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A coupling method of a homotopy technique and a perturbation technique for non-linear problems

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Abstract

In this paper, a coupling method of a homotopy technique and a perturbation technique is proposed to solve non-linear problems. In contrast to the traditional perturbation methods, the proposed method does not require a small parameter in the equation. In this method, according to the homotopy technique, a homotopy with an imbedding parameter $p \in [0, 1]$ is constructed, and the imbedding parameter is considered as a "small parameter". So the proposed method can take full advantage of the traditional perturbation methods. Some examples are given. The results reveal that the new method is very effective and simple.

Keywords: Perturbation technique; Homotopy; Non-linearity

1. Introduction

Until recently, non-linear analytical techniques for solving non-linear problems have been dominated by the perturbation methods, which have found wide applications in engineering. But, like other non-linear analytical techniques, perturbation methods have their own limitations. Firstly, almost all perturbation methods are based on small parameters so that the approximate solutions can be expressed in a series of small parameters. This so-called small parameter assumption greatly restricts applications of perturbation techniques, as is well known, an overwhelming majority of non-linear problems have no small parameters at all. Secondly, the determination of small parameters seems to be a special art requiring special techniques. An appropriate choice of small parameters leads to ideal results, however, an unsuitable choice of small parameters results in bad effects, sometimes seriously. Thirdly, even if there exist suitable parameters, the approximate solutions solved by the perturbation methods are valid, in most cases, only for the small values of the parameters. It is obvious that all these limitations come from the small parameter assumption. So it is very necessary to develop a kind of new non-linear analytical method which does not require small parameters at all.

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In 1997, Liu [1] proposed a new perturbation technique which is not based upon small parameters but upon artificial parameters, which are embedded in the equations. To illustrate Liu's basic idea, we consider the following example [1]:

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} + u^2(t) = 1 \tag{1}$$

with initial condition $u(0) = 0$.

Embedding an artificial parameter β in Eq. (1) results in

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = (1-u)(1+\beta u). \tag{2}$$

In Liu's method the embedding parameters are considered as small parameters, as a result, Liu obtained the following first-order approximation:

$$u(t,\beta) = u_0(t) + \beta u_1(t) = (1-\mathrm{e}^{-t}) + \beta \mathrm{e}^{-t}(\mathrm{e}^{-t} + t - 1). \tag{3}$$

The substitution $\beta = 1$ results in a good approximate solution of the original Eq. (1). In Liu's method, however, the artificial parameters are embedded much artificially or technically. In most cases, the method will fail to obtain a uniformly valid approximation. For example, if we embed the artificial parameters as follows:

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = (1-\beta u)(1+u) \tag{4}$$

or

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} + \beta u^2(t) = 1 \tag{5}$$

the approximate solutions obtained from Eq. (4) or Eq. (5) will not be uniformly valid. The problem lies on the fact that the artificial parameters can in no way be considered as small parameters! It thus becomes desirable to adjust the perturbation approach in such a manner that the embedding parameters are always small.

To this end, we will give a heuristical method based on the homotopy in topology [2, 3]. The homotopy technique, or the continuous mapping technique, embeds a parameter p that typically ranges from zero to one. When the embedding parameter is zero, the equation is one of a linear system, when it is one, the equation is the same as the original one. So the embedded parameter $p \in [0, 1]$ can be considered as a small parameter. The coupling method of the homotopy technique and the perturbation technique is called the homotopy perturbation method. Details will be discussed at below.

2. Basic idea of homotopy perturbation method

Homotopy is an important part of differential topology. Homotopy techniques are widely applied to find all roots of a non-linear algebraic equations (see [2, 3] and references cited therein). Here the technique will be used to construct a perturbation equation. To illustrate its basic ideas, we consider the following non-linear algebraic equation:

$$f(x) = 0, \quad x \in \mathscr{R}. \tag{6}$$

We construct a homotopy $\mathscr{R} \times [0,1] \to \mathscr{R}$ which satisfies

$$\mathscr{H}(\xi, p) = pf(\xi) + (1-p)[f(\xi) - f(x_0)] = 0, \quad x \in \mathscr{R}, \quad p \in [0,1] \tag{7a}$$

or

$$\mathscr{H}(\xi, p) = f(\xi) - f(x_0) + pf(x_0) = 0, \quad x \in \mathscr{R}, \quad p \in [0,1], \tag{7b}$$

where p is an imbedding parameter. x_0 is an initial approximation of Eq. (6).

