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Variational approach for nonlinear oscillators

Ji-Huan He *

College of Science, Donghua University, 1882 Yan-an Xilu Road, Shanghai 200051, China

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Abstract

We propose a novel variational approach for limit cycles of a kind of nonlinear oscillators. Some examples are given to illustrate the effectiveness and convenience of the method. The obtained results are valid for the whole solution domain with high accuracy.

1. Introduction

Generally speaking, there exist two basic ways to describe a physical problem [1–6]: (1) by differential equations (DE) with boundary or initial conditions; (2) by variational principles (VP). DE model requires strong local differentiability (smoothness) of the physical field, while its VP partner requires weaker local smoothness or only local integrability. The VP model has many advantages over its DE partner: simple and compact in form while comprehensive in content, encompassing implicitly almost all information characterizing the problem under consideration [7,8].

Variational methods have been, and continue to be, popular tools for nonlinear analysis. When contrasted with other approximate analytical methods, variational methods combine the following two advantages: (1) they provide physical insight into the nature of the solution of the problem; (2) the obtained solutions are the best among all the possible trial-functions.

Recently, some approximate variational methods, including approximate energy method [9–12] and variational iteration method [13–19], to soliton solution, bifurcation, limit cycle, and period solutions of nonlinear equations have been given much attention.

The approximate energy approach [9,10] can be applied not only to weakly nonlinear equations, but also strongly nonlinear ones. The so obtained results are valid for the whole solution domain [11,12].

Variational iteration method is based on a general Lagrange multiplier, and it can be applied to various nonlinear equations [20–22].

In [7,8], we applied the Ritz method to soliton solution of a nonlinear wave equation (see section 2.2 in Ref. [7]). In the present paper, we suggest a Ritz-like method for nonlinear oscillators.

* Tel.: +86 216 237 9917; fax: +86 216 237 8926.
E-mail address: jhhe@dhu.edu.cn

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2. A novel variational method

In the present paper, we consider a general nonlinear oscillator in the form

$$u'' + f(u) = 0 \tag{1}$$

Its variational principle can be easily established using the semi-inverse method [1]

$$J(u) = \int_0^{T/4} \left\{ -\frac{1}{2}u'^2 + F(u) \right\} \mathrm{d}t \tag{2}$$

where T is period of the nonlinear oscillator, $\partial F/\partial u = f$.

Assume that its solution can be expressed as

$$u(t) = A\cos\omega t, \tag{3}$$

where A and ω are the amplitude and frequency of the oscillator, respectively . Substituting (3) into (2) results in

$$\begin{aligned} J(A,\omega) &= \int_0^{T/4} \left\{ -\frac{1}{2}A^2\omega^2 \sin^2 \omega t + F(A\cos\omega t) \right\} \mathrm{d}t = \frac{1}{\omega} \int_0^{\pi/2} \left\{ -\frac{1}{2}A^2\omega^2 \sin^2 t + F(A\cos t) \right\} \mathrm{d}t \\ &= -\frac{1}{2}A^2\omega \int_0^{\pi/2} \sin^2 t\,\mathrm{d}t + \frac{1}{\omega}\int_0^{\pi/2} F(A\cos t)\,\mathrm{d}t \end{aligned} \tag{4}$$

Applying the Ritz method, we require

$$\frac{\partial J}{\partial A} = 0 \tag{5}$$

$$\frac{\partial J}{\partial \omega} = 0 \tag{6}$$

But by a careful inspection, for most cases we find that

$$\frac{\partial J}{\partial \omega} = -\frac{1}{2}A^2 \int_0^{\pi/2} \sin^2 t\,\mathrm{d}t - \frac{1}{\omega^2}\int_0^{\pi/2} F(A\cos t)\,\mathrm{d}t < 0 \tag{7}$$

Thus, we modify the conditions (5) and (6) into a more simply form:

$$\frac{\mathrm{d}J}{\mathrm{d}A} = 0 \tag{8}$$

from which the relationship between the amplitude and frequency of the oscillator can be obtained.

