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When Does a Category Built on a Lattice with a Monoidal Structure have a Monoidal Structure?

When Does a Category Built on a Lattice with a Monoidal Structure have a Monoidal Structure?

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Abstract

In a word, sometimes. And it gets harder if the structure on L *is* not commutative. In this paper we consider the question of what properties are needed on the lattice L equipped with an operation * for several different kinds of categories built using **Sets** and L to have monoidal and monoidal closed structures. This works best for the Goguen category **Set(L)** in which membership, but not equality, is made fuzzy and maps respect membership. Commutativity becomes critical if we make the equality fuzzy as well. This can be done several ways, so a progression of categories is considered. Using sets with an L-valued equality and functions which respect that equality gives a monoidal category which is closed if we use a strong form of the transitive law. If we use strict extensional total relations and a strong transitive law (and * is commutative and nearly idempotent), we get a monoidal structure. We also recall some constructions by Mulvey, Nawaz, and Hohle on quantales with properties making them commutative enough to have (non-symmetric) monoidal structures.

If $(L,$ A$, v)$ is a lattice, then we can consider it to be a category using the order. Hence we can form both the category of presheaves $\mathbf{Set}^{LOP}$ and the category of sheaves on L taking as covers of a the subsets C *of* L such that $V\ C = a$. This makes sense even if L is not complete, though we usually assume that L is a complete Heyting algebra. Both the category of presheaves and the category of sheaves are topoi, so they have a very nice logical structure and are not only cartesian closed but also locally cartesian closed. The logic which results is not, however, usually the kind encountered in fuzzy sets. It is intuitionistic.

So the question in the title becomes: If L has another connective * which is sufficiently well behaved (like a t-norm on *10,11,* for instance) what properties will be induced on the categories built on L. The presheaf and sheaf categories are not particularly good candidates for reflecting such a structure since their properties *tend* to come more from **Sets** than from the lattice properties, perhaps with sheafification needed. Looking at constructions like fuzzy sets and sets with a lattice valued equality seems more promising.

In general, the categories we consider have either L-valued membership or L-valued equality. In either case there is an obvious structure for A $\otimes$ $\boldsymbol{B}$—take the product set A x B and take the * operation on the relevant membership or equality functions. We will need to show that the resulting structure has any other properties we put on our objects

and that the maps needed are in fact maps in the relevant category. The isomorphisms in the definition of a monoidal category come from the isomorphisms for products in **Sets**, so the coherence conditions will be automatic.

The question becomes:

What properties do you need on $\star$ to get nice properties on $\otimes$?

1. Conditions on the lattice of truth values

We start with a lattice $(L, \wedge, \vee)$ which can be considered to be a category using the order to give the morphisms. There are standard additional conditions one may place on L:

1. Existence of a top element $\top$. This is a common way to give a value for full truth.
2. Existence of a least element $\bot$. This is a common way to give a value for full falsehood.
3. Completeness: existence of arbitrary suprema and infima, often needed to get a suitable quantification for first order logic.
4. Distributivity: For all $a, b, c \in L$ we have $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.
5. $\wedge \bigvee$-distributivity: $a \wedge \bigvee_{\lambda \in \Lambda} b_\lambda = \bigvee_{\lambda \in \Lambda}(a \wedge b_\lambda)$.
6. $\vee \bigwedge$-distributivity: $a \vee \bigwedge_{\lambda \in \Lambda} b_\lambda = \bigwedge_{\lambda \in \Lambda}(a \vee b_\lambda)$.

Giving an alternate conjunction $\star$ we usually ask for some of the following conditions:

1. Bifunctoriality: $\star$ is a bifunctor from $L \times L$ to L, that is if $a \leq b$ and $c \leq d$ then $a\star c \leq b\star d$.
2. Associativity: $a\star(b\star c) = (a\star b)\star c$.
3. Commutativity: $a\star b = b\star a$.
4. Tempered commutativity: $a\star b\star c = a\star c\star b$ (my term for the key property used by Mulvey and Nawaz in [9]).
5. Idempotence: $a\star a = a$ (which often forces $a\star b = a \wedge b$).
6. Nilpotence: for any a there is an n so that if we take n copies of a and combine them using $\star$ we get $a\star \cdots \star a = \bot$ (for t-norms this is characteristic of the Łukaciewicz t-norm).
7. If L has a top element $\top$ we can ask for
   (a) Right sided: $a\star\top \leq a$ with strict right sidedness if we get equality.
   (b) Left sided: $\top\star a \leq a$ with strict left sidedness if we get equality.
8. Units: a right unit u has $a\star u = a$ for all a and a left unit has $u\star a = a$. If an operation has both right and left units they will be the same. They may not be $\top$.
9. If L has arbitrary $\bigvee$ we can ask for distributive laws:
   (a) Left distributive: $a\star \bigvee_{\lambda \in \Lambda} b_\lambda = \bigvee_{\lambda \in \Lambda}(a\star b_\lambda)$.
   (b) Right distributive: $(\bigvee_{\lambda \in \Lambda} b_\lambda)\star a = \bigvee_{\lambda \in \Lambda}(b_\lambda\star a)$.
10. Autonomous or closed structures:
   (a) Left: $a\star - : L \to L$ has a right adjoint $a \searrow -$ with $a\star b \leq c$ if and only if $b \leq a \searrow c$. Note that this gives $a\star(a \searrow c) \leq c$ and tells us that $a\star -$ will preserve any joins which exist since it is a left adjoint.
   (b) Right: $-\star a : L \to L$ has a right adjoint $- \swarrow a$ with $b\star a \leq c$ if and only if $b \leq c \swarrow a$. Note that this gives $(c \swarrow a)\star a \leq c$ and tells us that $-\star a$ will preserve any joins which exist.