It is obvious that

$$\mathscr{H}(\xi, 0) = f(\xi) - f(x_0) = 0, \tag{8}$$

$$\mathscr{H}(\xi, 1) = f(\xi) = 0, \tag{9}$$

the changing process of p from zero to unity is just that of $\mathscr{H}(\xi, p)$ from $f(\xi) - f(x_0)$ to $f(\xi)$. In topology, this is called deformation, and $f(\xi) - f(x_0)$, $f(\xi)$ are called homotopic.

Due to the fact that $0 \leqslant p \leqslant 1$, so the embedding parameter can be considered as a small parameter.

Applying the perturbation technique [4], we can assume that the solution of Eqs. (7a) and (7b) can be expressed as a series in p

$$\xi = \xi_0 + p\xi_1 + p^2\xi_2 + p^3\xi_3 + \cdots . \tag{10}$$

To obtain its approximate solution of Eqs. (7a) and (7b), we, at first, expand $f(\xi)$ into a Taylor series

$$f(\xi) = f(\xi_0) + f'(\xi_0)(p\xi_1 + p^2\xi_2 + \cdots) + \frac{1}{2!}f(\xi_0)(p\xi_1 + p^2\xi_2 + \cdots)^2 + \cdots . \tag{11}$$

Substituting Eq. (11) into Eqs. (7a) and (7b), and equating the coefficients of like powers of p, we obtain

$$p^0: \quad f(\xi_0) - f(x_0) = 0, \tag{12}$$

$$p^1: \quad f'(\xi_0)\xi_1 + f(x_0) = 0, \tag{13}$$

$$p^2: \quad f'(\xi_0)\xi_2 + \frac{1}{2!}f''(\xi_0)\xi_1^2 = 0. \tag{14}$$

From Eq. (13), ξ_1 can be solved

$$\xi_1 = -\frac{f(x_0)}{f'(\xi_0)}. \tag{15}$$

If, for example, its first-order approximation is sufficient, then we have

$$\xi = \xi_0 - \frac{pf(\xi_0)}{f'(\xi_0)}. \tag{16}$$

The substitution $p = 1$ in Eq. (16) yields the first-order approximate solution of Eqs. (7a) and (7b):

$$x = \xi_0 - \frac{f(\xi_0)}{f'(\xi_0)}. \tag{17}$$

Using Eq. (17) as an initial approximation in Eq. (6) repeatedly, we have the following iteration formula:

$$x_{n+1} = \xi_n - \frac{f(\xi_n)}{f'(\xi_n)}. \tag{18}$$

From Eq. (12), we can obtain one of its solutions $\xi_0 = x_0$, under this condition, iteration formula (18) can be re-written down as follows:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{19}$$

which is the well-known Newton iteration formula.

By the same manipulation, from Eq. (14), ξ_2 can be solved, and the following iteration formula can be obtained:

$$x_{n+1} = \xi_n - \frac{f(\xi_n)}{f'(\xi_n)} - \frac{f''(\xi_n)}{2f'(\xi_n)}\left\{\frac{f(\xi_n)}{f'(\xi_n)}\right\}^2. \tag{20}$$

The iteration formula (20) is called the Newton-like iteration formula with second-order approximation. The approximate solution obtained by the above iteration formula (20) converges to its exact solution faster than the Newton iteration formula (18), for example,

$$f(x) = x^2 + x - 2 = 0. \tag{21}$$

Supposing $x_0 = 0$ be one of its initial approximate solutions, from Eq. (12), we have $\xi_0^{(1)} = 0$ and $\xi_0^{(2)} = -1$. By Newton-like iteration formula (20), we can immediately obtain its exact solutions $x_1^{(1)} = -2$ and $x_1^{(2)} = 1$ by only one iteration step.

3. Basic ideas of the proposed method

To illustrate the basic ideas of the new method for solving non-linear differential equations, we consider the following general non-linear differential equation:

$$A(u) + f(\mathbf{r}) = 0, \quad \mathbf{r} \in \Omega \tag{22}$$

with boundary conditions

$$B(u, \partial u/\partial n) = 0, \quad \mathbf{r} \in \Gamma, \tag{23}$$

where A is a general differential operator, B is a boundary operator, $f(\mathbf{r})$ is a known analytic function, Γ is the boundary of the domain Ω.

