, see Fig. 1, some other nonlinear interpolations can be found in detail in Refs. [6–8].

Furthermore, we have the following inequality

$$\frac{a}{b} < \frac{pa+qd}{pb+qc} < \frac{ma+nd}{mb+nc} < \frac{d}{c} \quad (\text{if } pn - qm > 0), \tag{6}$$

where $p, q, m$, and $n$ are weighting factors.

The proof of these inequalities is easy and straightforward.

$$\frac{pa+qd}{pb+qc} - \frac{ma+nd}{mb+nc} = \frac{(pa+qd)(mb+nc) - (ma+nd)(pb+qc)}{(pb+qc)(mb+nc)}$$
$$= \frac{(np-mq)(ac-bd)}{(pb+qc)(mb+nc)}.$$

Note that $ac - bd < 0$ and $pn - qm > 0$, so we have

$$\frac{pa+qd}{pb+qc} - \frac{ma+nd}{mb+nc} < 0. \tag{7}$$

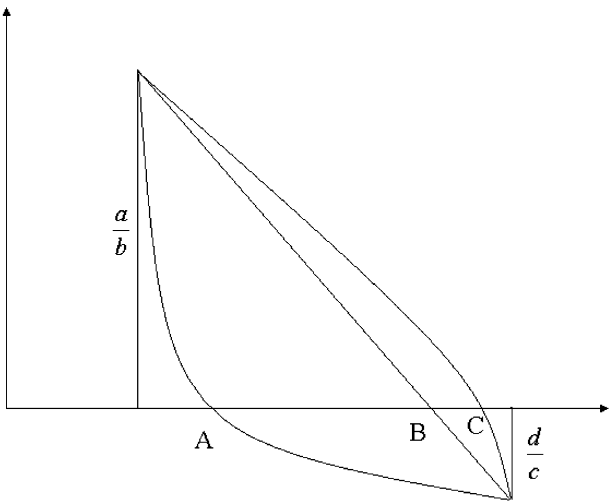


Fig. 1. An explanation of He Chengtian's inequality. $x_B = \frac{1}{2}\left(\frac{a}{b}+\frac{d}{c}\right)$ when $\frac{m}{n}=\frac{c}{b}$, point $A$ corresponds to the case $m > n$, and point $C$ $m < n$.

In He Chengtian's time, ancient Chinese mathematicians knew that $\frac{157}{50} < \pi < \frac{22}{7}$ ($\pi = 3.14$ was suggested by Liu Hui (263), and $\pi = 22/7$ was obtained by Zu Chongzhi and/or his son Zu Geng [5].)

Using the weighting factors 1 and 9 [1], we have

$$\pi = \frac{157 + 9 \cdot 22}{50 + 9 \cdot 7} = \frac{355}{113} = 3.1415929. \tag{8}$$

So we obtain

$$\frac{355}{113} < \pi < \frac{22}{7}, \tag{9}$$

which was obtained by Zu Chongzhi (430–501). □

## 3. King Fang's scales

Now we turn back to the problem of musical scale, we write

$$R(x, y) = 3^x - 2^{x+y}, \tag{10}$$

[1] 22/7 is more accurate than 157/50, so that value of weight factor for the former is bigger than that for the later.

where $R$ is the residual of the musical equation (1). It is obvious that $R(2,1) = 1$, and $R(5,3) = -13$, so the musical scale $M = x/y$ must locate at the domain [5/3, 2], i.e.,

$$\frac{5}{3} < M < \frac{2}{1}. \tag{11}$$

According to He's theory, the scale can be approximately expressed in the form

$$M(m,n) = \frac{5m + 2n}{3m + n}. \tag{12}$$

It is obvious that $M(2,1) = 12/7$, $M(7,3) = 41/24$, and $M(9,4) = 53/31$, which corresponds to King Fang's scales.

If King Fang used the same solution procedure, then He Chengtian's inequality dates back at least 40 BC. He Chengtian (369–447 AD) might apply successfully this inequality to the correction of lunar calendar. The dates of ancient Chinese works (for example the well-known *Jiu Zhang Suan Shu*, nine chapters on the Art of Mathematics [9] are very unclear, for example) is unclear. A given book may have been produced as a single work, or it may be a compilation of several earlier works. In either case, the material may be centuries older than the book and the preserved version of the book may be several centuries and editions later than its original form.

## 4. Conclusion

We, for the first time, re-build King Fang's scales by He Chengtian's method, which, in the authors' view, dates back before 40 BC.

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