A monoidal structure on a partially ordered set L (thought of as a category) is given by a $\star$ with bifunctoriality, associativity, and a two sided unit. It is a symmetric monoidal structure if it is commutative. It is monoidal closed if it satisfies autonomy.

If L is a complete lattice and $\star$ is distributive we have a simple formula for the closed structure:

$$a \searrow b = \bigvee\{c | a\star c \leq b\} \quad \text{and} \quad b \swarrow a = \bigvee\{c | c\star a \leq b\}$$

A long literature gives many names for structures satisfying different collections of these axioms (ordered semigroups, lattice ordered semigroups, c.l.o.s.g's, frames, quantales, Gelfand quantales, etc.). To avoid confusion I will specify which properties are being used in each construction. For Goguen's category very little is needed; for categories based on an L-valued equality quite a lot more is needed.

2. Goguen's Set(L)

Goguen's **Set**(L) from [3] makes only membership fuzzy and uses maps which respect degree of membership:

Definition 1. The category of L-valued fuzzy sets: **Set**(L) has

- Objects: pairs (A, α) where A is a set and $\alpha : A \to L$.
- Morphisms: maps $f : (A, \alpha) \to (B, \beta)$ are functions $f : A \to B$ such that $\beta(f(a)) \geq \alpha(a)$ for all $a \in A$.

This category is a quasitopos which is topological over **Sets**. The logic of the quasitopos is classical since strong subobjects have the form $(A', \alpha|_{A'}) \hookrightarrow (A, \alpha)$. To get the logic of interest in fuzzy set theory we look at the unbalanced subobjects: $\mathrm{id}_A : (A, \alpha') \longrightarrow (A, \alpha)$ with $\alpha'(a) \leq \alpha(a)$ for all $a \in A$. Following the notation of [12] the lattice of all such subobjects is $\mathcal{U}(A, \alpha)$. These subobjects result from maps which are both monomorphisms and epimorphisms but are not necessarily isomorphisms, showing that the category **Set**(L) is not balanced, hence is not a topos.

This category is sometimes criticized for not being fuzzy enough: it makes membership fuzzy, but not equality. Using upper cuts we can see how any such object gives a presheaf on L in which each transition map is a monomorphism:

$$(A, \alpha)^\uparrow(l) = \{a \in A \,|\, \alpha(a) \geq l\}$$

The morphisms of **Set**(L) give presheaf morphisms. Not all such presheaves arise from upper cuts, however.

Eytan [1] constructed a related category called **Fuzz**(H) where H is a Heyting algebra. Pitts [10] identified this category as equivalent to the category of subsheaves of constant sheaves on H, demonstrating that it is not a topos.

Looking at it as a sheaf or presheaf category does not help much with seeing the monoidal structure, but Goguen's approach does. Since it is the monoidal structure which reflects the logic used in fuzzy sets this is an advantage to the Goguen category.

2.1. Monoidal closed structures on **Set**(L)

For **Set**(L) existence of a monoidal structure follows quickly:

Theorem 1. *If L has a bifunctor $\star : L \times L \to L$ then there is a bifunctor $\otimes : \mathbf{Set}(L) \times \mathbf{Set}(L) \to \mathbf{Set}(L)$ given by $(A, \alpha) \otimes (B, \beta) = (A \times B, \alpha \star \beta)$. If in addition*

1. *$\star$ is associative, then so is $\otimes$, up to isomorphism.*
2. *$\star$ is commutative, then so is $\otimes$, up to isomorphism.*
3. *$\star$ has a unit u, then $(\{*\}, u)$ (the constant function) is a unit for $\otimes$ up to isomorphism. In general $(\{*\}, u)$ is an unbalanced subobject of the terminal.*
4. *$\star$ is left sided then the canonical map $A \to A \times \{*\}$ gives a map in $\mathbf{Set}(L)$ from $(A, \alpha) \to (A, \alpha) \otimes (\{*\}, \top)$. A similar situation holds if $\star$ is right sided.*
5. *L has $\bigwedge$ and $\star$ is left-autonomous and then $(A, \alpha) \otimes -$ has a right adjoint (giving a closed category structure on $\mathbf{Set}(L)$). Similarly, if $\star$ is right-distributive then $- \otimes (A, \alpha)$ has a right adjoint.*

Proof. Much of this is immediate. The isomorphisms for associativity, commutativity, unit, and one-sidedness come from those for the product and terminal in **Sets**. I will give the argument for the closed structures. The construction is essentially as given by Pultr [11].

Given (A, α) and (B, β) the object $(A, \alpha) \searrow (B, \beta) = (B^A, \xi)$ where B^A is the set of all functions from A to B and

$$\xi(f) = \bigwedge_{a \in A} (\alpha(a) \searrow \beta(f(a)))$$

Given a morphism $(C, \gamma) \xrightarrow{h} (A, \alpha) \searrow (B, \beta)$ we get $\hat{h} : A \times C \to B$ with $\hat{h}(a, c) = h(c)(a)$. We need to show that $\alpha(a) \star \gamma(c) \leq \beta(h(c)(a))$ for all $a \in A$ and $c \in C$. We know that

$$\gamma(c) \leq \bigwedge_{t \in A} (\alpha(t) \searrow \beta(h(c)(t)))$$

so

$$\alpha(a) \star \gamma(c) \leq \alpha(a) \star \bigwedge_{t \in A} (\alpha(t) \searrow \beta(h(c)(t)))$$

