A generalized optimal control problem

mechanism design and macroeconomics

Motivation

- Many economic problems reduce to optimization of an integral with respect to some measure subject to a dynamic constraint

- Examples:

1. mechanism design – maximize the principal's profits s.t. the envelope constraint on the consumer's surplus

2. the optimal growth – maximize discounted utility s.t. capital accumulation constraint

Motivation

- In the first example the principal's objective is represented as in integral with respect to a probability measure
- The probability measure can be arbitrary. There are some evolutionary reasons to believe it is supported by a fractal
- All papers consider either discrete measures or absolutely continuos measures

Motivation

- In the second example it seems that since the integration is taken with respect to time the natural measure is the Lebesgue measure

- Psychological research shows that the private experience of the passage of time is not captured by the physical time. A fractal measure can better capture it.

The problem

- Let **Σ** be the Borel sigma-algebra on **[0,1]**
- **ώ: Σ →*[0,1]*** a probability measure
- **f, g: *R×R×[0,1]→R*** be continuous functions
- U_{ad} be a compact subset of ***R***

Consider the following problem:

$$\max \int_0^1 f(x(t), u(t), t)\, d\omega(t)$$

$$s.t.\; x = x_0 + \int_0^t g(x(\tau), u(\tau), \tau)\, d\tau$$

Approach to the solution

The problem is solved in two steps:

1. Put a multiplier on a constraint. It will lead to a problem: maximize a sum of integrals with respect to different measures

2. Use Radon - Nikodym Theorem to solve the latter problem

Step one: putting a Lagrange multiplier on the constraint

- The state variable, *x(.)*, is a continuos function
- The costate variable, *the multiplier*, is an element of the dual space: it is a linear continuous functional defined on the space of the continuous functions
- Such functionals are known as distributions

Distributions

- **A distribution is a linear continuous functional defined on the space of the continuous functions.**
- **For any distribution φ its derivative is defined by:**

$$\varphi'(h) = -\varphi(h')$$

for any infinitely differentiable h

Example

- Let ξ : *[0,1]* → *R* be an integrable function

Consider the following functional:

$$\xi(h) = \int_0^1 \xi(t)h(t)dt$$

Its derivative is:

$$\xi'(h) = -\int_0^1 \xi(t)h'(t)dt$$

A useful observation

- Any integrable function is infinitely differentiable in the sense of distributions
- Below we will always use derivative in the latter sense

The Lagrangian

- The **Lagrangian** for the optimization problem is:

$$L = \int_0^1 f(x(t), u(t), t)\, d\omega(t) + \int_0^1 \gamma(t)\left(x(t) - x_0 - \int_0^t g(x(\tau), u(\tau), \tau)\, d\tau\right) dt$$

where γ is a distribution

- Introduce distribution λ by:

$$d\lambda/dt = -\gamma, \quad \lambda(1) = 0$$

The transformed Lagrangian

- **After some integration by parts the Lagrangian becomes:**

$$L = \int_0^1 f(x,u,t)\,d\omega(t) + \int_0^1 (\lambda g(x,u,t) + x\dot{\lambda})\,dt$$

A measure theoretic problem

- Let *(Ω, Σ)* be a measure space
- *$\{\mu_i\}_{i=1}^{n}$* be a family of measures on *Σ*
- *K* is a compact subset of the real line (or *n*-dimensional Euclidian space)
- *$f_i : \Omega \times K \to R$* measurable functions
- Consider a problem:

$$\max_{x(\bullet)} \sum_{i=1}^{n} \int_{\Omega} f_i(x,t)\,d\mu_i(t)$$

Note

The generalized optimal control problem is a

particular case

of the above measure-theoretic problem

Absolute Continuity of Measures

- Let λ and μ be two measures defined on the **same** sigma-algebra
- Measure λ is called **absolutely continuous** with respect to measure μ (denoted $\lambda << \mu$) if $\mu(A)=0$ implies $\lambda(A)=0$ for any measurable A

The Radon - Nykodim derivative

- **Let λ <<μ. There exists a measurable function h(t) such that**

$$\int_{\Omega} q(t)\,d\lambda(t) = \int_{\Omega} q(t)h(t)\,d\mu(t)$$

for any measurable function q(t).

- **h(t) is called a Radon-Nykodim derivative of λ w.r.t. μ**

A simple fact

- Let $\{\mu_i\}_{i=1}^n$ be a family of measures on Σ
- There exists a measure μ such that $\mu_i << \mu$
- *Proof. For any measurable set A define*

$$\mu(A) = \frac{1}{n}\sum_{i=1}^{n}\mu_i(A)$$

Solution to the measure-theoretic problem

There exist measurable functions

***$h_i : \Omega \to R$* such that**

$$x(t) \in \arg\max_{x \in K} \sum_{i=1}^{n} h_i(t) f_i(x,t)$$