The operator A can, generally speaking, be divided into two parts L and N, where L is linear, while N is non-linear, the Eq. (22), therefore, can be rewritten as follows:

$$L(u) + N(u) - f(\mathbf{r}) = 0. \tag{24}$$

We construct a homotopy $v(\mathbf{r}, p): \Omega \times [0, 1] \to \mathscr{R}$ which satisfies

$$\mathscr{H}(v, p) = (1 - p)[L(v) - L(u_0)] + p[A(v) - f(\mathbf{r})] = 0,$$

$$p \in [0, 1], \quad \mathbf{r} \in \Omega \tag{25a}$$

or

$$\mathscr{H}(v,p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(\mathbf{r})] = 0, \quad (25b)$$

where u_0 is an initial approximation of Eq. (22).

Eq. (25a) or Eq. (25b) is called the perturbation equation with an embedding parameter, and it can be solved by the traditional perturbation techniques using the embedding variable p as a "small parameter".

4. Numerical implementation

We illustrate the basic evaluation procedure of the newly proposed method by three examples.

Example 1 (Liao [2, 3]).

$$\frac{d^2u}{dt^2} + \omega^2 u + 4q^2u^2\frac{d^2u}{dt^2} + 4q^2u\left(\frac{du}{dt}\right)^2 = 0, \quad t\in\Omega \quad (26)$$

with initial conditions $u(0) = A$, and $u'(0) = 0$, where ω and q are known constants.

We construct a homotopy $\Omega\times[0,1]\to\mathscr{R}$ which satisfies

$$L(v) - L(u_0) + pL(u_0) + p\left\{4q^2v^2\frac{d^2v}{dt^2} + 4q^2v\left(\frac{dv}{dt}\right)^2\right\} = 0, \quad (27)$$

where $Lu = d^2u/dt^2 + \omega^2 u$.

Assuming the initial approximation of Eq. (26) has the form

$$u_0(t) = A\cos\alpha\omega t, \quad (28)$$

where $\alpha(q)$ is a non-zero unknown constant with $\alpha(0) = 1$.

Supposing the approximate solution of Eq. (27) has the form

$$v = v_0 + pv_1 + p^2v_2 + \cdots. \quad (29)$$

Substituting Eq. (29) into Eq. (27), and equating the terms with the identical powers of p, we have

$$L(v_0) - L(u_0) = 0, \quad v_0(0) = A, \quad v_0'(0) = 0, \quad (30)$$

$$L(v_1) + L(u_0) + 4q^2v_0^2\frac{d^2v_0}{dt^2} + 4q^2v_0\left(\frac{dv_0}{dt}\right)^2 = 0,$$
$$v_1'(0) = v_1(0) = 0. \quad (31)$$

We always set

$$v_0 = u_0 = A\cos\alpha t. \quad (32)$$

Substituting Eq. (32) into Eq. (31) results in

$$L(v_1) + (-\alpha^2 + 1 - 2q^2\alpha^2A^2)\omega^2A\cos\alpha\omega t - 2q^2\alpha^2\omega^2A^3\cos 3\alpha\omega t = 0. \quad (33)$$

The constant α can be identified by various methods such as the method of weighted residuals (least-square method, method of collocation, Galerkin method). Herein the Galerkin method is used to determine the unknown constant, setting

$$\int_0^{\pi/(\alpha\omega)} \sin\alpha\omega t\left\{L(u_0) + 4q^2u_0^2\frac{d^2u_0}{dt^2} + 4q^2u_0\left(\frac{du_0}{dt}\right)^2\right\}dt = 0 \quad (34a)$$

or

$$\int_0^{\pi/(\alpha\omega)} \sin\alpha\omega t\{(-\alpha^2 + 1 - 2q^2\alpha^2A^2)\omega^2A\cos\alpha\omega t - 2q^2\alpha^2\omega^2A^3\cos 3\alpha\omega t\}\,dt = 0 \quad (34b)$$

leads to

$$\alpha = 1/\sqrt{1 + 2q^2A^2}. \quad (35)$$

As a result, Eq. (33) reduces to

$$L(v_1) - 2q^2\alpha^2\omega^2A^3\cos 3\alpha\omega t = 0,\ v_1'(0) = v_1(0) = 0 \quad (36)$$

with α defined as Eq. (35).

