



ELSEVIER

Available at

[www.ElsevierMathematics.com](http://www.ElsevierMathematics.com)

POWERED BY SCIENCE @ DIRECT®

Applied Mathematics and Computation 152 (2004) 9–15

APPLIED  
MATHEMATICS  
AND  
COMPUTATION

[www.elsevier.com/locate/amc](http://www.elsevier.com/locate/amc)

# Zu-Geng's axiom vs Cavalieri's theory

Ji-Huan He

*College of Basic Science, Shanghai Donghua University, P.O. Box 471, 1882 Yan'an Xilu Road,  
Shanghai 200051, People's Republic of China*

---

## Abstract

When Cavalieri found his theory in 1635, Chinese mathematicians had used the theory for more than one millennium.

© 2003 Elsevier Inc. All rights reserved.

*Keywords:* Mathematical history; Cavalieri's theory; Ancient Chinese mathematics; Zu-Geng axiom; Liu Hui; Jiuzhang Suanshu

---

## 1. Introduction

Essentially nothing of a primary nature has come down to the West concerning ancient Chinese mathematics, little has been discussed on ancient Chinese mathematics in some of the most famous monographs on history of mathematics [1–3]. When Cavalieri [4] claimed that he found the way of finding the volume of a sphere in 1635, ancient Chinese mathematicians had used the theory for more than one millennium.

## 2. Zu-Gengs axiom

In later 5th century AD, Zu Chongzhi (429–500) and/or his son Zu Geng writes

---

*E-mail address:* [jhhe@dhu.edu.cn](mailto:jhhe@dhu.edu.cn) (J.-H. He).

叠棋成立积 (Die Qi Cheng Li Ji)  
 缘幂势既同 (Yuan Mi Shi Ji Tong)  
 则积不容异 (Ze Ji Bu Rong Yi)

The statement is rhymed and rather cryptic, and it is very difficult for our Western colleagues, who are unfamiliar with the Chinese language, to understand clearly its genuine meanings. Du Shiran [5] called the statement Zu-Geng's axiom. To begin our study, we write down first Wagner's translation [6]:

If blocks are piled up to form volumes, and corresponding areas are equal, then the volumes cannot be unequal.

This translation is understandable, and it is clear from the context that it is in fact a statement of Cavalieri 1635 theory, which reads

If two objects can be so oriented that there exists a plane such that any plane parallel to it intersects equal areas in both objects, then the volumes of the two objects are equal.

It was recognized in the past that the so-called Zu-Geng's axiom is equivalent to the Cavalieri theory [5–9]. The present paper reveals that the former naturally includes the later.

In the first line of the statement, the *Qi* is the cross-sections of non-zero thickness (See Fig. 1), and the *Qi* (thickness) would be reduced in thickness to a limit zero [6].

In the second line of the statement, a new mathematical term, the *Mi-Shi* (Lit. Area Power) is introduced. In ancient Chinese the *Mi* means area, and the *Shi* power, but here it should be explained as *Field*. So the *Mi-Shi* should be translated here as field of area. The *Tong* (Lit. identical, similar, same) should

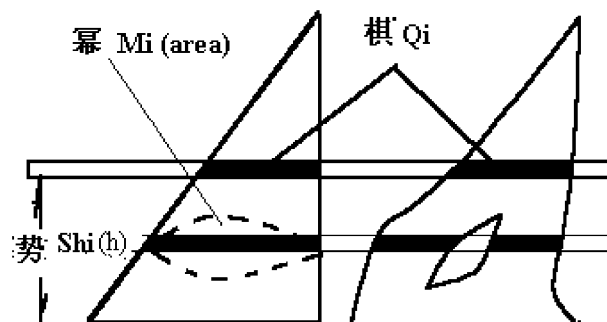


Fig. 1. The concepts of the *Qi* and the *Mi-Shi*.

be translated here as similar. So the present author re-translates the statement as follows

If infinite small blocks are piled up to form volumes, and the corresponding areas at same field of area are similar, then the volumes cannot be dissimilar.