$$\leq \alpha(a) \star (\alpha(a) \searrow \beta(h(c)(a)))$$

$$\leq \beta(h(c)(a))$$

Similarly if $\alpha(a) \star \gamma(c) \leq \beta(h(c)(a))$ for all pairs (a, c) then $\gamma(c) \leq \alpha(a) \searrow \beta(h(c)(a))$ for all a, so

$$\gamma(c) \leq \bigwedge_{t \in A} (\alpha(t) \searrow \beta(h(c)(t)))$$

A parallel proof switching sides shows that given (A, α) and (B, β) the object $(B, \beta) \swarrow (A, \alpha) = (B^A, \zeta)$ where B^A is the set of all functions from A to B and

$$\zeta(f) = \bigwedge_{a \in A} (\beta(f(a)) \swarrow \alpha(a)). \quad \square$$

Thus in particular if L has a monoidal closed structure given by $\star$ and the two right adjoints $\searrow$ and $\swarrow$ with unit u and L has $\bigwedge$ then $\mathbf{Set}(L)$ is monoidal closed. If $\star$ is commutative, then the monoidal structure is symmetric.

2.2. Monoidal structures on subobject lattices

There are three kinds of subobject lattices we might consider for this category: the strong subobjects considered in the quasitopos structure, the unbalanced subobjects studied in [12], and the lattices of L-valued fuzzy subsets of a crisp set. For the monoidal structure only the latter two are of interest.

If we always have $x \star x \leq x$ then the lattices of unbalanced subobjects $\mathcal{U}(A, \alpha)$ inherit a bifunctor structure from $\star$ fairly directly since if $\alpha' \leq \alpha$ and $\alpha'' \leq \alpha$ then $\alpha' \star \alpha'' \leq \alpha$. Commutativity, associativity, and right or left sidedness will follow from the same properties of $\star$. The existence of a unit is lost unless the unit for $\star$ is less than or equal to $\alpha(a)$ for all $a \in A$. Autonomy does follow, though: If (A, α') and (A, α'') are unbalanced subobjects of (A, α) then

$$\alpha' \searrow \alpha''(a) = \alpha(a) \wedge (\alpha'(a) \searrow \alpha''(a))$$

in $\mathcal{U}(A, \alpha)$. A further study of the properties of these lattices is in [13].

The lattice of fuzzy subsets of a crisp set with values in L will inherit all of the properties considered here of L, computed pointwise.

3. Constructions with L-valued equality

A lot of the development of categories of L-valued sets of one kind or another has been influenced by the construction of a category equivalent to the category of sheaves on a complete Heyting algebra first written down by Higgs (in the Boolean case) in a widely circulated unpublished paper from 1973 [4]. The first published version was in Loullis [7,8], who unfortunately died in his 20's almost immediately after his papers appeared. This approach is also in Fourman and Scott [2]. The generalization from Heyting algebras to more general residuated lattices (in particular quantales) has involved a number of technical modifications, so this section deals with several different constructions of what an L-valued equality should be and how a category of sets equipped with such an equality should be constructed. A straightforward analog of the Higgs construction works in the commutative case, but not if $\star$ is non-commutative. The aim of the constructions of Mulvey and Nawaz [9] and Höhle [5,6] has been to find parallels to the construction of the category of sheaves on a complete Heyting algebra, usually attempting to find classifiers for some kind of subobjects. Those constructions use non-commutative quantales with additional conditions sufficient to recover a monoidal structure; they give no construction for a closed structure.

3.1. Higgs constructions using ⋆

Perhaps the most straightforward approach (though not the one most often taken in the literature) to a $(L, \star)$ valued equality is:

Definition 2. A $(L, \star)$-valued equality on a set A is a function $\delta : A \times A \to L$ such that

1. Strictness: For all $a, a' \in A$ we have $\delta(a, a') \leq \delta(a, a)$.
2. Symmetry: For all $a, a' \in A$ we have $\delta(a, a') = \delta(a', a)$.
3. Transitivity: For all $a, a', a'' \in A$ we have $\delta(a, a') \star \delta(a', a'') \leq \delta(a, a'')$.

Strictness replaces reflexivity because we want to allow elements with incomplete existence. We can use $\varepsilon(a) = \delta(a, a)$ for the extent of existence of a.

Theorem 2. *If L is equipped with an associative, tempered commutative, bifunctorial operation $\star$ and (A, δ_A) and (B, δ_B) are sets equipped with an L-valued equality, then so is $(A \times B, \delta_A \star \delta_B)$.*

Proof. We need to show strictness, symmetry, and transitivity.
For strictness we look for any $a, a' \in A$ and $b, b' \in B$:

$$\delta_A(a, a') \leq \delta_A(a, a)$$
$$\delta_B(b, b') \leq \delta_B(b, b) \text{ so}$$
$$\delta_A(a, a') \star \delta_B(b, b') \leq \delta_A(a, a) \star \delta_B(b, b) \quad \text{by bifunctoriality}$$