Example 1. Consider a nonlinear oscillator with fractional potential:

$$u'' + \varepsilon u^{1/3} = 0 \tag{9}$$

Its variational formulation can be readily obtained as follows:

$$J(u) = \int_0^{T/4} \left\{ -\frac{1}{2}u'^2 + \frac{3}{4}\varepsilon u^{4/3} \right\} \mathrm{d}t \tag{10}$$

Substituting (3) into (10), we obtain

$$J(A) = \int_0^{T/4} \left\{ -\frac{1}{2}A^2\omega^2 \sin^2 \omega t + \frac{3}{4}A^{4/3}\varepsilon \cos^{4/3}\omega t \right\} \mathrm{d}t \tag{11}$$

The stationary condition with respect to A reads

$$\frac{\mathrm{d}J}{\mathrm{d}A} = \int_0^{T/4} \{ -A\omega^2 \sin^2 \omega t + A^{1/3}\varepsilon \cos^{4/3}\omega t \}\,\mathrm{d}t = 0 \tag{12}$$

which leads to the result

$$\omega^2 = \frac{\int_0^{T/4} \varepsilon \cos^{4/3}\omega t \mathrm{d}t}{A^{2/3}\int_0^{T/4} \sin^2 \omega t \mathrm{d}t} = \frac{\int_0^{\pi/2} \varepsilon \cos^{4/3} t \mathrm{d}t}{A^{2/3}\int_0^{\pi/2} \sin^2 t \mathrm{d}t} = \frac{0.163557892688082\pi^{3/2}\varepsilon}{A^{3/2}\pi/4} = \frac{1.15959526696393\varepsilon}{A^{3/2}}$$

or

$$\omega = 1.07684505243973\varepsilon^{1/2}A^{-1/3} \tag{13}$$

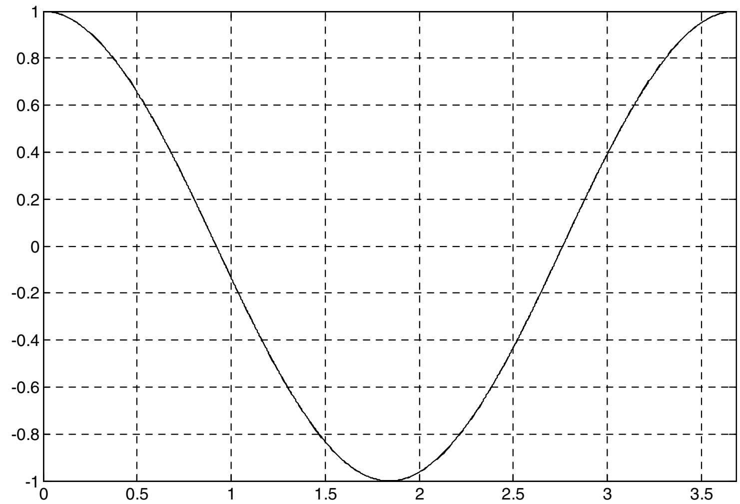
The exact frequency is $\omega = 1.070451\varepsilon^{1/2}A^{-1/3}$. The 0.597% accuracy is remarkably good considering the used crude trial solution, Eq. (3).

