ASYMPTOTICS EXAMPLES FROM THE NOTES OF TOMAS RAU

1. Problems

(1) (Amemiya 3.10) Suppose $y = y^* + u$ and $x = x^* + v$, where each variable is a vector of n components. Assume y^* and x^* are non-stochastic and (u_i, v_i) is a bivariate i.i.d. random variable with mean $\mathbf{0}$ and constant variances σ_u^2, σ_v^2, respectively and covariance σ_{uv}. Assume $y^* = \beta x^*$, but y^* and x^* are not observable so that we must estimate β on the basis of y and x. Obtain the probability limit of $\hat{\beta} = (x'x)^{-1}x'y$ on the assumption that $\lim_{N\to\infty} n^{-1}x^{*'}x^* = M$.

Solution

$$\hat{\beta} = (x'x)^{-1}x'(y^* + u) = [(x^* + v)'(x^* + v)]^{-1}(x^* + v)'(y^* + u)$$

since the square bracket contain just a scalar product, we can write

$$\hat{\beta} = \frac{\beta x^{*'}x^*/n + x^{*'}u/n + v'x^*/n + v'u/n}{x^{*'}x^*/n + x^{*'}v/n + v'x^*/n + v'v/n}$$

By LLN and assumption we have

$$x^{*'}u/n \xrightarrow{p} 0 \quad v'x^*/n \xrightarrow{p} 0 \quad v'u/n \xrightarrow{p} \sigma_{vu} \quad v'v/n \xrightarrow{p} \sigma_v^2$$

hence

$$\hat{\beta} \xrightarrow{p} \frac{\beta M + \sigma_{uv}}{M + \sigma_v^2} \neq \beta$$

(2) (Amemiya 3.15) Let 1 be the vector of ones. Assuming $\lim N^{-1}1'x^* = C \neq 0$ in the model of problem 3, prove consistency of $\tilde{\beta} = 1'y/1'x$ and obtain its limiting distribution.

Solution

we proceed as follows,

$$\tilde{\beta} = \frac{1'(y^* + u)}{1'(x^* + v)} = \frac{\beta 1'x^*/n + 1'u/n}{1'x^*/n + 1'v/n}$$

by the WLLN, $1'u/n \xrightarrow{p} 0$ and $1'v/n \xrightarrow{p} 0$, so

$$\tilde{\beta} \xrightarrow{p} \beta$$

Also,

$$\sqrt{n}(\tilde{\beta} - \beta) = \sqrt{n}\left(\frac{\beta 1'x^*/n + 1'u/n}{1'x^*/n + 1'v/n} - \beta\right)$$

$$\sqrt{n}(\tilde{\beta} - \beta) = \sqrt{n}\left(\frac{1'u/n - \beta 1'v/n}{1'x^*/n + 1'v/n}\right) = \frac{\sum_{i=1}^{n}(u_i - \beta v_i)/\sqrt{n}}{1'x^*/n + 1'v/n}$$

Now, the denominator converges in probability to C (using WLLN and assumption) and the numerator converge in distribution to $(0, V)$ by Lindeberg-Levy CLT.

Note that $\mathbb{E}(u_i - \beta v_i) = 0$ and $V = \mathbb{V}(u_i - \beta v_i) = \sigma_u^2 - 2\beta\sigma_{uv} + \beta^2\sigma_v^2$ and u_i, v_i are i.i.d.

$$\sqrt{n}(\tilde{\beta} - \beta) = \left(0, \frac{\sigma_u^2 - 2\beta\sigma_{uv} + \beta^2\sigma_v^2}{C^2}\right)$$