University of Texas at El Paso

From the SelectedWorks of Kien H Lim

September 2009

Mathematical habits of mind

Contact Author

Start Your Own SelectedWorks

Notify Me of New Work

Available at: http://works.bepress.com/kien_lim/13

MATHEMATICAL HABITS OF MIND

Kien Lim
University of Texas at El Paso
kienlim@utep.edu

Annie Selden
New Mexico State University
js9484@usit.net

The idea of "mathematical habits of mind" has been introduced to emphasize the need to help students think about mathematics "the way mathematicians do." There seems to be considerable interest among mathematics educators and mathematicians in helping students develop mathematical habits of mind. The objectives of this working group are: (a) to discuss various views and aspects of mathematical habits of mind, (b) to explore avenues for research, (c) to encourage research collaborations, and (d) to interest doctoral students in this topic. To facilitate the discussion during the working group meetings, we provide an overview of mathematical habits of mind, including concepts that are closely related to habits of mind—ways of thinking, mathematical practices, knowing-to act in the moment, cognitive disposition, and behavioral schemas. We invite mathematics educators who are interested in habits of mind, and especially those who have conducted research related to habits of mind, to share their work during the first working group meeting. If you would like to give a 10-minute presentation, please contact Kien Lim or Annie Selden in advance.

An Overview of Mathematical Habits of Mind

There are several terms and points of view in mathematics education that are somewhat similar or support each other, and might be brought together under the single phrase "mathematical habits of mind." We discuss several of these views that we see as related.

Habits of mind were introduced by Cuoco, Goldenberg, and Mark (1996) as an organizing principle for mathematics curricula in which high-school students and college students think about mathematics the way mathematicians do. They asserted:

> The goal is … to help high school students learn and adopt some of the ways that mathematicians *think* about problems. … A curriculum organized around habits of mind tries to close the gap between what the users and makers of mathematics *do* and what they *say*. … It is a curriculum that encourages false starts, calculations, experiments, and special cases. (p. 376)

They identified two broad classes of habits of mind: (a) general habits of mind that cuts across every discipline, and (b) content-specific habits of mind for the discipline of mathematics. General habits of mind include "pattern-sniffing," experimenting, formulating, "tinkering," inventing, visualizing, and conjecturing. Mathematical habits of mind, or mathematical approaches to things, include *talking big thinking small* (e.g., instantiating with examples), *talking small thinking big* (e.g., generalizing, abstracting), thinking in terms of functions, using multiple points of view, mixing deduction and experiment, and pushing the language (e.g., at first assuming the existence of things we want to exist, such as 2^0).

Habits of mind have two important characteristics: the "thinking" characteristic and the "habituated" characteristic. In addition, habits of mind are reflexively related to classroom practices. Below we discuss various related views of habits of mind.

The Thinking Characteristic

Harel's (2007, 2008) notion of *ways of thinking* underscores the thinking aspect of habits of mind. Harel (personal communication) regards habits of mind as internalized *ways of thinking*. In

Swars, S. L., Stinson, D. W., & Lemons-Smith, S. (Eds.). (2009). Proceedings of the 31st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Atlanta, GA: Georgia State University.

Harel's view, mathematics consists of two complementary subsets: (a) the first consists of institutionalized *ways of understanding*, which is a collection of established definitions, axioms, theorems, proofs, problems, and solutions that have been accepted by the mathematical community; and (b) the second is a collection of ways of thinking, which are conceptual tools that are useful for the generation of the first subset (Harel, 2008). The distinction between ways of thinking and ways of understanding underscores the importance of mathematical habits of mind, which tend to be neglected in traditional mathematics curricula.

According to Harel's *duality principle* (2007), "Students develop ways of thinking only through the construction of ways of understanding, and the ways of understanding they produce are determined by the ways of thinking they possess" (p. 272). This principle asserts that ways of thinking cannot be improved independently of ways of understanding, and vice versa. Hence, Harel advocates that both ways of understanding and ways of thinking should be incorporated as learning objectives for students.