Example 2. Consider a more complex example in the form

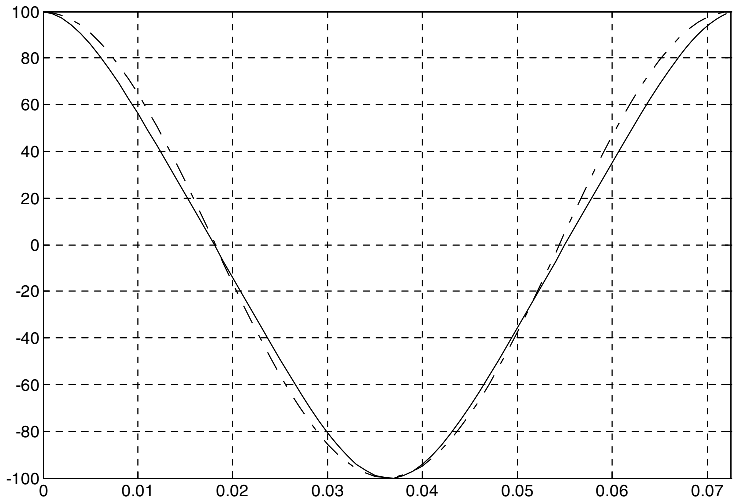
$$u'' + au + bu^3 + cu^{1/3} = 0 \tag{14}$$

Its variational form reads

$$J(u) = \int_0^{T/4} \left\{ -\frac{1}{2}u'^2 + \frac{1}{2}au^2 + \frac{1}{4}bu^4 + \frac{3}{4}cu^{4/3} \right\} \mathrm{d}t \tag{15}$$

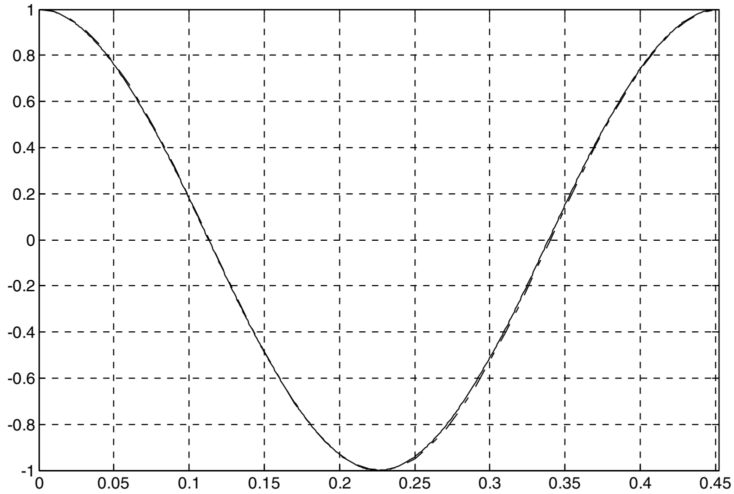


1) a=b=c=1, A=1

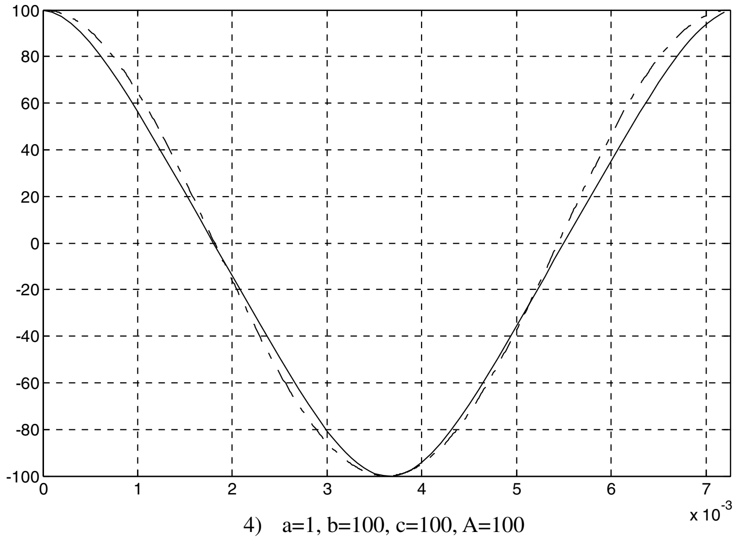


2) a=b=c=1, A=100

Fig. 1. Comparison of exact solution of Eq. (14) with approximate solution $u = A\cos\omega t$, where ω is defined by Eq. (18). Dashed line: approximate solution; continuous line: exact solution: (1) $a = b = c = 1$, $A = 1$; (2) $a = b = c = 1$, $A = 100$; (3) $a = 1$, $b = 100$, $c = 100$, $A = 1$; (4) $a = 1$, $b = 100$, $c = 100$, $A = 100$; (5) $a = 1$, $b = 100$, $c = 1000$, $A = 1$; (6) $a = 1$, $b = 100$, $c = 1000$, $A = 100$; (7) $a = 1$, $b = 1000$, $c = 1000$, $A = 1$; (8) $a = 1$, $b = 1000$, $c = 1000$, $A = 100$.



