

# Application of Parameter-expanding Method to Strongly Nonlinear Oscillators

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## Abstract

The parameter-expanding method is applied to a strongly nonlinear oscillator. The obtained frequency is of high accuracy which is valid for the whole solution domain. Comparison of the obtained solution with exact solution is also given.

**Keywords:** Parameter-expanding method, Nonlinear oscillator

## 1. Introduction

The study of nonlinear oscillators is of great interest to many researchers and various methods have been proposed, for example, variational iteration method[1-5], homotopy perturbation method[6-11], variational (energy) methods [12-15], and Exp-function method[16]. Surveys of the literature with numerous references and useful bibliographies have been given in [17,18]. In this paper will apply the parameter-expanding method proposed by J.H. He[19,17,18] to a strongly nonlinear oscillator.

## 2. Parameter-expanding Method

Recently parameter-expanding methods [18,20,21] including bookkeeping parameter method[19] and modified Lindstedt-Poincare methods[22-25] have been caught much attention, the parameter expansion can also be applied to homotopy perturbation method[26].

We consider the following nonlinear oscillator

$$u'' + \frac{u}{\sqrt{1+u^2}} = 0, u(0) = A, u'(0) = 0. \quad (1)$$

In view of bookkeeping parameter method, we re-write it in the form

$$u'' + 0 \cdot u + 1 \cdot \frac{u}{\sqrt{1+u^2}} = 0. \quad (2)$$

Assume that the solution can be expressed as a power series in p:

$$u = u_0 + pu_1 + p^2u_2 + \dots, \quad (3)$$

where p is a bookkeeping parameter.

We assume that the coefficients 0 and 1 in the left side of Eq. (2) can be respectively expanded to a series in p:

$$0 = \omega^2 + p\omega_1 + p^2\omega_2 + \dots, \quad (4)$$

$$1 = a_1p + a_2p^2 + \dots. \quad (5)$$

Substituting Eqs.(4) and (5) into Eq.(2) and equating the terms with the identical powers of p, we have

$$p^0 : u_0'' + \omega^2 u_0 = 0, u_0(0) = A, u_0'(0) = 0, \quad (6)$$

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$$p^1 : u_1'' + \omega^2 u_1 + \omega_1 u_0 + a_1 \cdot \frac{u_0}{\sqrt{1+u_0^2}} = 0. \quad (7)$$

The solution of Eq. (6) can be easily obtained

$$u_0 = A \cos \omega t. \quad (8)$$

Substituting the result into Eq. (7) yields

$$u_1'' + \omega^2 u_1 + \omega_1 A \cos \omega t + a_1 \cdot \frac{A \cos \omega t}{\sqrt{1+(A \cos \omega t)^2}} = 0 \quad (9)$$

Using Fourier series expansion, we have

$$\begin{aligned} \frac{\cos \omega t}{\sqrt{1+(A \cos \omega t)^2}} &= \sum_{n=0}^{\infty} b_{2n+1} \cos[(2n+1)\omega t] \\ &= b_1 \cos \omega t + b_3 \cos 3\omega t + \dots \end{aligned} \quad (10)$$

where

$$b_1 = \frac{2}{\pi} \int_0^{\pi} \frac{\cos \tau}{\sqrt{1+(A \cos \tau)^2}} \cdot \cos \tau d\tau, \quad \tau = \omega t. \quad (11)$$

No secular terms in  $u_1$  requires

$$\omega_1 A + a_1 b_1 A = 0. \quad (12)$$

If the first-order approximation is enough, then set  $p = 1$  and from (4) and (5) we have

$$0 = \omega^2 + \omega_1, \quad (13)$$

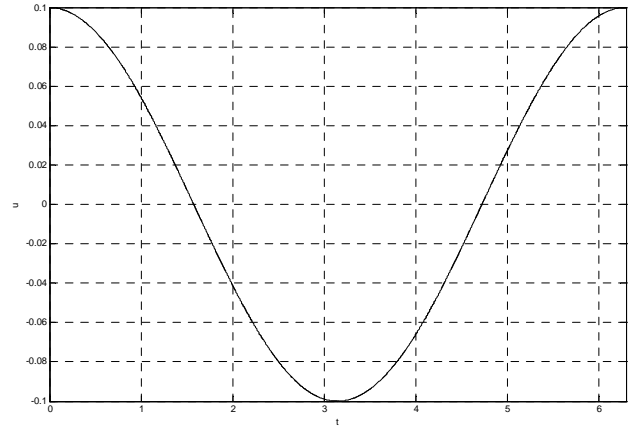
$$1 = a_1. \quad (14)$$

From Eqs.(12)~(14),we obtain

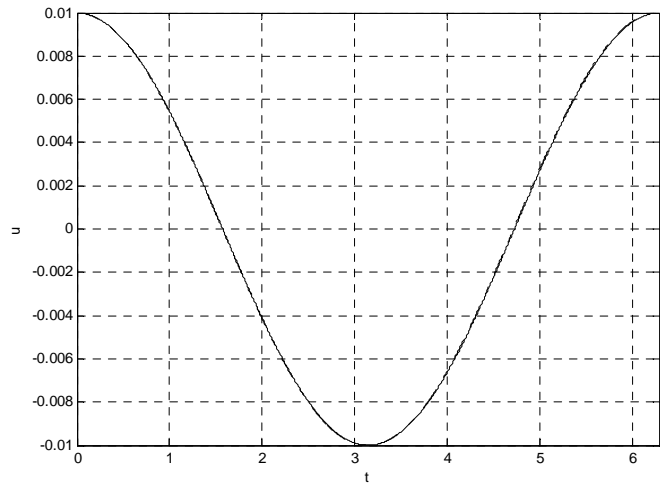
$$\omega = \sqrt{b_1}, \quad (15)$$

where  $b_1$  is defined by Eq.(11).

