TIME SERIES EXAMPLES FROM THE NOTES OF TOMAS RAU

1. Problems

(1) Let the $\{y_t\}_{t=1}^T$ be a stationary stochastic process such that $y_t = \alpha y_{t-1} + \epsilon_t$ and $\epsilon_t \quad i.i.d \sim (0, \sigma^2)$. Show that the maximum likelihood estimator (conditional on y_1) of α is equivalent to the LS of y_t on y_{t-1}.

Solution

the conditional density of $Y_2|Y_1$ is given by,

$$f_{Y_2|Y_1}(y_2|y_1; \alpha, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2}\frac{(y_2 - \alpha y_1)^2}{\sigma^2}\right)$$

and,

$$f_{Y_3|Y_2,Y_1}(y_3|y_2, y_1; \alpha, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2}\frac{(y_3 - \alpha y_2)^2}{\sigma^2}\right)$$

note that in general $f_{Y_t|Y_{t-1},...,Y_1} = f_{Y_t|Y_{t-1}}$
hence the joint can be decompsed as follow,

$$\begin{aligned}
f_{Y_T,Y_{T-1},...Y_2,Y_1} &= f_{Y_T|Y_{T-1},...,Y_1} \cdot f_{Y_{T-1},..,Y_1} \\
&= f_{Y_T|Y_{T-1}} \cdot f_{Y_{T-1}|Y_{T-2},...,Y_1} \cdot f_{Y_{T-2},..,Y_1} \\
&= . \\
&= . \\
&= f_{Y_T|Y_{T-1}} \cdot f_{Y_{T-1}|Y_{T-2}} ... \cdot f_{Y_2|Y_1} \cdot f_{Y_1}
\end{aligned}$$

and the conditional on Y_1 is just,

$$f_{Y_T,....,Y_2|Y_1} = \Pi_{t=2}^T f_{Y_t|Y_{t-1}}(y_t|y_{t-1})$$

so, the conditional loglikelihood is just,

$$\begin{aligned}
\log f_{Y_T,....,Y_2|Y_1}(y_T, .., y_2|y_1; \alpha, \sigma^2) &= \sum_{t=2}^T \log f_{Y_t|Y_{t-1}}(y_t|y_{t-1}) \\
&= -\tfrac{T-1}{2}\log(2\pi\sigma^2) - \sum_{t=2}^T \tfrac{(y_t - \alpha y_{t-1})^2}{2\sigma^2}
\end{aligned}$$

maximizing the loglikelihood function with respect to α we have the FOC,

$$-\sum_{t=2}^T (y_t - \hat{\alpha} y_{t-1}) y_{t-1} = 0$$

so,

$$\hat{\alpha}_{ml} = \frac{\sum_{t=2}^T y_t y_{t-1}}{\sum_{t=2}^T y_{t-1}^2}$$

which is the LS estimator.

(2) Assume that $\{\varepsilon_t\}$ and $\{\eta_t\}$ are i.i.d. sequences of random variables, independent of one another, with zero means, unit variances, and finite fourth moments. Let x_t be a weak stationary sequence satisfying the difference equation

$$x_t = \gamma x_{t-1} + \eta_t$$

where $|\gamma| < 1$; let y_t be the sequence given by

$$y_t = \beta x_t + \varepsilon_t$$

a) Show that the y_t are weak stationary.

b) Suppose you observe a sample of observations on (x_t, y_t) for $t = 1, .., T$ and regress y_t on x_t (without intercept). Assuming strong ergodicity for x_t find the limiting distribution of the least square regression coefficient. You can assume x_0 fixed and equal 0 for a) and b).