In their introductory article to a special issue on advanced mathematical thinking that considered symbolizing, mathematizing, algorithmatizing, defining, and reasoning, Selden and Selden (2005) stated:

> Sometimes referred to as "mathematical habits of mind" or "mathematical practices," these [aforementioned specific] ways of thinking about and doing mathematics may be fairly widely regarded as productive, but are often left to the implicit curriculum. (p. 1)

Also, according to Bass (2005), mathematical habits of mind are critical to many aspects of the educational process. He argued that:

> the knowledge, practices, and habits of mind of research mathematicians are not only relevant to school mathematics education, but that this mathematical sensibility and perspective is essential for maintaining the mathematical balance and integrity of the educational process—in curriculum development, teacher education, assessment, etc. (p. 418)

Bass (2008, January) has considered habits of mind as *practices*—things that mathematicians do. Such practices include asking 'natural' questions, seeking patterns or structure, consulting the literature and experts, making connections, using mathematical language with care and precision, seeking and analyzing proofs, generalizing, and exercising aesthetic sensibility and taste. Bass claims that children can, and should, cultivate these practices from their early school years on. By capitalizing on children's curiosity their inquisitive minds can be harnessed. Goldenberg (2009, January) offered some strategies that capitalize on children's phenomenal language-learning ability and abstracting-from-experience ability to develop certain algebraic ideas such as *breaking* [apart] *numbers and rearranging parts* (commutative property, associative property), and *breaking arrays and describing constituent parts* (distributive property). Goldenberg provided evidence to show that children can indeed use "algebra" as a language to describe a process or a pattern and to express what they already know.

For Leikin (2007), "employing habits of mind means inclination and ability to choose effective patterns of intellectual behavior" (p. 2333). With respect to the mental habit of solving problems in different ways, Leikin considers a problem-solving strategy as a habit of mind when it is within one's "personal solution spaces of many problems from different parts of [the] mathematical curriculum" (p. 2336). One goal of mathematical instruction is then to move solutions from students' *potential solution spaces* (containing solutions that are produced with the help of others; i.e., solutions that are within one's *zone of proximal development*) into their *personal solution spaces*.

The Habituated Characteristic

The *habituated* character of habits of mind is underscored in Goldenberg's description of habits of mind, which "one acquires so well, makes so natural, and incorporates so fully into one's repertoire, that they become mental habits—one not only can draw upon them easily, but one is likely to do so" (p. 13). Mason and Spence's (1999) notion of *knowing-to act in the moment* accentuates this habituated character. They have differentiated between two types of knowledge. The first type, referred to as *knowing-about*, consists of Ryle's (1949, cited in Mason & Spence) three classes of knowledge: *knowing-that* (factual knowledge), *knowing-how* (procedural skills), and *knowing-why* (personal stories to account for phenomena). The second type, referred to as *knowing-to*, is tacit knowledge that is context/situation dependent and becomes present in the moment when it is required. This distinction is important because "knowing to act when the moment comes requires more than having accumulated knowledge-about . . ." (Mason & Spence, 1999, p. 135).

> Knowing-about … forms the heart of institutionalized education: students can learn and be tested on it. But success in examinations gives little indication of whether that knowledge can be used or called upon when required, which is the essence of knowing-to. (p. 138)

Mason and Spence advocate the practice of reflection as a means to help students improve their knowing-to act in the moment. Students should be encouraged to reflect on (a) what they have done after an action, and (b) what they are doing while enacting it, which were termed by Schön (1983) *reflection-on-action* and *reflection-in-action* respectively. With respect to reflection-in-action, students should routinely ask themselves "What do I know?" and "What do I want?" (Mason & Spence, p. 154).