3) a=1, b=100, c=100, A=1



4) a=1, b=100, c=100, A=100

Fig. 1 (*continued*)

Substituting $u(t) = A\cos\omega t$ into (15) and making the resulted function stationary with respect to A, we obtain

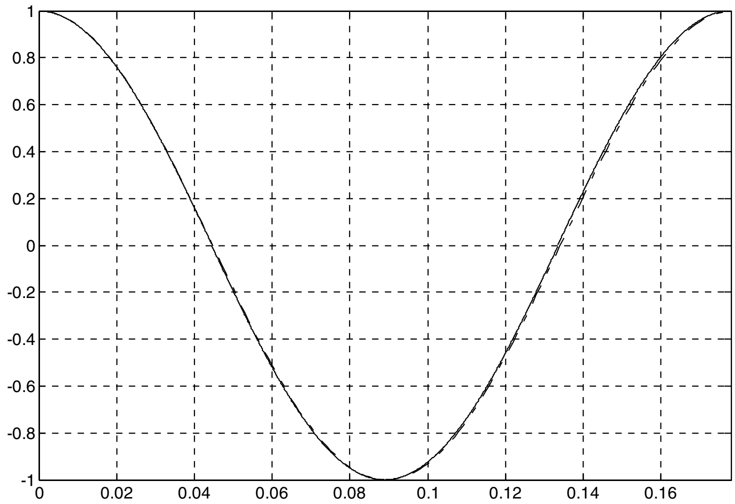
$$J(A) = \int_0^{T/4} \left\{ -\frac{1}{2}A^2\omega^2 \sin^2 \omega t + \frac{1}{2} aA^2 \cos^2 \omega t + \frac{1}{4} bA^4 \cos^4 \omega t + \frac{3}{4} cA^{4/3} \cos^{4/3} \omega t \right\} \mathrm{d}t \tag{16}$$

$$\frac{dJ}{dA} = \int_0^{T/4} \left\{ -A\omega^2 \sin^2 \omega t + aA\cos^2 \omega t + bA^3 \cos^4 \omega t + cA^{1/3} \cos^{4/3} \omega t \right\} \mathrm{d}t = 0 \tag{17}$$