Symmetry follows from $\star$ being well defined and the symmetry of δ_A and δ_B.
Transitivity needs tempered commutativity, associativity, and bifunctoriality: Given that $\delta_A(a, a') \star \delta_A(a', a'') \leq \delta_A(a, a'')$ and $\delta_B(b, b') \star \delta_B(b', a'') \leq \delta_B(b, b'')$ so we compute using associativity and tempered commutativity:

$$\begin{aligned}
(\delta_A(a, a') \star \delta_A(a', a'')) \star (\delta_B(b, b') \star \delta_B(b', a'')) &= \delta_A(a, a') \star (\delta_A(a', a'') \star \delta_B(b, b')) \star \delta_B(b', a'') \\
&= \delta_A(a, a') \star (\delta_B(b, b') \star \delta_A(a', a'')) \star \delta_B(b', a'') \\
&= (\delta_A(a, a') \star \delta_B(b, b')) \star (\delta_A(a', a'') \star \delta_B(b', a'')) \\
&\leq \delta_A(a, a'') \star \delta_B(b, b'')
\end{aligned}$$

using bifunctoriality in the last step. □

Definition 3. A $(L, \star)$-equality respecting map from (A, δ_A) to (B, δ_B) is a function $f : A \to B$ such that $\delta_B(f(a_1), f(a_2)) \geq \delta_A(a_1, a_2)$. We will call the category whose objects are sets with an L-valued equality and equality respecting maps $\hat{\mathcal{S}}(L)$.

There is a particularly nice forgetful functor from $U : \hat{\mathcal{S}}(L)$ to $\mathbf{Set}(L)$ taking (A, δ) to (A, ε), where $\varepsilon(a) = \delta(a, a)$. The conditions in the following theorem holds for t-norms and for two-sided quantales.

Theorem 3. *If L has $\bigwedge$ and we always have $a \star b \leq a \wedge b$, then the forgetful functor U is an initial structure functor: given any family of maps $f_\lambda : (B, \beta) \to U(A_\lambda, \delta_\lambda)$, where $\lambda \in \Lambda$, there is a largest structure δ_B with $U(B, \delta_B) = (B, \beta)$ making all of the maps $f_\lambda : (B, \delta_B) \to (A_\lambda, \delta_\lambda)$ equality respecting.*

Proof. We let

$$\delta_B(b, b') = \beta(b) \wedge \beta(b') \wedge \bigwedge_{\lambda \in \Lambda} \delta_\lambda(f_\lambda(b), f_\lambda(b'))$$

for the initial structure.
For each $\lambda \in \Lambda$ we have $\delta_\lambda(f_\lambda(b), f_\lambda(b')) \geq \bigwedge \delta_\lambda(f_\lambda(b), f_\lambda(b'))$ and thus $\delta_\lambda(f_\lambda(b), f_\lambda(b')) \geq \delta_B(b, b')$, so all of the f_λ are equality preserving.

Now if $U(B, \delta_B)$ has membership at b given by

$$\delta_B(b, b) = \beta(b) \wedge \beta(b) \wedge \bigwedge_{\lambda \in \Lambda} \delta_\lambda(f_\lambda(b), f_\lambda(b))$$

$$= \beta(b) \wedge \bigwedge_{\lambda \in \Lambda} \delta_\lambda(f_\lambda(b), f_\lambda(b))$$

$$= \beta(b)$$

since each λ has $\delta_\lambda(f_\lambda(b), f_\lambda(b)) \geq \beta(b)$.

We also need to show that δ_B is an L-valued equality:

- Strictness follows from

$$\delta_B(b, b') = \beta_B(b) \wedge \beta_B(b') \wedge \bigwedge_{\lambda \in \Lambda} \delta_\lambda(f_\lambda(b), f_\lambda(b'))$$

$$\leq \beta_B(b) \wedge \beta_B(b') \wedge \bigwedge_{\lambda \in \Lambda} \delta_\lambda(f_\lambda(b), f_\lambda(b))$$

$$\leq \beta_B(b) \wedge \bigwedge_{\lambda \in \Lambda} \delta_\lambda(f_\lambda(b), f_\lambda(b))$$

$$= \delta_B(b, b)$$

- Symmetry is obvious.
- Transitivity: Look at

$$\delta_B(b, b') \star \delta_B(b', b'') = \left(\beta_B(b) \wedge \beta_B(b') \wedge \bigwedge_{\lambda \in \Lambda} \delta_\lambda(f_\lambda(b), f_\lambda(b')) \right)$$

$$\star \left(\beta_B(b') \wedge \beta_B(b'') \wedge \bigwedge_{\lambda \in \Lambda} \delta_\lambda(f_\lambda(b'), f_\lambda(b'')) \right)$$

Now for each λ

$$\delta_B(b, b') \leq \delta_\lambda(f_\lambda(b), f_\lambda(b')$$

and

$$\delta_B(b', b'') \leq \delta_\lambda(f_\lambda(b'), f_\lambda(b'')$$

and

$$\delta_\lambda(f_\lambda(b), f_\lambda(b') \star \delta_\lambda(f_\lambda(b'), f_\lambda(b'') \leq \delta_\lambda(f_\lambda(b), f_\lambda(b'])$$

so

$$\delta_B(b, b') \star \delta_B(b', b'') \leq \bigwedge_{\lambda \in \Lambda} \delta_\lambda(f_\lambda(b), f_\lambda(b']).$$

We also have $\delta_B(b, b') \leq \beta(b)$ and $\delta_B(b', b'') \leq \beta(b'')$ so

$$\delta_B(b, b') \star \delta_B(b', b'') \leq \beta(b) \star \beta(b'') \leq \beta(b) \wedge \beta(b'')$$

which gives

$$\delta_B(b, b') \star \delta_B(b', b'') \leq \delta_B(b, b'').$$

Suppose that (B, γ) has $U(B, \gamma) = (B, \beta)$ and $f_\lambda : (B, \gamma) \to (A_\lambda, \delta_\lambda)$ are all equality preserving, then

$$\gamma(b, b') \le \gamma(b, b) = \beta(b)$$
$$\gamma(b, b') \le \gamma(b', b') = \beta(b')$$
$$\gamma(b.b') \le \delta_\lambda(f_\lambda(b), f_\lambda(b')) \text{ for each } \lambda \text{ so}$$
$$\gamma(b, b') \le \beta(b) \wedge \beta(b') \wedge \bigwedge_{\lambda \in \Lambda} \delta_\lambda(f_\lambda(b), f_\lambda(b'))$$
$$= \delta_B(b, b') \quad \square$$