The habituated character of habits of mind is also reflected in Lim's (2008) notion of *spontaneous anticipation* by a student—when he or she immediately anticipates and carries out an action for a situation based on the first idea that comes to mind. Whereas Cuoco, Goldenberg, and Mark's (1996) notion of habits of mind has a positive connotation, Lim's spontaneous anticipation can be either desirable or undesirable. *Interiorized anticipation* is desirable in that "one spontaneously proceeds with an idea without having to analyze the problem situation because one has interiorized the relevance of the anticipated action to the situation at hand" (p. 45). Interiorized anticipation is similar to Mason and Spence's notion of knowing-to. *Impulsive anticipation*, on the other hand, is undesirable in that "one spontaneously proceeds with an idea that comes to mind, without analyzing the problem situation and without considering the relevance of the anticipated action to the problem situation" (p. 44).

Lim notes that a habit of mind can also be regarded as a *cognitive disposition*—a tendency to act, mentally, in a certain way in response to certain situations. When a person has a particular habit of mind, he or she has a disposition to act according to that habit of mind. Lim (2009, January) uses the term *impulsive disposition* to refer to the proclivity of "doing whatever first comes to mind … or diving into the first approach that comes to mind" (Watson & Mason, 2007, p. 207). Lim (2009, January) offered the following strategies to address impulsive disposition: (a) do not teach algorithms and formulas prematurely; (b) pose problems that necessitate a particular algorithm or concept, that intrigue students, that require students to attend to the meaning of numbers and symbols, and that require students to explain and justify; (c) include contra-problems to promote skepticism; and (d) include superficially-similar-but-structurally-different problems on tests and examinations.

Selden and Selden (2009, January) have conceptualized (small) habits of mind as habitual situation-action pairs or *behavioral schemas*—"a form of (often tacit) procedural knowledge that

yields immediate (mental or physical) actions." They are developing this perspective in the context of proving in a design experiment with advanced undergraduate and beginning graduate students (Selden, McKee, & Selden, 2009), and in a teaching experiment with mid-level undergraduate real analysis students. Indeed, the entire proving process might be seen as a sequence of mental or physical actions (that cannot be fully reconstructed from the written proof). The individual actions often appear to be due to the enactment of behavioral schemas (that is, small, simple habits of mind). Here is an example of a common beneficial behavioral schema. The situation is having to prove a universally quantified statement such as, "For all real numbers *x*," and the linked action is writing into the proof something like, "Let *x* be a real number," meaning *x* is arbitrary but fixed. While some students are at first reluctant to write this, doing so can become habitual and automated, that is, become a behavioral schema and eventually just seems to be "the right thing to do." In contrast, a detrimental behavioral schema in proving is focusing on the hypotheses of a theorem too soon, and simply "forging ahead," without first examining the conclusion to see what is to be proved. Selden and Selden think it is likely that some larger, more complex, habits of mind can be decomposed into behavioral schemas. Also, they think this perspective would probably be useful in other kinds of reasoning, such as problem solving, and with K-12 students.

Selden and Selden think that focusing specifically on small habits of mind has two advantages. First, the uses, interactions, and origins of behavioral schemas are relatively easy to examine. For example, behavioral schemas tend to reduce the burden on working memory. Also, the process of enactment of a behavioral schema occurs outside of consciousness, but apparently the triggering situation must be conscious. Thus, such schemas cannot be "chained together" outside of consciousness with only the final action being conscious (Selden & Selden, 2008). For example, one cannot produce the solution to a linear equation without being conscious of the intervening steps. Second, this perspective is not only descriptive but also suggests concrete teaching actions, such as encouraging the writing of the formal-rhetorical parts of a proof at the beginning of the proving process (Selden & Selden, in press). In this way, it is fairly easy for a teacher to devise ways of helping a student strengthen a beneficial, or weaken a detrimental, behavioral schema.