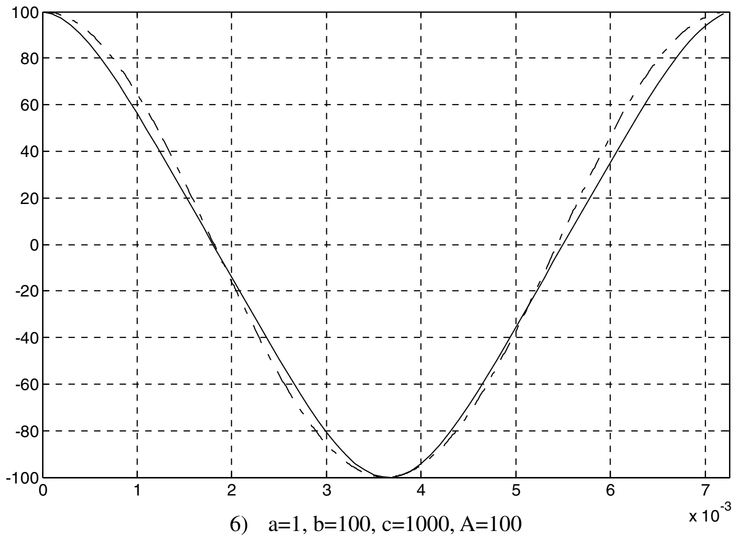
From (17), we have

$$\omega^2 = \frac{\int_0^{T/4} \{ a\cos^2 \omega t + bA^2 \cos^4 \omega t + cA^{-2/3} \cos^{4/3} \omega t \} \mathrm{d}t}{\int_0^{T/4} \sin^2 \omega t \mathrm{d}t} = \frac{\int_0^{\pi/2} \{ a\cos^2 t + bA^2 \cos^4 t + cA^{-2/3} \cos^{4/3} t \} dt}{\int_0^{\pi/2} \sin^2 t \mathrm{d}t}$$

$$= a + \frac{3}{4} bA^2 + 1.15959526696393 cA^{-2/3}$$



5) a=1, b=100, c=1000, A=1



6) a=1, b=100, c=1000, A=100

Fig. 1 (*continued*)

or

$$\omega = \sqrt{a + \frac{3}{4}bA^2 + 1.15959526696393cA^{-2/3}} \tag{18}$$

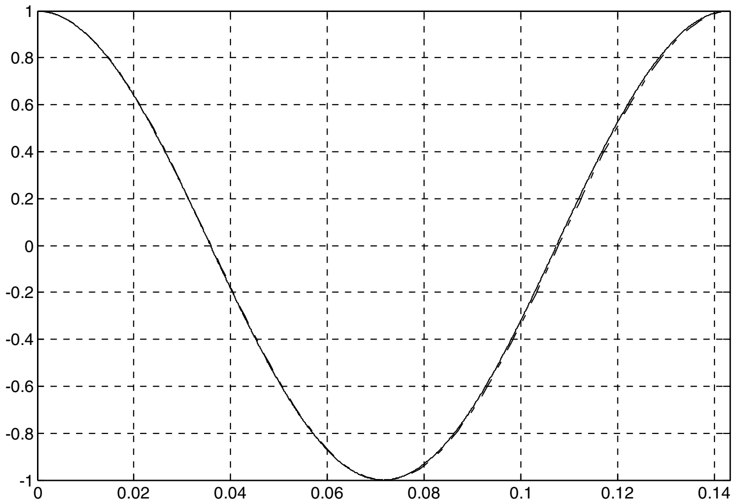
In case $a = 1$, $c = 0$, Eq. (14) reduces to the well-known Duffing equation, and its approximate frequency reads

$$\omega = \sqrt{1 + \frac{3}{4}bA^2} \tag{19}$$

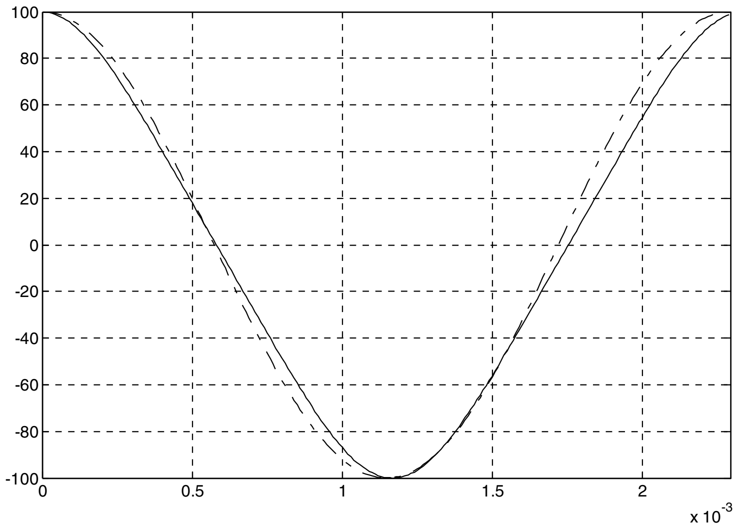
Observe that for small b, i.e. $0 < b \ll 1$, it follows that

$$\omega = 1 + \frac{3}{8}bA^2. \tag{20}$$

Consequently, in this limit, the present method gives exactly the same results as the standard perturbation method [7,8]. To illustrate the remarkable accuracy of the obtained result, we compare the approximate periods