Putting a monoidal structure on this category is straightforward if the $\star$ on L is tempered commutative, though none of this works if $\star$ is not tempered commutative:

Theorem 4. *If L has a tempered commutative, associative, bifunctorial connective $\star$ then $\hat{S}(L)$ has an up to coherent isomorphism associative bifunctor $\otimes$. If $\star$ is commutative then so is $\otimes$. If u is a unit for $\star$, then $\otimes$ has a unit as well so $\hat{S}(L)$ has a symmetric monoidal structure.*

Proof. On objects the bifunctor is defined as in Theorem 2.

The isomorphism involved in the commutativity and associativity is the one from the product structure on **Sets**. In order to get the bifunctoriality we start with $f : A \to C$ and $g : B \to D$ which respect the equality and see that the induced map $f \times g : A \times B \to C \times D$ also respects equality since $\delta_A(a, a') \le \delta_C(f(a), f(a'))$ and $\delta_B(b, b') \le \delta_D(g(b), g(b'))$ give $\delta_A(a, a')\star\delta_B(b, b') \le \delta_C(f(a), f(a'))\star\delta_D(g(b), g(b'))$. If u is a unit then $(\{*\}, u) \otimes (A, \delta_A) \approx (A, \delta_A)$. $\square$

In order to get a monoidal closed structure we need to impose a stronger form of transitivity on our structures.

Definition 4. An L-valued strong equality on a set A is a function $\delta : A \times A \to L$ such that

1. Strictness: For all $a, a' \in A$ we have $\delta(a, a') \le \delta(a, a)$.
2. Symmetry: For all $a, a' \in A$ we have $\delta(a, a') = \delta(a', a)$.
3. Strong transitivity: For every $a, a' \in A$,

$$\bigvee_{a'} (\delta(a, a')\star\delta(a', a'')) = \delta(a, a'')$$

If $\star = \wedge$ strong transitivity will follow for L-valued equalities (it is a consequence of strictness, transitivity, and $\delta(a, a') \wedge \delta(a, a) = \delta(a, a')$).

The category of sets with an L-valued strong equality and equality respecting functions has a monoidal structure provided $\star$ is tempered commutative. The obvious choice for a closed structure has an L-valued strong equality if we use strong transitivity on δ but it need not satisfy the transitive law if the δ is only transitive.

Theorem 5. *If L is complete and $\star$ is associative, tempered commutative, and distributes on both sides over $\bigvee$ then $(A, \delta_A) \otimes (B, \delta_B)$ satisfies the strong transitive law if both (A, δ_A) and (B, δ_B) do.*

Proof. Given that

$$\bigvee_{a'} (\delta_A(a, a')\star\delta_A(a', a'')) = \delta_A(a, a'')$$

and

$$\bigvee_{b'} (\delta_B(b, b')\star\delta_B(b', a'')) = \delta_B(b, b'')$$

we compute using associativity and tempered commutativity:

$$\bigvee_{a',b'} ((\delta_A(a,a')\star\delta_B(b,b'))\star(\delta_A(a',a'')\star\delta_B(b',b''))) = \bigvee_{a',b'} (\delta_A(a,a')\star(\delta_B(b,b')\star\delta_A(a',a''))\star\delta_B(b',a''))$$

$$= \bigvee_{a',b'} (\delta_A(a,a')\star(\delta_A(a',a'')\star\delta_B(b,b'))\star\delta_B(b',a''))$$

$$= \bigvee_{a',b'} (\delta_A(a,a')\star\delta_A(a',a''))\star(\delta_B(b,b')\star\delta_B(b',a''))$$

and then using the left distributive law for $\star$ over

$$\bigvee = \bigvee_{a'} \left((\delta_A(a,a')\star\delta_A(a',a''))\star\left(\bigvee_{b'} \delta_B(b,b')\star\delta_B(b',b'')\right)\right)$$

Now use the right distributive law to get

$$= \left(\bigvee_{a'}(\delta_A(a,a')\star\delta_A(a',a''))\right) \star \left(\bigvee_{b'}(\delta_B(b,b')\star\delta_B(b',b''))\right)$$

Now using strong transitivity of each part we get $\delta_A(a,a'')\star\delta_B(b,b'')$. $\square$

Notice that the strong transitive law has as a consequence that

$$\delta(a,a)\star\delta(a,a) = \delta(a,a),$$

severely restricting the choices for $\star$. In particular this excludes the possibility of non-crisp truth values for $\delta(a,a)$ in the cases where $\star$ is given by product or the Łukasiewicz t-norm on the lattice [0,1], hence ruling out this construction for important examples in fuzzy set theory.

Theorem 6. *If $(L,\star)$ is tempered commutative, associative, autonomous, strictly two sided, and complete then the category of sets with an L-valued strong equality and equality respecting maps is monoidal closed.*

Proof. The monoidal closed structure uses

$$\mathcal{H}om((A,\delta_A),(B,\delta_B)) = (\{f\in B^A | f \text{ respects equality}\}, \xi)$$

where

$$\xi(f,g) = \bigwedge_{a,a'\in A} (\delta_A(a,a')\to\delta_B(f(a),g(a'))).$$

This gives an $(L,\star)$-valued strong equality.