Relating Habits of Mind and Classroom Practices

In *Fostering Algebraic Thinking: A Guide for Teachers Grades 6-10*, Driscoll (1999) views habits of mind as ways of thinking, that when used habitually, can lead to successful learning of algebra. He stresses the development of three algebraic habits of mind: (a) *doing/undoing* which involves reversing mathematical processes; (b) *building rules to represent functions* which involves pattern-recognition and generalization; and (c) *abstracting from computation* which involves thinking about computations structurally without being tied to specific numbers, such as recognizing the equivalence of 5% of 7000 and 7% of 5000. He and his colleagues later developed a four-module toolkit for educators to work with teachers to learn how to foster these algebraic habits of mind in their classrooms (see Driscoll et al., 2001). Subsequently in *Fostering Geometric Thinking: A guide for teachers grades 5-10*, Driscoll, DiMatteo, Nikula, and Egan (2007) promote four geometric habits of mind: (a) reasoning with relationships, (b) generalizing geometric ideas, (c) investigating invariants, and (d) sustaining reasoned exploration by trying different approaches and stepping back to reflect while solving a problem. *The Fostering Geometric Thinking Toolkit* was published a year later (see Driscoll et al., 2008).

Cuoco (2008, January) has advocated making mathematical habits of mind a key component of the syllabus because "without explicit attention to mathematical ways of thinking, the goals of

'intellectual sophistication' and 'higher order thinking skills' will remain elusive." He offered some suggestions for helping students cultivate desirable habits of mind: (a) working on problems with students, (b) being explicit about one's own thinking, and (c) making thought experiments an integral part of the learning experience. Rasmussen (2009, January) emphasized the need for teachers to be deliberate about initiating and sustaining particular classroom norms so as to promote certain desirable habits of mind and effect students' beliefs and values.

The RAND Mathematics Study Panel (2003) referred to "mathematical know-how—what successful mathematicians and mathematics users *do*" (p. 29) as *mathematical practices*. They also identified mathematical practices as one of the three foci for a proposed research and development program aimed at improving mathematical proficiency among U.S. school students. The Panel stated:

- "A focus on understanding these practices and how they are learned could greatly enhance our capacity to create significant gains in student achievement, especially among currently low-achieving students who may have had fewer opportunities to develop these practices" (p. 29)
- "These practices are not, for the most part, explicitly addressed in schools. Hence, whether people somehow acquire these practices is part of what differentiates those who are successful with mathematics from those who are not" (p. 32-33)

The Panel recommended the following lines of research: (a) developing an understanding of specific mathematical practices, and their interactions, along the domains of representation, justification, and generalization; (b) examining the use of these mathematical practices in different settings (e.g., in school, at home, at work); and (c) investigating ways for developing these practices in classrooms. Further, the Panel stated that "such [mathematical] practices must be deliberately cultivated and developed, and therefore research and development should be devoted to addressing this challenge." (p. 40)

Many theoretical ideas and pedagogical suggestions related to habits of mind have been raised. However, the research on this topic is still relatively thin. Using Cobb and Yackel's (1996) emergent perspective in which "learning is a constructive process that occurs while participating in and contributing to the practices of the local community" (p. 185), we regard mathematical habits of mind as individual dispositions that are reflexively related to mathematical practices of a classroom community. Cobb and Yackel suggested that "analysis whose primary purpose is psychological should be conducted against the background of an interactionist analysis of the social situation in which the student is acting" (p. 188). Hence, we encourage research on understanding the interaction between individual mathematical habits of mind and classroom mathematical practices, in addition to research on how students develop mathematical habits of mind.

Purpose of this Working Group

This working group is a follow-up to two panel-discussion sessions on "Helping Students Develop Mathematical Habits of Mind" at two consecutive Joint Mathematics Meetings (JMM) of the American Mathematical Society and the Mathematical Association of America held in 2008 and 2009. The presenters-cum-panelists at the JMM 2008 session in San Diego included Hyman Bass, Al Cuoco, Guershon Harel, and Annie Selden. The presenters-cum-panelists at the JMM 2009 session in Washington DC included Hyman Bass, Paul Goldenberg, Kien Lim, Chris Rasmussen, Annie Selden, and John Selden. Both sessions were well attended and well received by the audience. The second session was in fact an encore of the first session. Based on

attendance reactions to these two sessions, there seems to be considerable interest among mathematics educators and mathematicians in this topic. This PME-NA working group can offer a platform for mathematics educators who are interested in this topic to explore research opportunities.