7) a=1, b=1000, c=1000, A=1



8) a=1, b=1000, c=1000, A=100

Fig. 1 (*continued*)

$$T = \frac{2\pi}{\sqrt{1 + 3bA^2/4}} \tag{21}$$

with the exact one

$$T_{ex} = \frac{4}{\sqrt{1 + bA^2}} \int_0^{\pi/2} \frac{dx}{\sqrt{1 - k\sin^2 x}}, \tag{22}$$

where $k = 0.5bA^2/(1 + bA^2)$.

What is rather surprising about the remarkable range of validity of (21) is that the approximate period, Eq. (21), as $b \to \infty$ is also of high accuracy.

$$\lim_{b\to\infty} \frac{T_{ex}}{T} = \frac{2\sqrt{3/4}}{\pi} \int_0^{\pi/2} \frac{dx}{\sqrt{1 - 0.5\sin^2 x}} = 0.9294$$

Therefore, for any value of $b>0$, it can be easily proved that the maximal relative error of the period (21) is less than 7.6%, i.e. $|T-T_{ex}|/T_{ex}<7.6\%$.

In case $a=0,c=0$, Eq. (14) becomes

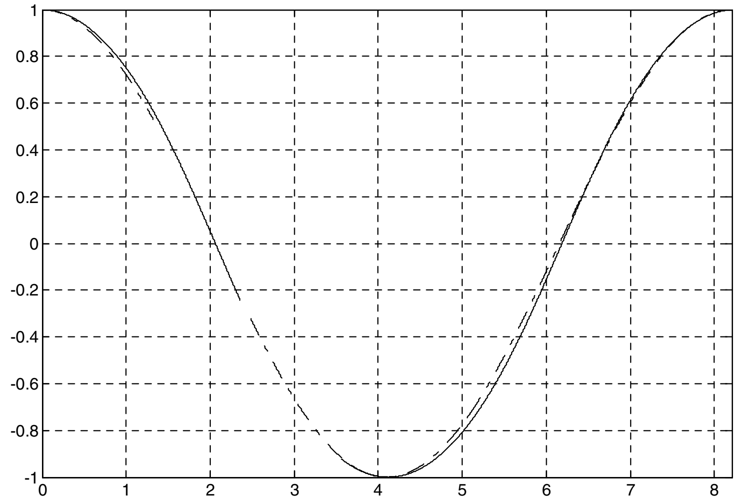
$$u''+bu^3=0, \tag{23}$$

Its frequency, then, reads

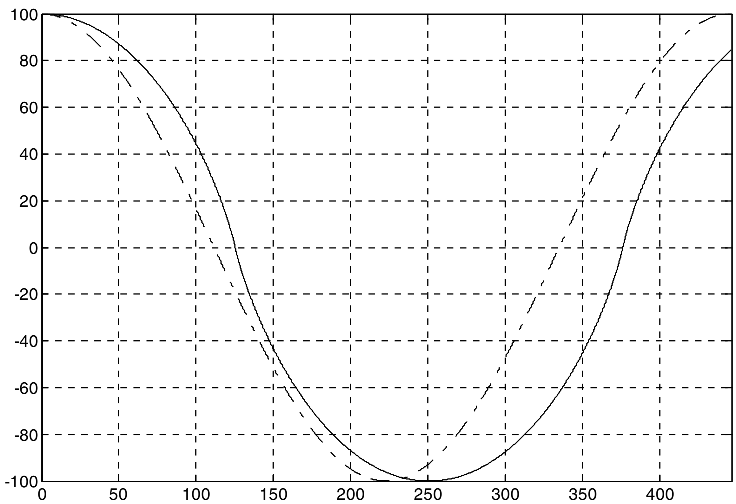
$$\omega=\sqrt{\frac{3}{4}bA^2}=0.866b^{1/2}A \tag{24}$$

Its exact frequency [8] is $\omega_{ex}=0.8472b^{1/2}A$. Therefore, its accuracy reaches 2.2%.