We need to show that ξ is strict, symmetric, and strongly transitive. Notice that since f respects the equality we get

$$\xi(f,f) = \bigwedge_{a,a'\in A} \delta_A(a,a')\to\delta_B(f(a),f(a')) = \top$$

Thus only the strong transitive law is non-trivial. We have

$$\xi(f,g)\star\xi(g,h) = \left(\bigwedge_{a,a'}(\delta_A(a,a')\to\delta_B(f(a),g(a')))\right)\star\left(\bigwedge_{a'',a'''}(\delta_A(a'',a''')\to\delta_B(g(a''),h(a''')))\right)$$

$$\leq \bigwedge_{a,a',a''} ((\delta_A(a,a')\to\delta_B(f(a),g(a')))\star(\delta_A(a',a'')\to\delta_B(g(a'),h(a''))))$$

Now for each triple a, a', a'' we have

$$\delta_A(a,a'') \geq \delta_A(a,a')\star\delta_A(a',a'')$$

and

$$(\delta_A(a, a') \to \delta_B(f(a), g(a'))) \star (\delta_A(a', a'') \to \delta_B(g(a'), h(a''))$$
$$\leq (\delta_A(a, a') \star \delta_A(a', a'') \to (\delta_B(f(a), g(a')) \star \delta_B(g(a'), h(a''))$$
$$\leq (\delta_A(a, a') \star \delta_A(a', a'') \to \delta_B(f(a), h(a'')))$$

Thus

$$\xi(f, g) \star \xi(g, h) \leq \bigwedge_{a, a', a''} (\delta_A(a, a') \star \delta_A(a', a'') \to \delta_B(f(a), h(a'')))$$

$$= \bigwedge_{a, a''} \left(\left(\bigvee_{a'} (\delta_A(a, a') \star \delta_A(a', a'')) \right) \to \delta_B(f(a), h(a'')) \right)$$

$$= \bigwedge_{a, a''} (\delta_A(a, a'') \to \delta_B(f(a), h(a'')))$$

$$= \xi(f, h)$$

Now because f respects equality we have $\xi(f, f) = \top$, so $\xi(f, f) \star \xi(f, h) = \xi(f, h)$, which shows that

$$\bigvee_g \xi(f, g) \star \xi(g, h) = \xi(f, h)$$

so we get strong transitivity of ξ.

Given $h : (C, \delta_C) \otimes (A, \delta_A) \to (B, \delta_B)$, we get the map $\hat{h} : (C, \delta_C) \to \mathcal{H}om((A, \delta_A), (B, \delta_B))$ with $\hat{h}(c)(a) = h(c, a)$ as in **Sets**. Given $k : (C, \delta_C) \to \mathcal{H}om((A, \delta_A), (B, \delta_B))$, we get $\underline{k} : (C, \delta_C) \otimes (A, \delta_A) \to (B, \delta_B)$ with $\underline{k}(c, a) = k(c)(a)$ again as in **Sets**. The equations and naturality for the monoidal closed structure adjunctions follow from those for **Sets**. What we need to show is that these maps are equality preserving.

Now $h : C \otimes A \to B$ respects equality when for all (a, a') we get

$$\delta_C(c, c') \star \delta_A(a, a') \leq \delta_B(h(c, a), h(c', a'))$$

or equivalently

$$\delta_C(c, c') \leq \delta_A(a, a') \to \delta(h(c, a), h(c', a'))$$

Since this holds for all (a, a') this is equivalent to

$$\delta_C(c, c') \leq \bigwedge_{a, a' \in A} (\delta_A(a, a') \to \delta_B(h(c, a), h(c', a'))) = \xi(\hat{h}(c), \hat{h}(c'))$$

so $\hat{h}$ respects equality if and only if h does. $\square$

Because of close ties to sheaf theory, it is more common to use L-valued relations which are single valued, total and respect the strong equalities as maps. We compose these using convolution. This puts some constraints on the possible $\star$ beyond associativity and distributivity over $\bigvee$. In order for the identity map on A, given by δ_A, to be an identity for composition we will need

$$\bigvee_{a' \in A} \delta_A(a, a') \star \delta_A(a', a'') = \delta(a, a'').$$

So we need strong transitivity and not just transitivity of δ in order for this to give a category. Indeed we need a strong form of extensionality as given by the first two conditions in the following definition:

Definition 5. A map of sets with an L-valued strong equality is an L-valued relation $f : A \times B \to L$ which

1. Composes on the left with identity: $\bigvee_{a'} \delta_A(a, a') \star f(a', b) = f(a, b)$.
2. Composes on the right with identity: $\bigvee_{b'} (f(a, b') \star \delta_B(b', b)) = f(a, b)$.

3. Is single valued: $f(a, b) \star f(a, b') \leq \delta_B(b, b')$.
4. Is total: $\bigvee_{b \in B} f(a, b) = \delta_A(a, a)$.

The category $(L, \star) - \mathbf{Set}$ has object sets with an $(L, \star)$-valued strong equality and has such relations as morphisms with composition given by convolution:

$$f \circ g(a, c) = \bigvee_{b \in B} f(a, b) \star g(b, c)$$

Loullis notes in [7] that such a relation $f : A \times B \to L$ can be represented by a function $\hat{f} : A \to B$ using $f(a, b) = \varepsilon_A(a) \wedge \delta_B(\hat{f}(a), b)$ when $\hat{f}$ respects equality and $\star = \wedge$. The somewhat weaker looking conditions he uses imply the strong ones we use in the complete Heyting algebra case he is considering.