The primary purpose of this working group is to generate interest among mathematics educators for conducting research related to mathematical habits of mind. The second purpose is to encourage research collaborations. The objectives of this working group are:

- To discuss various views and aspects of mathematical habits of mind.
- To explore avenues for future research.
- To facilitate mathematics educators with similar research interests to form research groups.
- To motivate doctoral students who may plan to work on this topic for their dissertations.

Proposed Activities for this Working Group

Meeting 1

- An overview on mathematical habits of mind.
- Individual presentations, if any, on research related to habits of mind.
- An open forum to discuss theoretical and pedagogical issues related to mathematical habits of mind.
- A brainstorming session to identify worthwhile avenues of research.

Meeting 2

- Small-group breakout sessions to identify research opportunities, formulate research questions, and discuss research designs.

Meeting 3

- Small-group presentations of plans for research.
- Discussion of next steps.

Anticipated Follow-up

We anticipate that promising avenues for research related to mathematical habits of mind will be identified. The working group may broaden the scope of research for some mathematics educators by integrating their existing research with research on mathematical habits of mind. This working group is likely to continue if there are groups of researchers who plan to conduct collaborative research on this topic. There may be a possibility of eventually having a special issue of a journal dedicated to mathematical habits of mind.

References

Bass, H. (2005). Mathematics, mathematicians, and mathematics education. *Bulletin of the American Mathematical Society, 42* (4), 417-430.

Bass, H. (2008, January). *Mathematical practices*. Paper presented at a Project NExT Session on Helping Students Develop Mathematical Habits of Mind, Joint Mathematics Meetings, San Diego, CA. Powerpoint available at http://www2.edc.org/CME/showcase.html.

Cobb, P., & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational Psychologist, 31* (3/4), 175-190.

Cuoco, A. (2008, January). *Mathematical habits of mind: An organizing principle for curriculum design*. Paper presented at a Project NExT Session on Helping Students Develop

Mathematical Habits on Mind, Joint Mathematics Meetings, San Diego, CA. Powerpoint available at http://www2.edc.org/CME/showcase.html.

Cuoco, A. , Goldenberg, E. P., and Mark, J. (1996). Habits of mind: An organizing principle for a mathematics curriculum. *Journal of Mathematical Behavior, 15* (4), 375-402.

Driscoll, M. (1999). *Fostering algebraic thinking: A guide for teachers grades 6-10*. Portsmouth, NH: Heinemann.

Driscoll, M., DiMatteo, R. W., Nikula, J. E., & Egan, M. (2007). *Fostering geometric thinking: A guide for teachers grades 5-10*. Portsmouth, NH: Heinemann.

Driscoll, M., DiMatteo, R. W., Nikula, J., Egan, M., June, M., & Kelemanik, G. (2008). *The fostering algebraic thinking toolkit: A guide for staff development*. Portsmouth, NH: Heinemann.

Driscoll, M., Zawojewski, J., Humez, A., Nikula, J. E., Goldsmith, L. T., & Hammerman, J. (2001). *The fostering algebraic thinking toolkit: A guide for staff development*. Portsmouth, NH: Heinemann.

Goldenberg, P. (1996). Habits of mind as an organizer for the curriculum. *Journal of Education, 178*(1), 13-34.

Goldenberg, P. (2009, January). *Mathematical habits of mind and the language-learning brain: Algebra as a second language*. Paper presented at an AMS-MAA-MER Special Session on Mathematics and Education Reform, Joint Mathematics Meetings, Washington, DC. Powerpoint available at http://www.math.utep.edu/Faculty/kienlim/hom.html.