The solution of Eq. (36) can be readily obtained by the so-called variational iteration method [5, 6]:

$$v_1(t) = \frac{1}{\omega}\int_0^t \sin\omega(\tau - t) \times(-2q^2\alpha^2\omega^2A^3\cos 3\alpha\omega\tau)\,d\tau = -\frac{2q^2\alpha^2A^3}{9\alpha^2 - 1}(\cos 3\alpha t - \cos t). \quad (37)$$

We, therefore, obtain its first-order approximation:

$$u_1 = v_0 + v_1 = A\cos\alpha\omega t - \frac{2q^2\alpha^2A^3}{9\alpha^2-1}\times(\cos 3\alpha t - \cos t) \tag{38}$$

with α defined as Eq. (35).

Its period can be approximately expressed as follows:

$$T_{\text{app}} = \frac{2\pi}{\omega}\sqrt{1+2q^2A^2} \tag{39}$$

while the period obtained by the perturbation method reads [4]

$$T_{\text{pert}} = \frac{2\pi}{\omega}(1+q^2A^2) \tag{40}$$

and the exact one is [4]

$$T_{\text{ex}} = \frac{2}{\pi}\int_0^{\pi/2}\sqrt{1+4q^2A^2\cos^2 t}\,\mathrm{d}t \tag{41}$$

Formula (40) is valid only for the case when $q^2A^2 \ll 1$, while formula (39) obtained by the proposed method is valid for a very large region $0 < q^2A^2 < \infty$. Even in case $q^2A^2 \to \infty$, we have

$$\lim_{|qA|\to\infty}\frac{T_{\text{ex}}}{T_{\text{app}}} = \lim_{|qA|\to\infty}\frac{\frac{2}{\pi}\int_0^{\pi/2}\sqrt{1+4q^2A^2\cos^2 t}\,\mathrm{d}t}{\sqrt{1+2q^2A^2}} = \frac{2\sqrt{2}}{\pi} = 0.900. \tag{42}$$

Therefore, for any value of q^2A^2, it can be easily proved that

$$0 \leqslant \frac{|T_{\text{ex}} - T_{\text{app}}|}{T_{\text{app}}} \leqslant 10\%. \tag{43}$$

Example 2 (Nayfeh [4]).

$$\frac{\mathrm{d}^2u}{\mathrm{d}t^2} + \frac{\omega^2u}{1+\varepsilon u^2} = 0, \quad t\in\Omega,$$

$$u(0) = A, \quad u'(0) = 0, \quad t\in\Omega. \tag{44}$$

We construct a homotopy $\Omega\times[0,1]\to\mathscr{R}$ which satisfies

$$(1-p)[L(v)-L(u_0)] + p[(1+\varepsilon v^2)v'' + \omega^2 v] = 0, \tag{45}$$

where $Lu = \mathrm{d}^2u/\mathrm{d}t^2 + \omega^2u$.

Assuming the initial approximation of Eq. (44) has the form

$$u_0(t) = A\cos\alpha\omega t, \tag{46}$$

where $\alpha(\varepsilon)$ is a non-zero unknown constant with $\alpha(0) = 1$.

By the same manipulation as the above example, we have

$$L(v_0) - L(u_0) = 0, \quad v_0(0) = A, \quad v_0'(0) = 0, \tag{47}$$

$$L(v_1) - L(v_0) + L(u_0) + (1+\varepsilon v_0^2)v_0'' + \omega^2 v_0 = 0,$$

$$v_1'(0) = v_1(0) = 0. \tag{48}$$

Setting $v_0 = u_0 = A\cos\alpha\omega t$, the unknown α can be determined by the Galerkin method:

$$\int_0^{\pi/(\alpha\omega)}\sin\alpha\omega t\,\{(1+\varepsilon u_0^2)u_0'' + \omega^2u_0\}\mathrm{d}t = 0 \tag{49a}$$

or

$$\int_0^{\pi/(\alpha\omega)}\sin\alpha\omega t\left\{\omega^2A\left(-\alpha^2 - \frac{3\varepsilon A^2}{4}\alpha^2 + 1\right)\times\cos\alpha\omega t - \alpha^2\omega^2\frac{\varepsilon A^3}{4}\cos 3\alpha\omega t\right\}\mathrm{d}t = 0. \tag{49b}$$

The unknown α therefore can be identified

$$\alpha = 1/\sqrt{1+3\varepsilon A^2/4}. \tag{50}$$

As a result, from Eq. (48), we obtain

$$L(v_1) + \omega^2A\left(-\alpha^2 - \frac{3\varepsilon A^2}{4}\alpha^2 + 1\right)\cos\alpha\omega t - \alpha^2\omega^2\frac{\varepsilon A^3}{4}\cos 3\alpha\omega t = 0 \tag{51a}$$

or

$$v_1'' + \omega^2 v_1 - \alpha^2\omega^2\frac{\varepsilon A^3}{4}\cos 3\alpha\omega t = 0. \tag{51b}$$