We can get a monoidal structure here as well:

Theorem 7. *If L is complete and has a tempered commutative, associative bifunctorial connective $\star$ which distributes over $\bigvee$ on both sides then $(L, \star)$-**Set** has an up to isomorphism commutative, associative, bifunctor $\otimes$. If there is a unit for $\star$, then $\otimes$ has a unit as well.*

Proof. On objects the bifunctor is defined as in Theorem 2.

We need to see how the tensor product treats maps. Suppose that f is an $(L, \star) - \mathbf{Set}$ map from (A, δ_A) to (B, δ_B) and g is an $(L, \star) - \mathbf{Set}$ map from (C, δ_C) to (D, δ_D) then we get a map from $(A, \delta_A) \otimes (C, \delta_C)$ to $(B, \delta_B) \otimes (D, \delta_D)$ by $f \otimes g : (A \times C) \times (B \times D) \to L$ with

$$f \otimes g(((a, c), (b, d))) = f(a, b) \star g(c, d).$$

The bifunctoriality and the associativity and commutativity up to isomorphism are then straightforward.

We still need to show that this is indeed a map. What we need is

- $\bigvee_{a', c'} (\delta_A(a, a') \star \delta_C(c, c') \star f \otimes g((a, c), (b, d))) = f \otimes g(((a', c'), (b, d)))$.
  This is a computation which makes heavy use of associativity and tempered commutativity of $\star$ and distributivity over $\bigvee$ on both sides:

$$\bigvee_{a', c'} (\delta_A(a, a') \star \delta_C(c, c') \star f \otimes g((a, c), (b, d))) = \bigvee_{a', c'} ((\delta_A(a, a') \star \delta_C(c, c')) \star (f(a, b) \star g(c, d)))$$

$$= \bigvee_{a', c'} ((\delta_A(a, a') \star f(a, b)) \star (\delta_C(c, c') \star g(c, d)))$$

$$= \bigvee_{a'} (\delta_A(a, a') \star f(a, b)) \star \bigvee_{c'} (\delta_C(c, c') \star g(c, d)))$$

$$= f(a', b) \star f(c', d)$$

$$= f \otimes g(((a', c'), (b, d)))$$

- The computation for

$$\bigvee_{b', d'} (f \otimes g(((a, c), (b, d))) \star \delta_B(b, b') \star \delta_D(d, d')) = f \otimes g(((a, c), (b', d')))$$

  is similar but puts the δ's on the other side.
- $f \otimes g(((a, c), (b, d))) \star f \otimes g(((a, c), (b', d'))) \leq \delta_B(b, b') \star \delta_D(d, d')$.
  Here we need bifunctoriality, associativity, and tempered commutativity of $\star$ again

$$f \otimes g(((a, c), (b, d))) \star f \otimes g(((a, c), (b', d'))) = (f(a, b) \star g(c, d)) \star (f(a, b') \star g(c, d'))$$

$$= (f(a, b) \star f(a, b')) \star (g(c, d) \star g(c, d'))$$

$$\leq \delta_B(b, b') \star \delta_D(d, d')$$

- $\bigvee_{(b,d) \in B \times D} f \otimes g(((a, c), (b, d))) = \delta_A(a, a) \star \delta_C(c, c)$

This one only needs distributivity of $\star$ over $\bigvee$ on both sides:

$$\bigvee_{(b,d)\in B\times D} f\otimes g(((a,c),(b,d))) = \bigvee_{b\in B}\bigvee_{d\in D} f(a,b)\star g(c,d)$$

$$= \bigvee_{b\in B}\left(f(a,b)\star\bigvee_{d\in D} g(c,d)\right)$$

$$= \bigvee_{b\in B}(f(a,b)\star\delta_C(c,c))$$

$$= \left(\bigvee_{b\in B} f(a,b)\right)\star\delta_C(c,c)$$

$$= \delta_A(a,a)\star\delta_C(c,c)$$

To show that $D\otimes -$ behaves properly with respect to composition we need

$$\delta_D(d,d'')\star\bigvee_{b\in B}(f(a,b)\star g(b,c))$$

$$= \bigvee_{(d',b)\in D\times B}((\delta_D(d,d')\star f(a,b))\star(\delta_D(d',d'')\star g(b,c))).$$

This follows using tempered commutativity, associativity, distributivity and the composition property of the identity:

$$\bigvee_{(d',b)\in D\times B}((\delta_D(d,d')\star f(a,b))\star(\delta_D(d',d'')\star g(b,c)))$$

$$= \bigvee_{(d',b)\in D\times B}\delta_D(d,d')\star\delta_D(d',d'')\star f(a,b)\star g(b,c)$$

$$= \bigvee_{d'\in D}(\delta_D(d,d')\star\delta_D(d',d''))\star\bigvee_{b\in B}(f(a,b)\star g(b,c))$$

$$= \delta_D(d,d'')\star\bigvee_{b\in B}(f(a,b)\star g(b,c))$$

If u is a unit for $\star$ then $(\{*\}, u)$ will be a unit for $\otimes$. $\quad\square$

It has been known since the work of Higgs and Loullis that this category is a topos when L is a complete Heyting algebra and $\star = \wedge$. It is not clear how to get a monoidal closed structure in the (slightly) more general case considered here.

I would like to thank the referee for the help in finding the subtleties in the constructions in this section and for catching errors in an earlier draft.