Harel, G. (2007). The DNR system as a conceptual framework for curriculum development and instruction. In R. Lesh, J. Kaput, E. Hamilton (Eds.), *Foundations for the future in mathematics education* (pp. 263-280). Mahwah, NJ: Lawrence Erlbaum Associates.

Harel, G. (2008). What is mathematics? A pedagogical answer to a philosophical question. In B. Gold & R. Simons (Eds.), *Current issues in the philosophy of mathematics from the perspective of mathematicians*. Washington, DC: Mathematical American Association.

Leikin, R. (2007). Habits of mind associated with advanced mathematical thinking and solution spaces of mathematical tasks. In the *Proceedings of the Fifth Conference of the European Society for Research in Mathematics Education* (pp. 2330-2339). Larnaca, Cyprus.

Lim, K. H. (2008). *Students' mental acts of anticipating: Foreseeing and predicting while solving problems involving algebraic inequalities and equations*. Saarbrücken, Germany: VDM Verlag Dr. Müller.

Lim, K. H. (2009, January). *Undesirable habits of mind of pre-service teachers: Strategies for addressing them*. Paper presented at an AMS-MAA-MER Special Session on Mathematics and Education Reform, Joint Mathematics Meetings, Washington, DC. Powerpoint available at http://www.math.utep.edu/Faculty/kienlim/hom.html.

Mason, J., & Spence, M. (1999). Beyond mere knowledge of mathematics: The importance of knowing-to act in the moment. *Educational Studies in Mathematics, 38*, 135-161.

RAND Mathematics Study Panel. (2003). *Mathematical proficiency for all students: Toward a strategic research and development program in mathematics education*. Santa Monica, CA: RAND Corporation MR-1643.0-OERI.

Rasmussen, C. (2009, January). *Mathematical proficiency: An example from the inquiry oriented differential equations project*. Paper presented at an AMS-MAA-MER Special Session on Mathematics and Education Reform, Joint Mathematics Meetings, Washington, DC. Powerpoint available at http://www.math.utep.edu/Faculty/kienlim/hom.html.

Ryle, G. (1949). *The concept of mind*. London: Hutchison.

Schön, D. (1983). *The reflective practitioner: How professionals think in action*. London: Temple Smith.

Selden, A., McKee, K., & Selden, J. (2009). *Affect, behavioral schemas, and the proving process*. Manuscript submitted for publication.

Selden, A., & Selden, J. (2005). Perspectives on advanced mathematical thinking. *Mathematical Thinking and Learning*, *7*, 1-13.

Selden, A., & Selden, J. (2009, January). *Habits of mind for proving*. Paper presented at an AMS-MAA-MER Special Session on Mathematics and Education Reform, Joint Mathematics Meetings, Washington, DC. Powerpoint available at http://www.math.utep.edu/Faculty/kienlim/hom.html.

Selden, J., & Selden, A. (2008). *Consciousness in enacting procedural knowledge*. *Proceedings of the Conference on Research in Undergraduate Mathematics Education* (2008). Retrieved February 15, 2008, from http://cresmet.asu.edu/crume2008/Proceedings/Proceedings.htmlH.

Selden, J., & Selden, A, (in press). *Teaching proving by coordinating aspects of proof*s *with students' abilities*. In M. Blanton, D. Stylianou, & E. Knuth (Eds.), *The Learning and Teaching of Proof Across the Grades*, Lawrence Erlbaum Associates/Taylor & Francis, London. An earlier version is available as Technical Report No. 2007-2 at Hhttp://www.tntech.edu/techreports/techreports.htmlH.

Watson, A. & Mason, J. (2007). Taken-as-shared: A review of common assumptions about mathematical tasks in teacher education. *Journal of Mathematics Teacher Education, 10*, 205-215.