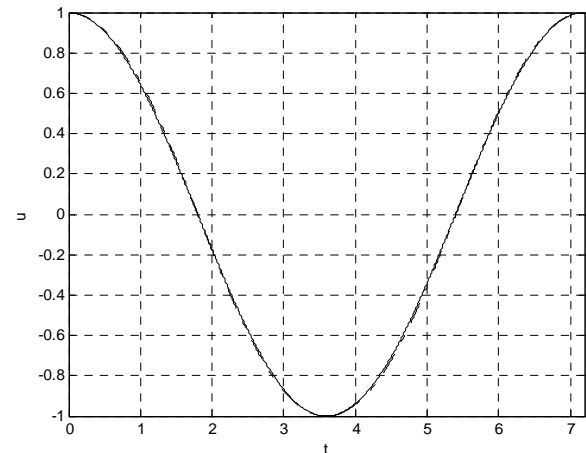
which agrees well with the exact solution. The obtained frequency is valid for all  $0 < A < \infty$ , see Fig.1.



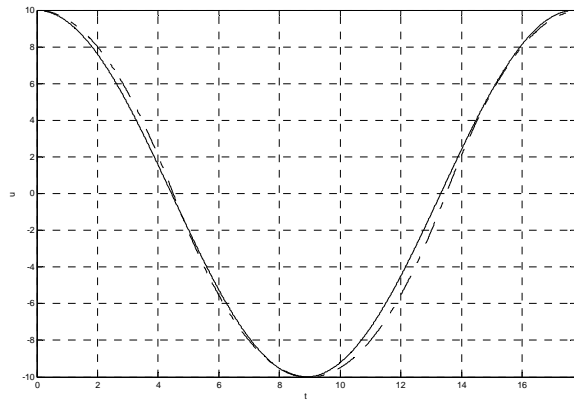
(a)  $A = 0.1, \omega_{app} = 0.998$



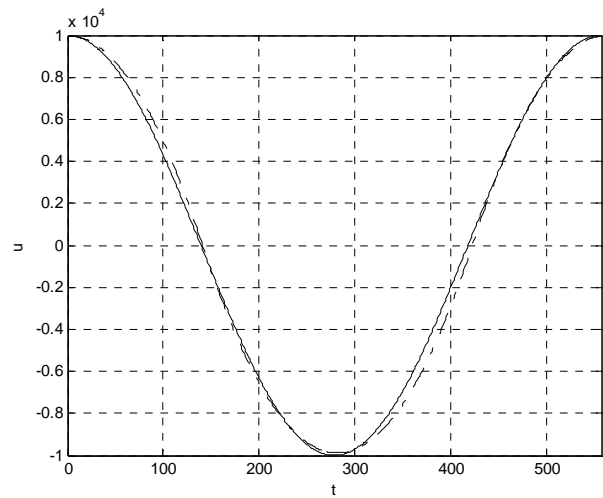
(b)  $A = 0.01, \omega_{app} = 0.99998$



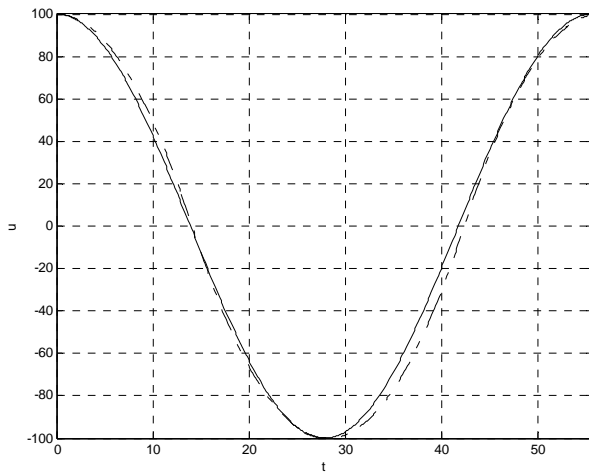
(c)  $A = 1, \omega_{app} = 0.873$



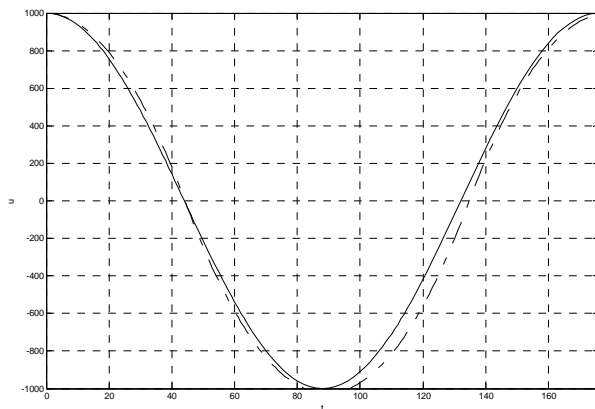
(d)  $A = 10, \omega_{app} = 0.354$



(g)  $A = 10000, \omega_{app} = 0.01128$



(e)  $A = 100, \omega_{app} = 0.1128$



(f)  $A = 1000, \omega_{app} = 0.03568$

Fig. 1 Comparison of exact solution with approximate solution  $u = A \cos \omega t$ . Dashed line: exact solution; continuous line: approximate solution.

### 3. Conclusion

The parameter-expanding method, which is proved to be a powerful mathematical tool to nonlinear oscillators, can be easily extended to any nonlinear oscillators, and the present short note can be used as paradigms for many other applications in searching for period or frequency of various nonlinear oscillators.

**Acknowledgments:** The work is supported by the Program for New Century Excellent Talents in University.

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