In case $a=b=0$, Eq. (14) turns out to be Eq. (9), it accuracy is 0.597%. Fig. 1 illustrates other various cases with different values of a, b, c, and A.

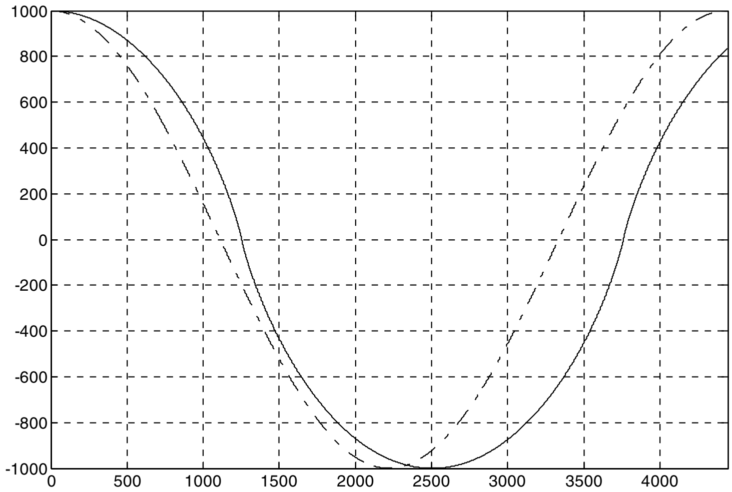


1) a=b=1, A=1, $\omega=0.76536686473018$



2) a=b=1, A=100, $\omega=0.01407125083255$

Fig. 2. Comparison of exact solution of Eq. (25) with approximate solution $u=A\cos\omega t$, where ω is defined by Eq. (29). Dashed line: approximate solution; continuous line: exact solution: (1) $a=b=1$, $A=1$, $\omega=0.76536686473018$; (2) $a=b=1, A=100$, $\omega=0.01407125083255$; and (3) $a=b=1, A=1000$, $\omega=0.00141350627908$.



3) a=b=1, A=1000, $\omega = 0.00141350627908$

Fig. 2 (*continued*)

Example 3. Consider the following nonlinear oscillator

$$u'' + \frac{u}{a + bu^2} = 0 \tag{25}$$

Its variational formulation is

$$J(u) = \int_0^{T/4} \left\{ -\frac{1}{2}u'^2 + \frac{1}{2b}\ln(a + bu^2) \right\} \mathrm{d}t \tag{26}$$

By a similar manipulation as illustrated in previous examples, we have

$$J(A) = \int_0^{T/4} \left\{ -\frac{1}{2}A^2\omega^2 \sin^2 \omega t + \frac{1}{2b}\ln(a + bA^2 \cos^2 \omega t) \right\} \mathrm{d}t \tag{27}$$

and

$$\frac{\mathrm{d}J}{\mathrm{d}A} = \int_0^{T/4} \left\{ -A\omega^2 \sin^2 \omega t + \frac{A\cos^2 \omega t}{a + bA^2 \cos^2 \omega t} \right\} \mathrm{d}t = 0 \tag{28}$$

From (28) we have

$$\omega = \sqrt{\frac{\int_0^{T/4} \frac{b\cos^2 \omega t}{a+bA^2\cos^2 \omega t}\mathrm{d}t}{\int_0^{T/4} \sin^2 \omega t \mathrm{d}t}} = \sqrt{\frac{\int_0^{\pi/2} \frac{\cos^2 t}{a+bA^2\cos^2 t \mathrm{d}t}}{\int_0^{\pi/2} \sin^2 t \mathrm{d}t}} \tag{29}$$

In case $a = 0$, Eq. (25) becomes

$$u'' + \frac{1}{bu} = 0 \tag{30}$$

and its approximate frequency is

$$\omega = \sqrt{2}b^{-1/2}A^{-1} \tag{31}$$

while its exact frequency [8] is $\omega_{\text{ex}} = 1.2533b^{-1/2}A^{-1}$. Fig. 2 reveals high accuracy of the obtained solution for other cases.