Solution

a) note that x_t can be written as follows

$$x_t = \gamma^t x_0 + \sum_{i=0}^{t-1} \gamma^i \eta_{t-i} = \sum_{i=0}^{t-1} \gamma^i \eta_{t-i}$$

since x_t is stationary and x_0 is fixed and equal to 0, we have that $\mathbb{E}(x_t) = 0$.(We also have that $\gamma^t \to 0$ since $|\gamma| < 1$ by stationarity).

$$\mathbb{E}(y_t) = \mathbb{E}(\beta x_t) = \mathbb{E}(\beta \sum_{i=0}^{t-1} \gamma^i \eta_{t-i}) = 0$$

$$\begin{aligned} \mathbb{V}(y_t) &= \mathbb{V}(\beta x_t + \varepsilon_t) = \beta^2 \mathbb{V}(\textstyle\sum_{i=0}^{t-1} \gamma^i \eta_{t-i}) + \mathbb{V}(\varepsilon_t) \\ &= \beta^2 \textstyle\sum_{i=0}^{t-1} \gamma^{2i} \sigma_\eta^2 + \sigma_\varepsilon^2 \\ &= \beta^2 \frac{1}{1-\gamma^2} + 1 \end{aligned}$$

note that x_t can be written as follows

$$x_t = \gamma^s x_{t-s} + \sum_{i=0}^{s-1} \gamma^i \eta_{t-i}$$

Now,

$$\begin{aligned} \mathbb{E}(y_t y_{t-s}) &= \mathbb{E}[(\beta x_t + \varepsilon_t)(\beta x_{t-s} + \varepsilon_{t-s})] \\ &= \mathbb{E}[(\beta^2 x_t x_{t-s})] \\ &= \mathbb{E}[\beta^2 \gamma^s x_{t-s}^2 + x_{t-s} \textstyle\sum_{i=0}^{s-1} \gamma^i \eta_{t-i}] \end{aligned}$$

note that we can express x_{t-s} as $\sum_{i=0}^{t-s-1} \gamma^i \eta_{t-s-i}$. Hence,

$$\begin{aligned} \mathbb{E}(y_t y_{t-s}) &= \mathbb{E}[\beta^2 \gamma^s x_{t-s}^2 + (\textstyle\sum_{i=0}^{t-s-1} \gamma^i \eta_{t-s-i})(\textstyle\sum_{i=0}^{s-1} \gamma^i \eta_{t-i})] \\ &= \frac{\beta^2 \gamma^s}{1-\gamma^2} \end{aligned}$$

Hence, y_t is covariance stationary.

b)

$$\hat{\beta}_{LS} = \frac{\sum y_t x_t}{\sum x_t^2} = \beta + \frac{\sum \varepsilon_t x_t}{\sum x_t^2}$$

$$\sqrt{T}(\hat{\beta}_{LS} - \beta) = \frac{\frac{1}{\sqrt{T}} \sum \varepsilon_t x_t}{\frac{1}{T} \sum x_t^2}$$

since ε_t is uncorrelated to x_t (recall that x_t can be written as a function of $\eta' s$), we have that

$$\mathbb{V}(\varepsilon_t x_t) = \mathbb{V}(\varepsilon_t)\mathbb{V}(x_t) = \frac{1}{1-\gamma^2}$$

$$\mathbb{E}(\varepsilon_t x_t) = \mathbb{E}(\varepsilon_t)\mathbb{E}(x_t) = 0$$

$$Cov(\varepsilon_t x_t, \varepsilon_{t-s} x_{t-s}) = \mathbb{E}(\varepsilon_t x_t \varepsilon_{t-s} x_{t-s}) = \mathbb{E}(\varepsilon_t \varepsilon_{t-s})\mathbb{E}(x_t x_{t-s}) = 0$$

We can apply dependent CLT, and strong ergodicity and Slutsky theorem

$$\frac{1}{\sqrt{T}} \sum \varepsilon_t x_t \xrightarrow{d} \left(0, \frac{1}{1-\gamma^2}\right)$$

$$\frac{1}{T} \sum x_t^2 \xrightarrow{p} \frac{1}{1-\gamma^2} = \mathbb{E}(x_t^2)$$

Hence,

$$\sqrt{T}(\hat{\beta}_{LS} - \beta) \xrightarrow{d} (0, 1-\gamma^2)$$

provided that $|\gamma| \neq 1$