In modern mathematical terms, we re-write the translation in the form:

If the corresponding areas at the same field of area have a similar characters, i.e.

$$S(h) = f[S'(h)],$$

where  $S$  and  $S'$  are areas of the two intersects respectively,  $h$  is the *Shi* (see Fig. 1), and  $f$  is a function, then their volumes also have the same characters, i.e.

$$V = f(V'),$$

where  $V$  and  $V'$  are volumes of the two objects respectively. Therefore, it is obvious that the so-called Zu-Geng's axiom naturally includes Cavalieri 1635 theory.

The work was done by either Zu Chongzhi (429–500 AD) or his son Zu Geng (Zu Gengzhi [6]). The former is famous for the calculation of pi ( $22/7 < \pi < 355/113$ ), which was the most accurate of his time.

Unfortunately all Zu's works were lost except for some odd fragments cited by some rescuers, for example, Li Chunfeng (602–670). The above statement might be included either in Zu Geng's *ZhuiShu* (5 or 6 volumes [9]), or in Zu Chongzhi's commentary on the *Jiu Zhang Suan Shu* (Nine Chapters on the Art of Mathematics) [7,10]. As it was pointed by Dauben [11] that the *Nine Chapters* can be regarded as a Chinese counterpart to Euclid's *Elements*, which dominated Western mathematics in the same the *Nine Chapters* came to be regarded as the seminal work of ancient Chinese mathematics for nearly two millennia. Its author and time have been the issue of polemics for a long time, one of the tradition says it was finished during the period of *Huang Di* (Yellow Emperor) in the 27th century BC, and the name then was called the *Huangdi Jiuzhang*. The *Jiuzhang Suanshu* appears to have reached its present form in the first century AD [8]. There are many commentors to the *Nine Chapters*, among which the names Liu Hui (third century AD), Zu Chongzhi (429–500 AD), and Li Chunfeng (602–670) are most famous.

Thanks to Li Chunfeng's annotated edition of *Jiuzhang Suanshu*, where a piece of Zu's works appears. And what Li Chunfeng is cited might be the only the conclusion of the theory or axiom.

### 3. Liu Hui's infinitesimal consideration

As it is now well-known that any treatment of the volume of a sphere must use infinitesimal considerations, which lead Newton (1642–1727) and Leibniz (1646–1716) to the concepts of fluxions and differential and integral calculus. The origin of infinitesimal calculus comes from a famous ancient Chinese mathematician, Liu Hui (third century AD).

Liu Hui is an applied mathematician in ancient China, he is an extremely important figure in the inheritance of *Nine Chapters* and other Chinese ancient mathematics classics before his time, and in the development of mathematics, yet our Western colleges know little about his mathematical achievements. Liu Hui is considered the one of few first mathematicians who attempted to be beyond practical calculation in *Nine Chapters* to a more abstract and reasoned mathematics, but not in Greek sense.

Liu Hui's immortal commentary (in about 263 AD) on the *Jiuzhang Suanshu* leads him to a worldwide famous scientist. He proved the Chinese *Guo-Gu* theory [12], known in the West as Pythagorean (569–475 BC) theorem, which had been widely applied in engineering before 1100 BC (see Zhou Bi Algorithm [13]), other achievements obtained by Liu can be found in details in Qian Baocong's history book [8].

Liu was the first to use the infinitesimals to obtain higher accuracy of pi. One of very sophisticated ideas obtained by Liu is that:

There is a line of one meter length, cut the line to half, leave one part away, and still cut the left half to half, the process can be continued infinitely.

In brief, and in a modern mathematical form, the procedure includes this

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} + \cdots = 1.$$

The last term  $1/(2^n)$ , in case  $n \gg 1$ , is an infinite small value different from zero.

Liu Hui uses this limit concept to calculate pi, this is described in his commentary on the *Nine Chapters* as follows

The finer we cut the segments, the less will be the loss in our calculation of the area of the circle. The exact area of the circle is obtained when such segments so cut off come to be infinitesimals.