3.2. Quantal sets

Mulvey and Nawaz [9] work over an idempotent right sided (generally non-commutative) quantale Q (a right Gelfand quantale). If Q is commutative and has a unit, this gives a locale and the construction gives the Higgs construction for Heyting-valued sets. Their definition of a Q-valued equality tempers symmetry with an extent of existence on the left:

Definition 6. If $(Q, \star)$ is a right Gelfand quantale then a quantal set (A, ε) is a set equipped with a function $\varepsilon : A\times A \to Q$ subject to the following conditions, where we write $E(a)$ for $\varepsilon(a,a)$:

1. $\varepsilon(a,a') \le E(a)\star E(a')$.
2. $E(a)\star\varepsilon(a',a) \le \varepsilon(a,a')$.
3. $\varepsilon(a,a')\star\varepsilon(a',a'') \le \varepsilon(a,a'')$.

Notice that this implies that $E(a)\star E(a) = E(a)$ using (1) and (3) with $a = a' = a''$.

They then define a map of quantal sets to be similarly tempered:

Definition 7. A map of quantal sets is a function $f : A \times B \to Q$ such that

1. $f(a, b) \le E(a)\star E(b)$.
2. $\varepsilon_A(a, a')\star f(a', b) \le f(a, b)$.
3. $f(a, b)\star\varepsilon_B(b, b') \le f(a, b')$.
4. $E(b)\star E(b')\star f(a, b)\star f(a, b') \le \varepsilon_B(b, b')$.
5. $E(a) \le \bigvee_b f(a, b)$.

Mulvey and Nawaz note (their Theorem 6) that in a right Gelfand quantale we always have the tempered commutativity $p\star q\star r = p\star r\star q$, so we are able to recover enough commutativity to define the monoidal structure in this case, largely because of the tempering with the existence predicates.

Proposition 8. *Given quantal sets* (A, ε_A) *and* (B, ε_B) *we get a quantal set*

$$(A, \varepsilon_A) \otimes (B, \varepsilon_B) = (A \times B, \varepsilon_A\star\varepsilon_B)$$

Proof. We need to check the properties of the equality relation on $A \times B$:

1. $\varepsilon_A(a, a')\star\varepsilon_B(b, b') \le E(a)\star E(b)\star E(a')\star E(b')$ follows from bifunctoriality of $\star$ since $\varepsilon_A(a, a') \le E(a)\star E(a')$ and $\varepsilon_B(b, b') \le E(b)\star E(b')$ so

$$\varepsilon_A(a, a')\star\varepsilon_B(b, b') \le E(a)\star E(a')\star E(b)\star E(b')$$
$$= E(a)\star E(b)\star E(a')\star E(b')$$

2. $E(a)\star E(b)\star\varepsilon(a', a)\star\varepsilon_B(b', b) \le \varepsilon_A(a, a')\star\varepsilon_B(b, b')$. This follows from bifunctoriality and tempered symmetry since

$$E(a)\star E(b)\star\varepsilon(a', a)\star\varepsilon_B(b', b) = E(a)\star\varepsilon(a', a)\star E(b)\star\varepsilon_B(b', b)$$

3. $\varepsilon_A(a, a')\star\varepsilon_B(b, b')\star\varepsilon_A(a', a'')\star\varepsilon_B(b', b'') \le \varepsilon(a, a'')\star\varepsilon_B(b, b'')$ This follows from bifunctoriality since

$$\varepsilon_A(a, a')\star\varepsilon_B(b, b')\star\varepsilon_A(a', a'')\star\varepsilon_B(b', b'')$$
$$= \varepsilon_A(a, a')\star\varepsilon_A(a', a'')\star\varepsilon_B(b, b')\star\varepsilon_B(b', b'') \qquad \square$$

In order to get a monoidal structure on the category of quantal sets we need maps of quantal sets for functoriality, associativity and unit (if Q has one). Again the tempering gives us the commutativity needed.

*3.3. Höhle's $Q-$ **Set***

In Höhle's [5] a different notion of quantale valued equivalence is used. He asks that Q be a strictly two sided commutative quantale and then makes the following definitions:

Definition 8. A quantale valued set is a set X equipped with a function $E : X \times X \to Q$ such that

1. $E(x, x)$ is a divisor of $E(x, y)$ with respect to $\star$.
2. $E(x, y) = E(y, x)$.
3. $E(x, y)\star(E(y, y) \to E(y, z)) \le E(x, z)$.

A morphism of quantale valued sets is then a structure preserving map $\phi : X \to Y$ such that

1. (Invariance of extent of existence) $E(x, x) = E(\phi(x), \phi(x))$.
2. (Preservation of equality) $E(x_1, x_2) \le E(\phi(x_1), \phi(x_2))$.

Höhle notes (p.11) that this gives a monoidal category with

$$E \otimes F((x_1, y_1), (x_2, y_2)) = E(x_1, x_2)\star F(y_1, y_2)$$

Transitivity of the resulting structure is bit subtle and makes use of the divisibility condition. He does not address the problem of whether or not there is a right adjoint to $-\otimes (Y, F)$ giving a closed structure.

He also shows that keeping only the extent of existence gives an underlying map to the comma category (which he writes $\mathbf{SET} \downarrow Q$) of sets with a map to Q and morphism preserving that map. Using that underlying functor we get that the category of quantale sets is topological over $\mathbf{SET} \downarrow Q$. This is not the category of fuzzy sets $\mathbf{Set}(Q)$ since that category allows extent of existence to rise under the action of a map.

Höhle's sheaves on involutive unital quantales [6] build an involution into this definition. He does not give a monoidal structure, but rather concentrates on the algebra structure for the singleton monad, noting that in the Heyting algebra case this gives a way of constructing sheaves.

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