Example 4. As a last example, we consider the following nonlinear oscillator

$$u'' + au + \frac{bu}{\sqrt{1+u^2}} = 0 \quad (32)$$

Its variational formulation is

$$J(u) = \int_0^{T/4} \left\{ -\frac{1}{2}u'^2 + \frac{1}{2}au^2 + b\sqrt{1+u^2} \right\} dt \quad (33)$$

Proceeding in a similar way as before, we have

$$J(A) = \int_0^{T/4} \left\{ -\frac{1}{2}A^2\omega^2 \sin^2 \omega t + \frac{1}{2}aA^2 \cos^2 \omega t + b\sqrt{1+A^2\cos^2 \omega t} \right\} dt \quad (34)$$

and

$$\frac{dJ}{dA} = \int_0^{T/4} \left\{ -A\omega^2 \sin^2 \omega t + aA\cos^2 \omega t + \frac{bA\cos^2 \omega t}{\sqrt{1+A^2\cos^2 \omega t}} \right\} dt = 0 \quad (35)$$

From (35) we obtain

$$\omega^2 = \frac{\int_0^{T/4} \left\{ a\cos^2 \omega t + \frac{b\cos^2 \omega t}{\sqrt{1+A^2\cos^2 \omega t}} \right\} dt}{\int_0^{T/4} \sin^2 \omega t dt} = \frac{\int_0^{\pi/2} \left\{ a\cos^2 t + \frac{b\cos^2 t}{\sqrt{1+A^2\cos^2 t}} \right\} dt}{\int_0^{\pi/2} \sin^2 t dt} \quad (36)$$

In case $a = 1$, $b = -\lambda$, Eq. (32) reduces to

$$u'' + u - \frac{\lambda u}{\sqrt{1+u^2}} = 0 \quad (37)$$

Its approximate frequency is

$$\omega = \sqrt{\frac{\int_0^{\pi/2} \left\{ \cos^2 t - \frac{\lambda\cos^2 t}{\sqrt{1+A^2\cos^2 t}} \right\} dt}{\int_0^{\pi/2} \sin^2 t dt}}, \quad (38)$$

For the sake of comparison, we write down its exact frequency, which reads

$$\omega_e(A) = \frac{\pi}{2A} \left[\int_0^1 \frac{du}{\sqrt{A^2(1-u^2) - 2\lambda(\sqrt{1+A^2} - \sqrt{1+A^2u^2})}} \right] \quad (39)$$

Comparison of the approximate frequency with exact one is shown in Table 1.

Table 1
Comparison between approximate frequency and exact frequency of Eq. (37)

(A, λ)	ω	ω_e	Accuracy (%)
(1, 0.1)	0.96112904412516	0.98893067352701	2.81
(1, 0.5)	0.78666714517941	0.80849935859609	2.70
(1, 1)	0.48753501885450	0.49326576850212	1.16
(10, 0.1)	0.99371843657127	1.02253969127683	2.82
(10, 0.5)	0.96818472199692	0.99603091278763	2.79
(10, 1)	0.93528782298098	0.96171841039433	2.75
(100, 0.1)	0.99936367085476	1.02838778973372	2.82
(100, 0.5)	0.99681429219364	1.02574662698185	2.82
(100, 1)	0.99361837052412	1.02243421806133	2.82
(1000, 0.1)	0.99993636807484	1.02898096862826	2.82
(1000, 0.5)	0.99968179987111	1.02871730218752	2.82
(1000, 1)	0.99936349842641	1.02838761077250	2.82

3. Conclusion

We give a very simple but effective new method for nonlinear oscillators. The first-order approximate solutions are of a high accuracy. Of course the accuracy can be improved if higher order approximate solutions are required.

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