Liu Hui first obtains a value of pi using a 192-sided polygon in a circle with a diameter of 100 m, which gives the perimeter of 192-sided polygon  $S_{192} = 314\frac{64}{625}$  (about 314.1024) meter. Using the so-called Liu Hui inequality [14]

$$S_{2n} < S_{\text{circle}} < S_{2n} + (S_{2n} - S_n).$$

He obtains [14]

$$S_{\text{circle}} = S_{192} + \frac{36}{105}(S_{192} - S_{96}) = 314 \frac{4}{25}$$

His approximation of pi ( $\pi = 3.14 \frac{4}{25}$ ) was the most accurate of his time [15].

It seems that Liu has used the following iteration formula

$$S_{n+1} = S_n + \omega(S_n - S_{n-1})$$

where  $\omega$  is the relaxation factor, the optimal value for Liu's problem is  $\omega_{\text{opt}} = 1/3$ . It is still a puzzle for mathematicians how Liu chooses such a good value of the relaxation factor  $\omega = 36/105 \approx 35/105 = 1/3$ .

In his commentary on the *Nine Chapters*, Liu Hui writes

If an object with circular cross-section is inscribed in an object with square cross-section, and every circular cross-section is inscribed in the corresponding square cross-section, then the ratio of the volumes of the objects is equal to the ratio of the areas of a circle and a circumscribed square, i.e.  $\pi:4$ .

Archimedes (285–212 BC) obtained a similar result [16]. In his tombstone, a figure of a sphere is inscribed inside a cylinder, and the 2:3 ratio of the volumes between them is indicated. It is said the solution to the problem is one of his greatest achievements.

Liu Hui, therefore, can be considered one of the greatest mathematician of all time along with Archimedes and Newton.

Some examples of Liu Hui's are translated and discussed in details by Wagner [17].

If Liu is an applied mathematicians, then Zu Chongzhi and his son must be both applied and pure (abstract) mathematician. The present author guesses that the Zu's works were lost due to their pure forms of mathematics. Many classics are preserved after Zu's time, such as the *WuCao Suanjing* in later fifth century AD, the *Jigu Suanshu* in early 6th century AD, and others in practical forms are all preserved.

#### 4. The transmission of ancient Chinese mathematics to the west

In view of Chinese ancient mathematics, ancient Chinese mathematicians paid special attention to the practical problems and skills (or algorithms) for their solutions. A glimpse at the above statement, the present author guesses

that Zu's works were of relative austerity of their axiomatic methods, proof to the theory might have given in the lost *Zhuishu* (note that there are 5 or 6 volumes!). Due to its pure mathematics, there existed no rescuers after Zu.

Zu's works had been transmitted to Japan and Korea [14], if the works were transmitted to the West via Arabian version, its effect were beyond imagining. It was guessed that Bonaventura Cavalieri (1598–1647), a famous Italian mathematician, had read some of Chinese ancient mathematics classics, and might know the works of Liu Hui, Zu Chongzhi and Zu Geng. Unfortunately all works by Zu have lost, no comparison can be made between Zu's and Cavalieri's works. It was also guessed [18] that there might be some transmission of Chinese ancient mathematical classics including the *Nine Chapters*, the *Zhuishu*, and others over the Silk Road or other central Asian trade routes which could have brought some information directly or indirectly to Europe, bypassing the Indians and Arabs. For example, some examples in the *Nine Chapters* are appeared in an Indian Mathematical book named *Lilavati* (ca. 1150),<sup>1</sup> and in Fibonacci Leonardo's *Liber Abaci* (1202).<sup>2</sup> So we have the very reason that the *Zhuishu* like the *Jiuzhang Suanshu* also transmitted to Japan, Korea, India, Arab and even to the West.

The lost *Zhuishu* is considered a greatest mathematical classic with a pure form, and it can also be regarded as another Chinese counterpart to Euclid's *Elements*. The *Zhuishu* and other lost Zu's mathematic books are generally believed to have been concerned with successive approximations, infinitesimal considerations, and other abstract (pure) mathematics.