By the variational iteration method [5, 6], we have

$$v_1(t) = \frac{1}{\omega}\int_0^t \sin\omega(\tau - t)\left(-\alpha^2\omega^2\frac{\varepsilon A^3}{4}\cos 3\alpha\omega\tau\right)d\tau$$

$$= -\frac{\alpha^2\omega^2\varepsilon A^3}{4(9\alpha^2-1)}(\cos 3\alpha t - \cos t) \qquad (52)$$

with α defined as Eq. (50).

If, for example, the first-order approximation is sufficient, then we have

$$u_1(t) = \lim_{p\to 1} v_1(t) = v_0(t) + v_1(t)$$

$$= A\cos\alpha\omega t - \frac{\alpha^2\omega^2\varepsilon A^3}{4(9\alpha^2-1)}(\cos 3\alpha t - \cos t) \qquad (53)$$

with α defined as Eq. (50).

The period of the solution can be expressed as follows:

$$T = \frac{2\pi}{\omega}\sqrt{1 + 3\varepsilon A^2/4} \qquad (54)$$

while the approximate solution and its period obtained by the traditional perturbation method read

$$u = A\cos(1 - \tfrac{3}{8}\varepsilon A^2)\,\omega t, \quad T_{\text{pert}} = \frac{2\pi}{\omega(1 - 3\varepsilon A^2/8)}. \qquad (55)$$

It is also interesting to point out that Eq. (55) are valid only for small εA^2, while Eqs. (53) and (54) for a very large region $0 \leqslant \varepsilon A^2 < \infty$, furthermore the approximations obtained by the proposed new method are of high accuracy.

Now we turn back to Eq. (1), which can be rewritten as follows:

$$\frac{du(t)}{dt} + \beta u(t) + u^2(t) - \beta u(t) = 1, \qquad (56)$$

where β is a non-zero constant.

A homotopy can be constructed as follows:

$$(1-p)\left(\frac{dv}{dt} + \beta v - \frac{du_0}{dt} - \beta u_0\right) + p\left(\frac{dv}{dt} + v^2 - 1\right) = 0. \qquad (57)$$

By the same manipulation, we obtain

$$\frac{dv_0}{dt} + \beta v_0 - \frac{du_0}{dt} - \beta u_0 = 0, \quad v_0(0) = 0, \qquad (58)$$

$$\frac{dv_1}{dt} + \beta v_1 + \left(\frac{dv_0}{dt} + v_0^2 - 1\right) = 0, \quad v_1(0) = 0, \qquad (59)$$

we set $u_0(t) = u(0) = 0$, in view of Eq. (58), we obtain

$$v_0 = 1 - e^{-\beta t}. \qquad (60)$$

The substitution of Eq. (60) into (59) results in

$$\frac{dv_1}{dt} + \beta v_1 + (\beta - 2)e^{-\beta t} + e^{-2\beta t} = 0. \qquad (61)$$

In order to eliminate the secular term ($te^{-\beta t}$), we set $\beta = 2$, as a result, we get

$$v_1 = \tfrac{1}{2}(e^{-4t} - e^{-2t}). \qquad (62)$$

So we have following first-order approximation:

$$u = v_0(t) + v_1(t) = 1 - e^{-2t} + \tfrac{1}{2}(e^{-4t} - e^{-2t}), \qquad (63)$$

while its exact one reads

$$u_{\text{ex}}(t) = \frac{1 - e^{-2t}}{1 + e^{-2t}}. \qquad (64)$$

So we can see clearly that its first-order approximation is of high accuracy.

5. Conclusions

In this paper we have studied few problems with or without a small parameter with the homotopy perturbation technique. The results show that

(1) A perturbation equation can be easily constructed by homotopy in topology, the embedding parameter $p \in [0, 1]$ is considered as a "perturbation parameter". The novel method can take full advantage of the traditional perturbation techniques.

(2) The initial approximation can be freely selected with unknown constants, which can be identified via various methods.

(3) The approximations obtained by this method are valid not only for small parameters, but also for very large parameters, furthermore their first-order approximations are of extreme accuracy.

Although the examples given in this paper are non-linear differential equations, it can be applicable to non-linear partial differential equations.

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