## 5. Conclusion

Some Chinese scientists suggest that the theory is called as Liu–Zu theory in memory of Liu Hui, Zu Chongzhi and Zu Geng, The present author agrees with this issue, or at least we call the theory Zu–Cavalieri theory, or Liu–Zu–Cavalieri theory.

<sup>1</sup> In an ancient Indian mathematic book “*Lilavat*” two examples (i.e examples 148 and 153) are exactly same as those examples in the 9th Chapter of Gou-Gu theory in *Nine Chapters*, see [19].

<sup>2</sup> In “*Liber Abaci*” (1202) by Fibonacci Leonardo, the ancient Chinese Algorithm called the *Ying Buzu Shu* is introduced, and called it *De Regulis el-Chatayn* (literally rule of China), known in the West as the Rule of Double False Position. The problem discussed in section 23.8 is originated from an ancient book named the *Sun Zi Algorithms*, in about 4th century AD: *There are things of an unknown number which when divided by 3 leave 2, by 5 leave 3, and by 7 leave 2. What is the smallest number?* The solution technique of this problem is the famous *Chinese Remainder Theory* of elementary number theory, see [19].

## References

- [1] M. Kline, *Mathematical Thought from Ancient to Modern Times*, Oxford University Press, New York, 1972.
- [2] H. Eves, *An Introduction to the History of Mathematics*, fifth ed., CBS College Publishing, 1983.
- [3] H. Eves, *Great Moments in Mathematics*, The Mathematical Association of America, 1983.
- [4] B. Cavalieri, *Geometria Indivisibilibus Continuum Nova Quadam Ratione Promota*, Bologna, 1635.
- [5] S. Du, The axiom of Zu Geng, *Shuxue Tongbao* 3 (1954) 9–11.
- [6] D.B. Wagner, Liu Hui and Zu Gengzhi on the volume of a sphere, *Chinese Sci.* 3 (1978) 59–79.
- [7] S. Bai, Annotated Edition of *Jiuzhang Suanshu*, Science Press, Beijing, 1983.
- [8] B.C. Qian, *History of Chinese Mathematics*, Science Publisher, Beijing, 1992.
- [9] Y. Li, S. Du, *Chinese Mathematics a Concise History*, Clarendon Press, Oxford, 1987 (Tr. by J.N. Crossley and A.W.-C. Lun).
- [10] K. Vogel, *Neun Bucher Arithmetischer Technik*, Vieweg and Sohn, Braunschweig, 1968.
- [11] J.W. Dauben, Ancient Chinese mathematics: the *Jiu Zhang Suan Shu* vs Euclid's Elements. Aspects of proof and the linguistic limits of knowledge, *Int. J. Eng. Sci.* 36 (1998) 1339–1359.
- [12] D.B. Wagner, A proof of the Pythagorean theorem by Liu Hui (third century AD), *Historia Mathematica* 12 (1985) 71–73.
- [13] C. Cullen, *Astronomy and Mathematics in Ancient China: The Zhou Bi Suan Jing*, Cambridge University Press, New York, 1996.
- [14] Z. Ji, *Mathematics in the Northern–Southern, Sui and Tang Dynasties*, Hebei Science and Technology Publishing House, Shijiazhuang, 1999.
- [15] Archimedes (287–212 BC):  $3 \frac{1}{2} < \pi < 3 \frac{10}{71}$ .
- [16] E.T. Bell, *Men of Mathematics*, Simon and Schuster, New York, 1937, pp. 28–34.
- [17] D.B. Wagner, An early Chinese derivation of the volume of a pyramid: Liu Hui, third century AD, *Historia Mathematica* 6 (1979) 164–188.
- [18] Singmaster D., Queries on oriental sources in recreational mathematics, email letter, also available from <http://anduin.eldar.org/~problemi/singmast/chinarec.html>.
- [19] W.L. Li, *Moments in Mathematics*, Science Publisher, Beijing, 1998.